III

Gauss: Theorema Egregium

①

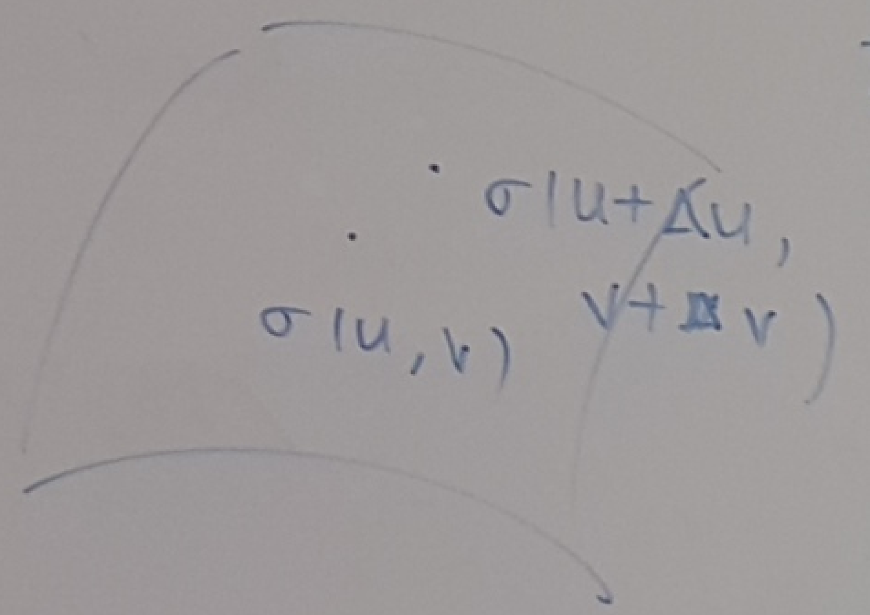
Hatırlatma

$\sigma : U \longrightarrow \mathbb{R}^3$ Düzgün Regular Yüzey

Regular: σ_u, σ_v bağımsız

$\sim$ $\sigma_u \times \sigma_v \neq 0$

Eğriliği



$$\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v) \simeq \sigma_u(u,v)\,\Delta u + \sigma_v(u,v)\,\Delta v + \sigma_{uu}(u,v)(\Delta u)^2 + 2\sigma_{uv}(u,v)\,\Delta u\,\Delta v + \sigma_{vv}(u,v)(\Delta v)^2$$

$$\overline{N}\cdot\left(\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)\right) = \overline{N}\cdot\sigma_{uu}(\Delta u)^2 + 2\overline{N}\cdot\sigma_{uv}(\Delta u)(\Delta v) + \overline{N}\cdot\sigma_{vv}(\Delta v)^2$$

