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3. Torsion (Burulma)

$\gamma(s)$: Birim Hız eğri
(Formüller kolay oluyor: Bu durumda)

$t := \dot{\gamma}$ (Teğet / Tangent vektör)

Birim Hız $\Rightarrow \|\dot{\gamma}\| = 1$
$\|t\| = 1$

$\langle t, t \rangle = 1$
$\Rightarrow \langle \dot{t}, t \rangle = 0$

n : Eğrilik vektör

$n := \dfrac{\dot{t}}{\kappa}$

Not: $\dot{t} = \ddot{\gamma}$
$\Rightarrow \|n\| = \dfrac{\|\dot{t}\|}{\kappa} = \dfrac{\|\ddot{\gamma}\|}{\kappa} = 1$

Tanım (Binormal vektör)

γ : Birim Hız. O zaman,

$b := t \times n$

b ← Bi normal vektör.

Not a) t, n dik $\Rightarrow \|b\| = 1$

b) Eğer $\gamma \subset \mathbb{R}^2 \Rightarrow$ Hem $t, n \subset \mathbb{R}^2$
$\Rightarrow b = \pm e_3 \Rightarrow \dot{b} \equiv 0$.

İki Boyutlu eğri $\Rightarrow$ Burulma / Torsion $\equiv 0$

$\Rightarrow \langle \dot{t}, t \rangle = 0$

$\Rightarrow b = \pm e_3 \Rightarrow b \equiv 0.$ $\equiv 0$

İki Boyutlu eğri $\Rightarrow$ Burulma / Torsion

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Tartışma

Torsion / Burulma incelemek için $\dot{b}$ incelememiz lazım.

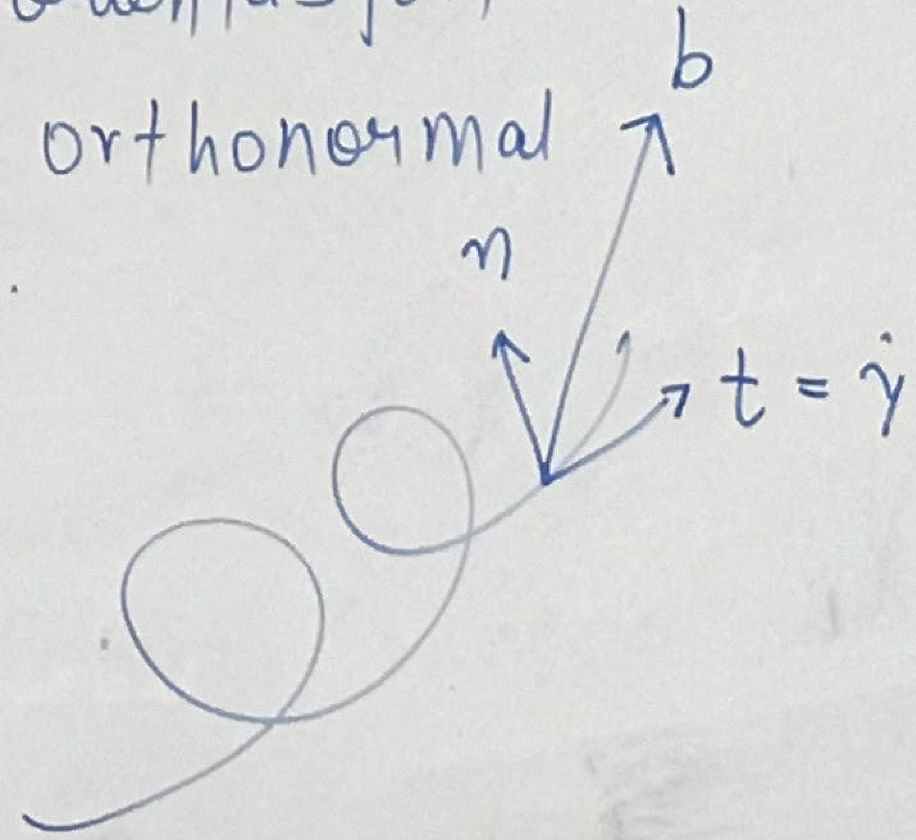
Hesaplama

$\{t, n, b\}$ sağ oryantasyon olan orthonormal baz.