$L = \sigma_{uu}\cdot\overline{N}$

$M = \sigma_{uv}\cdot\overline{N}$

$N = \sigma_{vv}\cdot\overline{N}$

$\overline{N}$: Birim Normal vektör

İkinci Temel Formu: $L\,du^2 + 2M\,du\,dv + N(dv)^2$

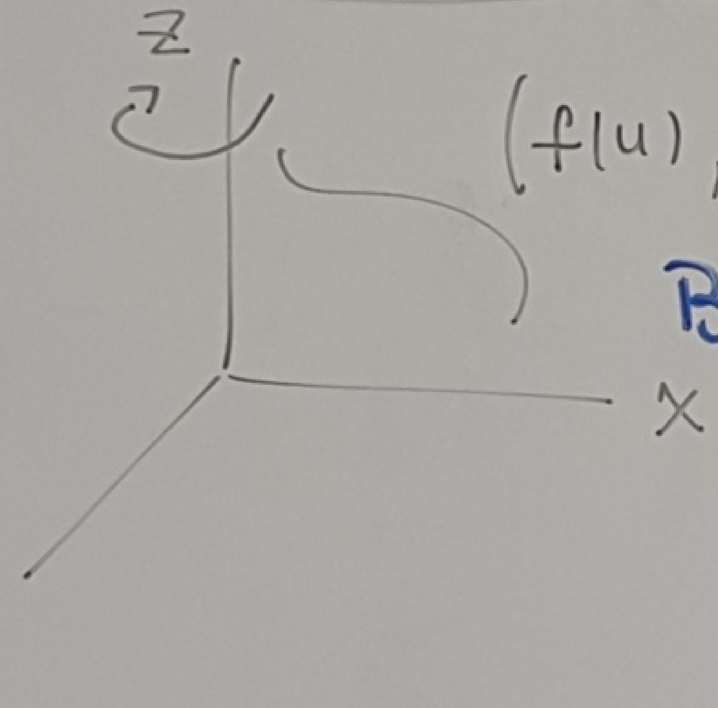
$\sim$ Teğet Düzlem üzerinde iç çarpım

$$\langle\langle \alpha_1\sigma_u + \beta_1\sigma_v, \alpha_2\sigma_u + \beta_2\sigma_v \rangle\rangle := \alpha_1\alpha_2 L + (\alpha_1\beta_2 + \beta_1\alpha_2) M + \beta_1\beta_2 N$$

$B_1\ B_2\ N$

- Parallel Transport
- Soap Bubbles : Sabun köpüğü
- Gauss – Bonnet Teoremi (Atiyah – Singer Teoremi)

Örnek Çevirilmiş yüzeyler



$(f(u), 0, g(u))$

$\sigma(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u))$

Birinci Temel formu

$\sigma_u = (f'(u)\cos v,\ f'(u)\sin v,\ g'(u))$

$\sigma_v = (-f(u)\sin v,\ f(u)\cos v,\ 0)$

$E\,du^2 + 2F\,du\,dv + G\,dv^2$

$E = \sigma_u \cdot \sigma_u \qquad F = \sigma_u \cdot \sigma_v \qquad G = \sigma_v \cdot \sigma_v$

Birinci Temel formu

$$\boxed{(\dot f^2 + \dot g^2)\,du^2 + f^2 dv^2 = du^2 + f^2 dv^2}$$

İkinci Temel formu (Birim hız $\Rightarrow$ $\dot f^2 + \dot g^2 = 1$)

$$\bar N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

Birim hız $\sigma_u \times \sigma_v = (-\dot g f \cos v,\ -\dot g f \sin v,\ f \dot f)$

$\|\sigma_u \times \sigma_v\| = \sqrt{\dot g^2 f^2 + \dot f^2 f^2} = f$

$\bar N = (-\dot g \cos v,\ -\dot g \sin v,\ \dot f)$

$\sigma_{uu} = (\ddot f \cos v,\ \ddot f \sin v,\ \ddot g)$

$\sigma_{uv} = (-\dot f \sin v,\ \dot f \cos v,\ 0)$

$\sigma_{vv} = (-f \cos v,\ -f \sin v,\ 0)$

$L = \sigma_{uu} \cdot \overline{N} = -\ddot{f}\dot{g} + \dot{f}\ddot{g}$

$M = \sigma_{uv} \cdot \overline{N} = 0$

$N = \sigma_{vv} \cdot \overline{N} = f\dot{g}$

İkinci Temel formu

$$\boxed{(\dot{f}\ddot{g} - \ddot{f}\dot{g})\, du^2 + f\dot{g}\, dv^2}$$

Birinci Temel Formu $\longrightarrow$ Mesafeler, Açılar, Alanlar.

Teorem 1 $\sigma(u,v)$ S_1 için bir chart

$\tilde{\sigma}(u,v)$ S_2 için bir chart

$(\sigma, \tilde{\sigma} : U \longrightarrow \mathbb{R}^3)$

$\sigma(u,v) \xrightarrow{\Phi} \tilde{\sigma}(u,v)$

$\Phi : S_1 \longrightarrow S_2$

O zaman bunlar denktir

1. Φ mesafeler koruyor
2. Birinci Temel formlar aynı.