$b = t \times n$

$\dot{b} = \dot{t} \times n + t \times \dot{n}$

$$\boxed{\begin{array}{l} t = \dot{\gamma}. \\ n = \dfrac{\dot{t}}{\kappa} = \dfrac{\ddot{\gamma}}{\kappa} \end{array}}$$



$\dot{b} = \ddot{\gamma} \times \dfrac{\ddot{\gamma}}{k} + t \times \dot{n} = t \times \dot{n}$

b : Uzunluk 1 $\quad \|b\| \equiv 1$

$\Rightarrow \langle b, \dot{b} \rangle \equiv 0$

Çünkü

$\langle b, b \rangle \equiv 1$

$\Rightarrow \dfrac{d}{ds} \langle b, b \rangle \equiv 0$

$\Rightarrow \langle \dot{b}, b \rangle \equiv 0$

Demek ki,

$\dot{b}$: Hem b, Hem t ile dik.

Demek ki,

$\dot{b}$, n ile paraleldir.

$\dot{b} = -\tau n$ şeklinde yazabiliriz.

τ: Torsion Denir.

Tanım 3.3

γ : Birim Hız eğri.

$b := t \times n$

O zaman bir sayı τ

için $b' = -\tau n$ dır.

τ : Torsion denir.

Örnek 3.4

Heliks $\gamma(t) = (a\cos t, a\sin t, bt)$

Birim Hız değil. Ama

$$\gamma(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$$

Birim Hız dır.

$$t = \dot{\gamma} = \left(\frac{-a\sin\frac{s}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}}, \frac{a\cos\frac{s}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$$

$$n = \frac{\ddot{\gamma}}{\|\ddot{\gamma}\|}$$

$$\ddot{\gamma} = \left(\frac{-a\cos\frac{s}{\sqrt{a^2+b^2}}}{a^2+b^2}, \frac{-a\sin\frac{s}{\sqrt{a^2+b^2}}}{a^2+b^2}, 0\right)$$

$$\|\ddot{\gamma}\| = \frac{a}{a^2+b^2}$$

$$\Rightarrow n = \left(-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$$

Birim hız dır.

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$$b = t \times n$$

$$= \frac{1}{\sqrt{a^2+b^2}} \begin{vmatrix} i & j & k \\ -a\sin(\cdot) & a\cos(\cdot) & b \\ -\cos(\cdot) & -\sin(\cdot) & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{a^2+b^2}} \left(b\sin(\cdot), -b\cos(\cdot), a \right)$$

$$\dot{b} = \frac{1}{a^2+b^2} \left(b\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), b\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0 \right)$$

$$= \frac{b}{a^2+b^2} \left(\cos\frac{s}{\sqrt{a^2+b^2}}, \sin\frac{s}{\sqrt{a^2+b^2}}, 0 \right)$$

$$\dot{b} = -\tau n \Rightarrow \tau = \frac{b}{a^2+b^2}$$

Heliks için τ sabittir.

Heliks için

$$\boxed{\kappa = \frac{a}{a^2+b^2} \qquad \tau = \frac{b}{a^2+b^2}}$$

Theorem 3.5

γ_1 ve γ_2 iki
birim hız eğri olsun.
Eğer
$$\kappa_1(s) \equiv \kappa_2(s)$$
ve $\tau_1(s) \equiv \tau_2(s)$ ise
o zaman bir uzaklık koruyan dönüşüm T altında,
$$\gamma_1 = T\gamma_2$$

Theorem 3.6 (τ için bir formula)

Her regular eğri için,
$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$$

İspat İlk olarak, γ birim hız olsun.

$$\Rightarrow \dot{b} = -\tau n \qquad \{t, n, b\}$$
$$\Rightarrow \tau = -n \cdot \dot{b}$$
$$= -n \cdot (t \times \dot{n})$$

Hatırlatma: $n = \dfrac{\ddot{\gamma}}{\kappa}$ $\quad \dot{n} = \dfrac{\dddot{\gamma}}{\kappa} - \dfrac{\ddot{\gamma}}{\kappa^2}\dfrac{d}{ds}(\kappa)$

$\gamma_1 = T\gamma_2$ Hatırlatma: $n = \gamma$ ...