Teorem 2 $\sigma(u,v) \xrightarrow{\Phi} \tilde{\sigma}(u,v)$

Bunlar denk

1. Φ açılar koruyor
2. $\tilde{E}\, d\tilde{u}^2 + 2\tilde{F}\, d\tilde{u}\, d\tilde{v} + \tilde{G}\, d\tilde{v}^2 = \lambda(p)\left[E\, du^2 + 2F\, du\, dv + G\, dv^2\right]$

$E\,du^2 + 2F\,du\,dv + G\,dv^2$

Teorem 3 $\quad \sigma(u,v) \xrightarrow{\Phi} \tilde{\sigma}(u,v)$

Bunlar denk

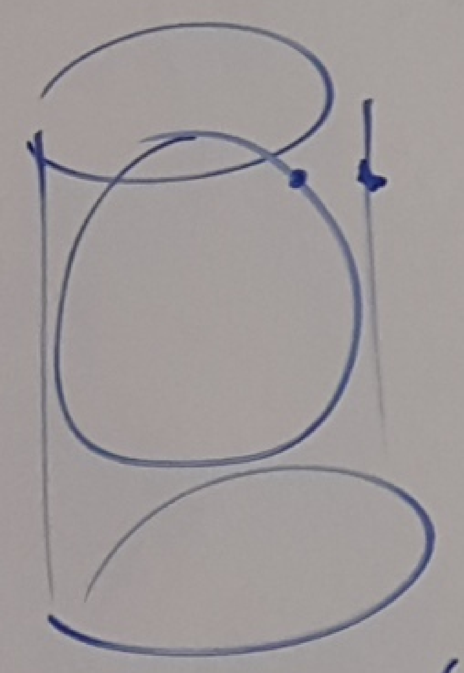
1. Φ alanları koruyor
2. $EG - F^2 = \tilde{E}\tilde{G} - \tilde{F}^2$

$$|\sigma(A)| = \int_A \|\sigma_u \times \sigma_v\| \, du\, dv$$

$$\|\sigma_u \times \sigma_v\|^2 = (\sigma_u \cdot \sigma_u)(\sigma_v \cdot \sigma_v) - (\sigma_u \cdot \sigma_v)^2 = EG - F^2$$

Uygulama

$S^2: \quad du^2 + \cos^2 u\, dv^2$ (BTF)



Archimedes dönüşüm

$$\sigma(u,v) \longrightarrow (\cos v,\ \sin v,\ \sin u)$$

$(\cos u \cos v,\ \cos u \sin v,\ \sin u)$

$\tilde{\sigma}(u,v) = (\cos v,\ \sin v,\ \sin u)$ ④

Buna göre BTF

$\tilde{\sigma}_u = (0, 0, \cos u)$

$\tilde{\sigma}_v = (-\sin v, \cos v, 0)$

$$\tilde{E}\,du^2 + 2\tilde{F}\,du\,dv + \tilde{G}\,dv^2 = \cos^2 u\, du^2 + dv^2$$

Not $\quad \left. EG - F^2 = \tilde{E}\tilde{G} - \tilde{F}^2 \right\}$ Archimedes Teoremi

$\downarrow \qquad\qquad \downarrow$

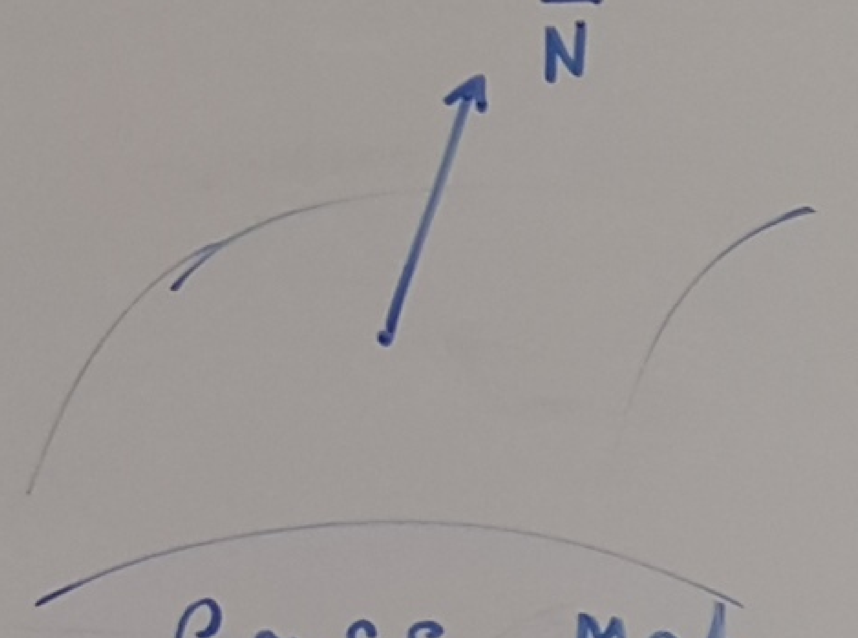
$S^2 \qquad\quad S^1 \times [-1, 1]$

... geometrik
şeklinde anlamak mümkün.

$aF + bG = \mu$
$cF + dG = N$

(5)

III . 2 İkinci Temel formu için bir Geometrik ifadesi



Gauss Map $\mathcal{G}$

$$\sigma(u,v) \longrightarrow \bar{N}(\sigma(u,v))$$

$$S \longrightarrow S^2$$

$\sigma(u,v)$ noktadaki teğet düzlem

$\bar{N}$ noktadaki (S^2 içinde) teğet düzlem

$T_{\sigma(u,v)}$ ve $T_{\bar{N}}$ aynı düzlemdir.

$\Rightarrow$ $\mathcal{W} := -D\mathcal{G}$ bunu $T_{\sigma(u,v)} \longrightarrow T_{\sigma(u,v)}$ üzerinde bir lineer dönüşüm olarak düşünebiliriz.

WEINGARTEN

$$\mathcal{W}(\sigma_u) = -\bar{N}_u$$
$$\mathcal{W}(\sigma_v) = -\bar{N}_v$$

Teorem $\langle\langle \xi, \eta \rangle\rangle = \langle \mathcal{W}\xi, \eta \rangle$ $\quad \xi, \eta \in T_{\sigma(u,v)}$

İspat Bunu göstermek yeter

$$L = \langle\langle \sigma_u, \sigma_u \rangle\rangle = \langle \mathcal{W}\sigma_u, \sigma_u \rangle$$
$$M = \langle\langle \sigma_u, \sigma_v \rangle\rangle = \langle \mathcal{W}\sigma_v, \sigma_v \rangle$$
$$N = \langle\langle \sigma_v, \sigma_v \rangle\rangle = \langle \mathcal{W}\sigma_v, \sigma_v \rangle$$

$N = \langle\langle \sigma_v, \sigma_v \rangle\rangle = \langle \mathcal{W}\sigma_v, \sigma_v \rangle$

$$\mathcal{W}(\sigma_u) = -\overline{N}_u$$

$$\langle \mathcal{W}(\sigma_u), \sigma_u \rangle = -\overline{N}_u \cdot \sigma_u$$

$$\sigma_u \cdot \overline{N} = 0$$

$$\Rightarrow \frac{d}{du}(\sigma_u \cdot \overline{N}) = 0$$

$$\Rightarrow \sigma_{uu} \cdot \overline{N} + \sigma_u \cdot \overline{N}_u = 0$$

$$\ni \quad -\overline{N}_u \cdot \sigma_u = \sigma_{uu} \cdot \overline{N} = L$$

i.e

$$L = \langle \sigma_{uu}, \overline{N} \rangle = \langle -\overline{N}_u, \sigma_u \rangle = \langle \mathcal{W}(\sigma_u), \sigma_u \rangle$$

$$M = \sigma_{uv} \cdot \overline{N} = -\sigma_v \cdot \overline{N}_u = \langle \mathcal{W}(\sigma_u), \sigma_v \rangle$$

$$\langle\langle \sigma_v, \sigma_v \rangle\rangle = N = \sigma_{vv} \cdot \overline{N} = -\sigma_v \cdot \overline{N}_v = \langle \mathcal{W}(\sigma_v), \sigma_v \rangle$$

Not İkinci Temel formu Analitik şeklinde tanımlamıştık. Ama bunu Geometrik şeklinde anlamak mümkün.