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$$\Rightarrow \tau = -\frac{\ddot{\gamma}}{\kappa}\cdot\left[t \times \left(\frac{\dddot{\gamma}}{\kappa} - \frac{\ddot{\gamma}}{\kappa^2}\frac{d\kappa}{ds}\right)\right]$$

$$= -\frac{\ddot{\gamma}}{\kappa^2}\cdot\left[t \times \dddot{\gamma}\right]$$

$$= -\frac{\ddot{\gamma}\cdot\left[\dot{\gamma} \times \dddot{\gamma}\right]}{\kappa^2}$$

Not: γ birim hız olduğu için

$\|\dot{\gamma}\| = 1 \Rightarrow \dot{\gamma} \perp \ddot{\gamma}$

$\Rightarrow \|\dot{\gamma} \times \ddot{\gamma}\| = \|\ddot{\gamma}\|\|\dot{\gamma}\| = \|\ddot{\gamma}\| = \kappa$

$$\Rightarrow \tau = -\frac{\ddot{\gamma}\cdot\left[\dot{\gamma} \times \dddot{\gamma}\right]}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$$

$\boxed{\ddot{\gamma} = \gamma^{(2)}}$

İddia:

$-\ddot{\gamma}\cdot\left[\dot{\gamma} \times \dddot{\gamma}\right] = (\dot{\gamma} \times \ddot{\gamma})\cdot\dddot{\gamma}$

Not: $\ddot{\gamma}\cdot\left[\dot{\gamma} \times \ddot{\gamma}\right] = 0$

Türev alalım,

$\dddot{\gamma}\cdot\left[\dot{\gamma} \times \ddot{\gamma}\right] + \underbrace{\ddot{\gamma}\cdot\left[\ddot{\gamma} \times \ddot{\gamma}\right]}_{=0} + \ddot{\gamma}\cdot\left[\dot{\gamma} \times \dddot{\gamma}\right] = 0$

$\Rightarrow \gamma^{(3)}\cdot\left[\gamma^{(1)} \times \gamma^{(2)}\right] = -\gamma^{(2)}\cdot\left[\gamma^{(1)} \times \gamma^{(3)}\right]$

Sonuç $$\tau = \frac{(\gamma^{(1)} \times \gamma^{(2)})\cdot\gamma^{(3)}}{\|\gamma^{(1)} \times \gamma^{(2)}\|^2}$$ Birim hız ise

$\|\dot{\gamma}\|^3$

e θ

γ Birim Hız değilse. $\gamma = \gamma(t)$

s: Birim Hız Parametrizasyon olsun

$$\frac{d\gamma}{dt} = \frac{d\gamma}{ds}\,\frac{ds}{dt}$$

$$\frac{d^2\gamma}{dt^2} = \frac{d}{dt}\left(\frac{d\gamma}{ds}\,\frac{ds}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{d\gamma}{ds}\right)\frac{ds}{dt} + \frac{d\gamma}{ds}\,\frac{d^2s}{dt^2}$$

$$= \frac{\frac{d}{ds}\left(\frac{d\gamma}{ds}\right)}{dt/ds}\,\frac{ds}{dt} + \frac{d\gamma}{ds}\,\frac{d^2s}{dt^2}$$

$$\Rightarrow \frac{d^2\gamma}{dt^2} = \frac{d^2\gamma}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\gamma}{ds}\,\frac{d^2s}{dt^2}$$

$$\frac{d^3\gamma}{dt^3} = \left(\frac{ds}{dt}\right)^3\frac{d^3\gamma}{ds^3} + 3\,\frac{ds}{dt}\,\frac{d^2s}{dt^2}\,\frac{d^2\gamma}{ds^2} + \frac{d^3s}{dt^3}\,\frac{d\gamma}{ds}$$

$$\Rightarrow \left(\frac{d\gamma}{dt} \times \frac{d^2\gamma}{dt^2}\right) = \left(\frac{ds}{dt}\right)^3\left(\frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2}\right)$$

$$\Rightarrow \frac{\frac{d^3\gamma}{dt^3}\cdot\left(\frac{d\gamma}{dt} \times \frac{d^2\gamma}{dt^2}\right)}{\left\|\frac{d\gamma}{dt} \times \frac{d^2\gamma}{dt^2}\right\|^2} = \frac{\cancel{\left(\frac{ds}{dt}\right)^6}\,\frac{d^3\gamma}{ds^3}\cdot\left(\frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2}\right)}{\cancel{\left(\frac{ds}{dt}\right)^6}\left\|\frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2}\right\|^2}$$

$$= \tau \quad \square$$

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$$\tau = \frac{(\gamma^{(1)} \times \gamma^{(2)}) \cdot \gamma^{(3)}}{\| \gamma^{(1)} \times \gamma^{(2)} \|^2}$$

örnek 3.7

$$\gamma(t) = (1, t, t^2)$$

$$\left(\begin{array}{c} z = y^2 \\ x = 1 \end{array} \cap \right)$$

$$k = \frac{\| \dot\gamma \times \ddot\gamma \|}{\| \dot\gamma \|^3}$$

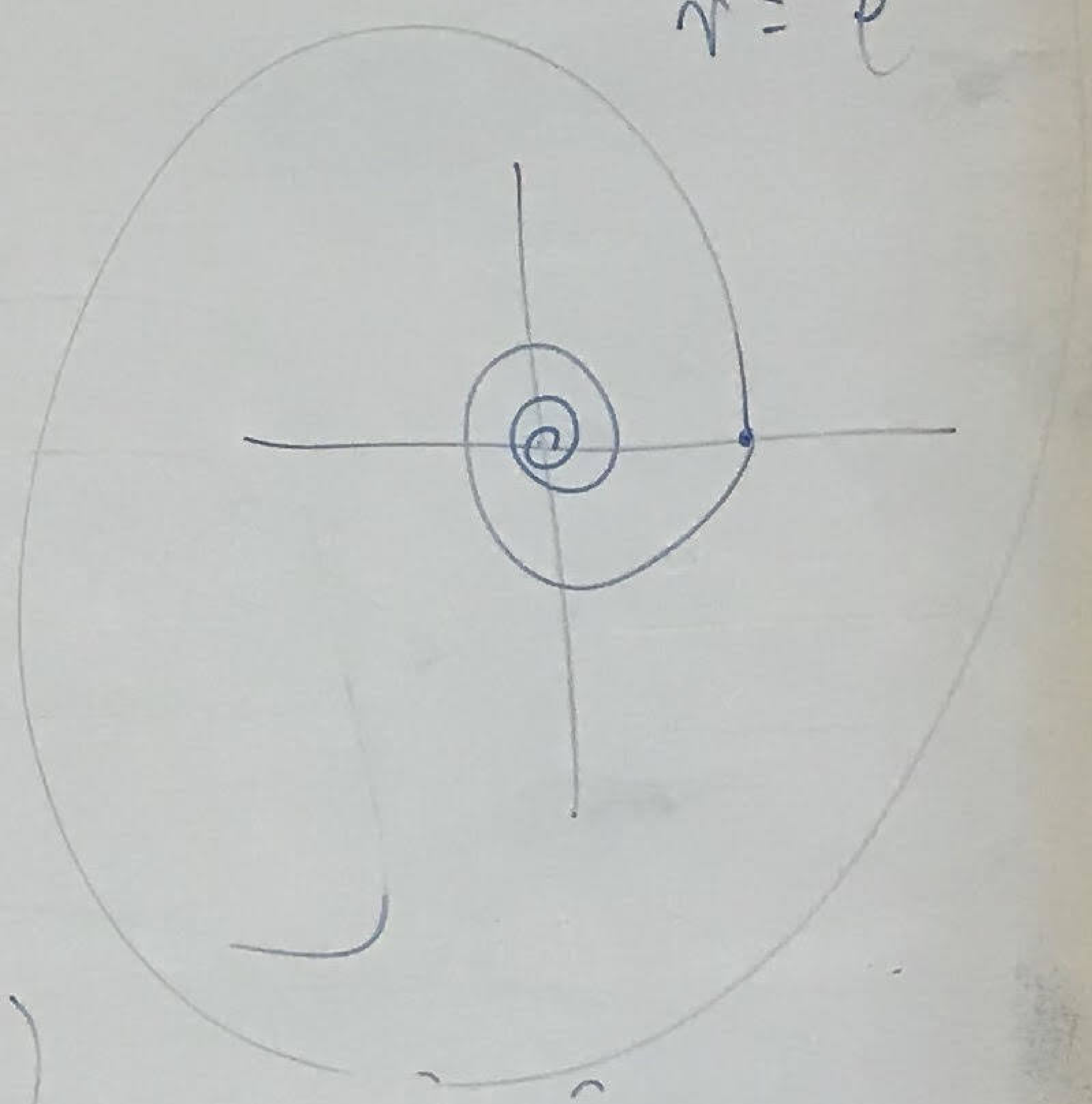
$$\dot\gamma(t) = (0, 1, 2t) \qquad \Rightarrow \quad k = \frac{2}{(1+4t^2)^{3/2}}$$