Tartışma II.2.2

Weingarten dönüşüm için bir matris formula.

$$\mathcal{W} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

i.e

$$\left.\begin{aligned} \mathcal{W}(\sigma_u) &= a\sigma_u + b\sigma_v \\ \mathcal{W}(\sigma_v) &= c\sigma_u + d\sigma_v \end{aligned}\right\} \text{ } \sigma_u, \sigma_v \text{ ile iç çarpım alalım}$$

$$\mathcal{W}(\sigma_u) = -N_u$$

$$\mathcal{W}(\sigma_u) \cdot \sigma_u = -N_u \cdot \sigma_u = N \cdot \sigma_{uu} = L$$

$$\|$$

$$a\,\sigma_u \cdot \sigma_u + b\,\sigma_v \cdot \sigma_u$$

$$\Rightarrow \boxed{\begin{aligned} aE + bF &= L \\ cE + dF &= M \\ aF + bG &= M \\ cF + dG &= N \end{aligned}}$$

$$\Rightarrow \underbrace{\begin{pmatrix} L & M \\ M & N \end{pmatrix}}_{\mathcal{F}_{II}} = \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{\mathcal{F}_{I}} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \mathcal{W} = \mathcal{F}_{I}^{-1} \mathcal{F}_{II}$$

$$= \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Tanım III.2.3

Gaussian Eğriliği $:= \det(\mathcal{W})$

$$= \frac{LN - M^2}{EG - F^2}$$

Γ^{i}_{jk} : Christoffel Katsayıları

$$\Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_v}{2(EG-F^2)},$$

$$\Rightarrow \underbrace{\begin{pmatrix} L & M \\ M & N \end{pmatrix}}_{\mathcal{F}_{II}} = \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{\mathcal{F}_{I}} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \mathcal{W} = \mathcal{F}_I^{-1} \, \mathcal{F}_{II}$$

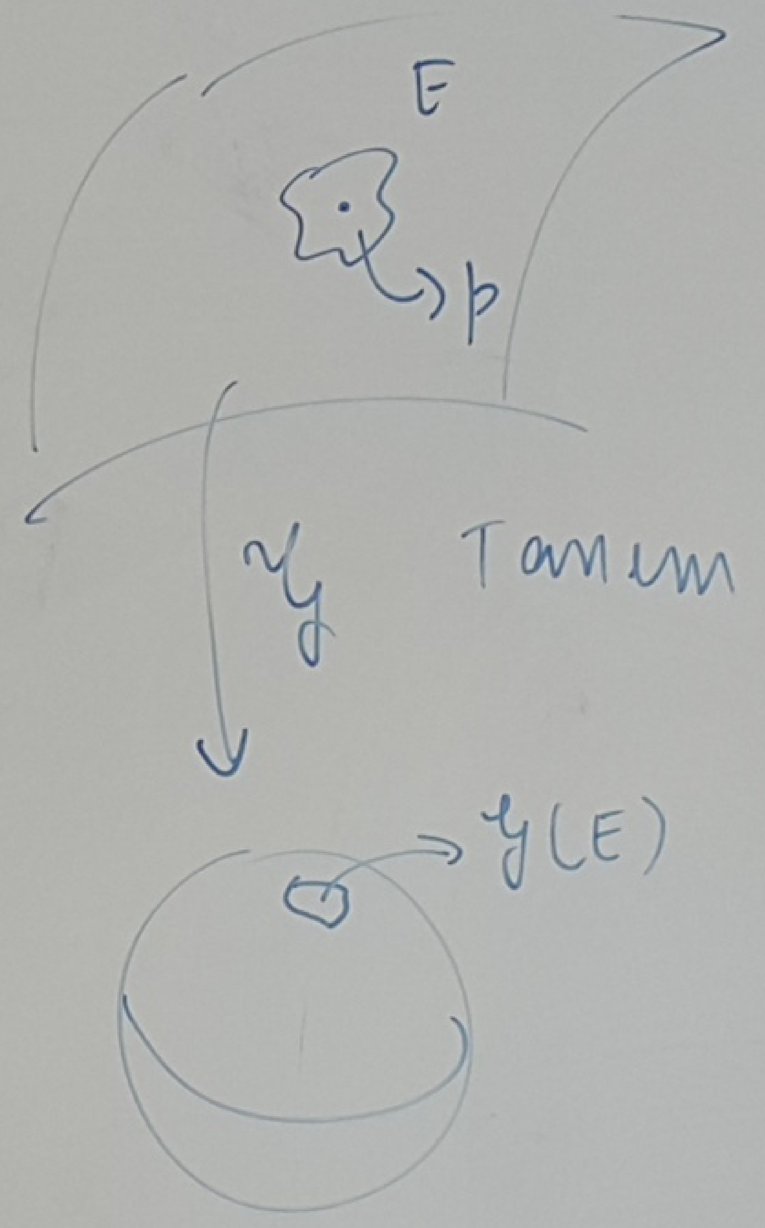
$$= \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Tanım III.2.3

Gaussian Eğriliği $:= \det(\mathcal{W})$

$$= \frac{LN - M^2}{EG - F^2}$$

Klasik Tanım (Gauss) ⑦



E : p ye içeren küçük bölge

$\gamma(E) \subset S^2$

Tanım: $K := \lim\limits_{|E| \to 0} \dfrac{|\gamma(E)|}{|E|}$

Not Düzlem için $K \equiv 0$

RS^2 için $K \equiv \dfrac{1}{R^2}$

$$EG - F^2$$

III.2.4 (Theorema Egregium)

Gaussian eğriliği yerel izometriler altında sabit kalır.

Ya da

Gaussian eğriliği (K) ikinci temel form ile bağımsız.

İspat Birinci adım

$$\begin{bmatrix} \sigma_{uu} \\ \sigma_{uv} \\ \sigma_{vv} \end{bmatrix} = \begin{bmatrix} \Gamma^1_{11} & \Gamma^2_{11} & L \\ \Gamma^1_{12} & \Gamma^2_{12} & M \\ \Gamma^1_{22} & \Gamma^2_{22} & N \end{bmatrix} \begin{bmatrix} \sigma_u \\ \sigma_v \\ \overline{N} \end{bmatrix}$$

Γ^i_{jk} : Christoffel katsayıları

Burada

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}$$

$$\Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}$$

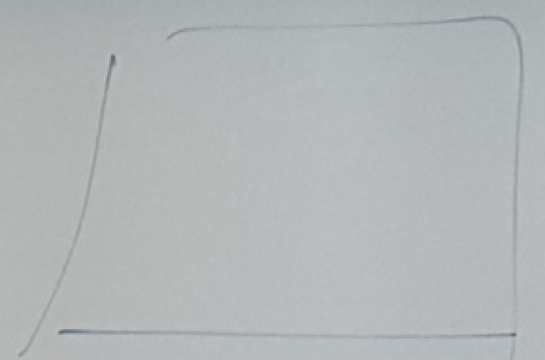
$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)}$$

$$\Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_v}{2(EG - F^2)}$$

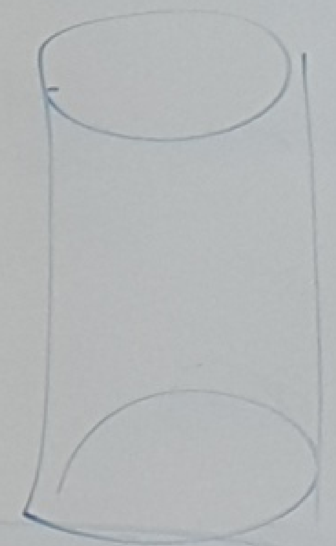
Düzlem



$du^2 + dv^2$ 8

Aynı

$S^1 \times \mathbb{R}$



$du^2 + dv^2$

$(\cos u, \sin u, v)$

Hepsi

SADECE

Birinci temel formu ile bağımlı.

İspat $\{\sigma_u, \sigma_v, \overline{N}\}$ bir baz

$\Rightarrow \sigma_{uu} = a\sigma_u + b\sigma_v + c\overline{N}$ şeklinde yazabiliriz

$\Rightarrow L = \sigma_{uu} \cdot \overline{N} = c$

$\sigma_{uu} \cdot \sigma_u = aE + bF$

$\sigma_{uu} \cdot \sigma_v = aF + bG$

a, b nedir?