$$\ddot\gamma(t) = (0, 0, 2)$$

$$\gamma^{(3)}(t) = 0.$$

$$\Rightarrow \tau = 0.$$

$$r = e^{\theta}$$



örnek 3.8

$$\gamma(\theta) = (e^{\theta}\cos\theta, e^{\theta}\sin\theta, e^{\theta})$$

$$\dot\gamma(\theta) = \big(e^{\theta}(\cos\theta - \sin\theta), \; e^{\theta}(\sin\theta + \cos\theta), \; e^{\theta}\big)$$

$$\ddot\gamma(\theta) = (-2e^{\theta}\sin\theta, 2e^{\theta}\cos\theta, e^{\theta})$$

$$\dddot\gamma(\theta) = \big(-2e^{\theta}(\sin\theta + \cos\theta), -2e^{\theta}(\sin\theta - \cos\theta), e^{\theta}\big)$$

Sonuç $\tau = \dfrac{\ldots}{\| y^{(1)} \times y^{(2)} \|^2}$ HIZ ise

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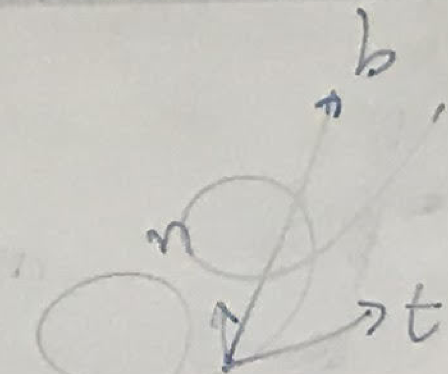
3.9 Frenet - Serret Matris.

$\{t, n, b\}$



$$\begin{bmatrix} \dot{t} \\ \dot{n} \\ \dot{b} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}}_{\text{Frenet - Serret Matris}} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

$A^t = -A$ Anti-Simetrik Matris.

$n = \dfrac{\dot{t}}{\kappa} \Rightarrow \dot{t} = \kappa n$

$\dot{b} = -\tau n$

Soru

$\dot{n} = ?$

$n = b \times t$

$\Rightarrow \dot{n} = \dot{b} \times t + b \times \dot{t}$

$= -\tau n \times t + \kappa b \times n$

$= \tau b - \kappa t$

Uygulama

Bir Eğri ne zaman kürenin üzerinde olacak?