$\sigma_u \cdot \sigma_u = E \Rightarrow \sigma_u \cdot \sigma_{uu} = \frac{E_u}{2}$ (*)

$\Rightarrow \sigma_u \cdot \sigma_{uv} = \frac{E_v}{2}$

$\sigma_u \cdot \sigma_v = F \Rightarrow \sigma_{uu} \cdot \sigma_v + \sigma_u \cdot \sigma_{uv} = F_u$

$\Rightarrow \sigma_{uu} \cdot \sigma_u = \frac{E_u}{2} = aE + bF$

$\sigma_{uu} \cdot \sigma_v = F_u - \frac{E_v}{2} = aF + bG$

$$\Rightarrow \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} E_u/2 \\ F_u - E_v/2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} E_u/2 \\ F_u - E_v/2 \end{pmatrix}$$

$$\Rightarrow a = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \quad (\Gamma^1_{11})$$

$$b = \frac{-FE_u + 2EF_u - EE_v}{2(EG - F^2)} \quad (\Gamma^2_{11})$$

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İkinci Adım

Codazzi - Minardi Denklemler

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- $L_v - M$

$\overline{N}_v : \overline{N}$ ile dik.

$\overline{N}$ deki katsayılar eşit olmak lazım.

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İspat $\{\sigma_u, \sigma_v, \overline{N}\}$ bir baz

$\Rightarrow \sigma_{uu} = a\sigma_u + b\sigma_v + c\overline{N}$ şeklinde yazabiliriz.

$\Rightarrow L = \sigma_{uu}\cdot\overline{N} = c$

$\sigma_{uu}\cdot\sigma_u = aE + bF$

$\sigma_{uu}\cdot\sigma_v = aF + bG$

a, b nedir?

$\sigma_u\cdot\sigma_u = E \Rightarrow \sigma_u\cdot\sigma_{uu} = \dfrac{E_u}{2}$ $(*)$

$\Rightarrow \sigma_u\cdot\sigma_{uv} = \dfrac{E_v}{2}$

$\sigma_u\cdot\sigma_v = F \Rightarrow \sigma_{uu}\cdot\sigma_v + \sigma_u\cdot\sigma_{uv} = F_u$

$\Rightarrow \sigma_{uu}\cdot\sigma_v = F_u - \dfrac{E_v}{2}$ $(**)$

$\Rightarrow \sigma_{uu}\cdot\sigma_u = \dfrac{E_u}{2} = aE + bF$

$\sigma_{uu}\cdot\sigma_v = F_u - \dfrac{E_v}{2} = aF + bG$

$\Rightarrow \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} E_u/2 \\ F_u - E_v/2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \dfrac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} E_u/2 \\ F_u - E_v/2 \end{pmatrix}$

$\Rightarrow a = \dfrac{GE_u - 2FF_u + FE_v}{2(EG-F^2)}$ (Γ^1_{11})

$b = \dfrac{-FE_u + 2EF_u - EE_v}{2(EG-F^2)}$ (Γ^2_{11})

Başka hesaplamalar aynı.

$\Rightarrow \sigma_{uu} \cdot \sigma_v = F_u - \frac{E_v}{2}$ (* *)

Başka hesaplamalar aynı.

(12)

İkinci Adım

Codazzi - Minardi Denklemler

- $L_v - M_u = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}$
- $M_v - N_u = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}$

Lütfen hesaplamak için çalışınız

Fikir

$$(\sigma_{uu})_v = (\sigma_{uv})_u$$

$$\Rightarrow (\Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + L\overline{N})_v = (\Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + M\overline{N})_u$$

$\sigma_{uv}, \sigma_{vv} \longrightarrow$ Matris kullanınız

$\overline{N}_v$: $\overline{N}$ ile dik.

$\overline{N}$ deki katsayılar eşit olmak lazım.

$$\overline{N}_v = a\sigma_u + b\sigma_v + \underset{\underset{0}{\uparrow}}{c}\overline{N}$$