

Hatırlatma

$E \subset \mathbb{R}^n$ Jordan Bölgesi

$f: E \rightarrow \mathbb{R}$, f E dışında
'0' olarak uzatalım.

Eğer her $\epsilon > 0$ için

E' 'ye kapsayan bir dik-dörtgen
 R üzerinde

$$U(f; \mathcal{E}) - L(f; \mathcal{E}) < \epsilon$$

Sağlayan bir ϵ varsa, f
İntegrelenbilir denir.

$$U(f; \mathcal{E}) = \sum_{R_j \cap E \neq \emptyset} M_j |R_j|$$

$$R_j \cap E \neq \emptyset$$

$$L(f; \mathcal{E}) = \sum_{R_j \cap E \neq \emptyset} m_j |R_j|$$

$$R_j \cap E \neq \emptyset$$

$$M_j = \sup_{x \in R_j} f(x)$$

$$x \in R_j$$

$$m_j = \inf_{x \in R_j} f(x)$$

$$x \in R_j$$

Integral

$$\int_E f$$

ya da yazabiliriz.

$$U(f+g; \mathcal{E}) \leq U(f; \mathcal{E}) + U(g; \mathcal{E})$$

$$L(f+g; \mathcal{E}) \geq L(f; \mathcal{E}) + L(g; \mathcal{E})$$

$$\mathcal{E} = \{R_1, \dots, R_p\} \quad (1)$$

Sağlayan bir ağ varsa, f
İntegrelenebilir denir.

Integral $\int_E f$ ya da $\int f$

Not ① $f \in \mathbb{R}$ üzerinde df için
Sürekli ise, f E üzerinde
İntegrelenebilir.

② $\mathbb{R}$: Jordan. 0 Zaman

$$\int_E 1 = \text{Vol}(E)$$

Integral'ın Bazı Özellikleri ②

Theorem $f, g: E \rightarrow \mathbb{R}$ İntegrelenebilir

$\hookrightarrow$ Jordan

ise $f + \alpha g$ ($\alpha \in \mathbb{R}$) İntegrelenebilir.

İspat $U(f; \mathcal{P}) - L(f; \mathcal{P}) < \frac{\epsilon}{2}$

ve $U(g; \mathcal{P}) - L(g; \mathcal{P}) < \frac{\epsilon}{2}$ Sağlayan

$\mathcal{P} \supset E$ üzerinde bir ağ olsun.

0 Zaman,

$$U(f+g; \mathcal{P}) \leq U(f; \mathcal{P}) + U(g; \mathcal{P})$$

$$L(f+g; \mathcal{P}) \geq L(f; \mathcal{P}) + L(g; \mathcal{P})$$

Aynı en

$$L_{E_1 \cup E_2}(f; \eta) \geq -\frac{\epsilon}{6} + L_{E_1}(f; \eta) + L_{E_2}(f; \eta)$$

$$+ [U_{E_2}(f; \eta) - L_{E_2}(f; \eta)]$$

$\leq \epsilon$

□

Theorem

$E_1, E_2 \subset \mathbb{R} \subset \mathbb{R}^n$ iki

Jordan Bölg. $E_1 \cap E_2$

Sıfır hacimli olsun.

f E_1 ve E_2 üzerinde

İntegrellenebilir olsun.

O zaman f $E_1 \cup E_2$ üzerinde
İntegrellenebilir ve

$$\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$$

İspat $\epsilon > 0$ olsun. Oylar bir η

Seçelim ki

$$U_{E_1}(f; \eta) - L_{E_1}(f; \eta) < \frac{\epsilon}{3}$$

$$U_{E_2}(f; \eta) - L_{E_2}(f; \eta) < \frac{\epsilon}{3}$$

$$V(E_1 \cap E_2; \eta) - v(E_1 \cap E_2; \eta) < \frac{\epsilon}{6M}$$

$\eta = \{R_1, \dots, R_p\}$ olsun.

$$M = \sup_{x \in E_1 \cup E_2} |f(x)|$$

③

$$\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$$

$$U_{E_1 \cup E_2}(f, \varphi) = \sum_{R_j \cap (E_1 \cap E_2) \neq \emptyset} M_j |R_j| + \sum_{R_j \cap (E_1 \cap E_2) \neq \emptyset} M_j |R_j|$$

$$\leq \sum_{R_j \cap (E_1 \cap E_2) \neq \emptyset} M_j |R_j| + \sum_{R_j \cap E_1 \neq \emptyset} M_j |R_j| + \sum_{R_j \cap E_2 \neq \emptyset} M_j |R_j|$$

$$\leq M \sum_{R_j \cap (E_1 \cap E_2) \neq \emptyset} |R_j| + U_{E_1}(f, \varphi) + U_{E_2}(f, \varphi)$$

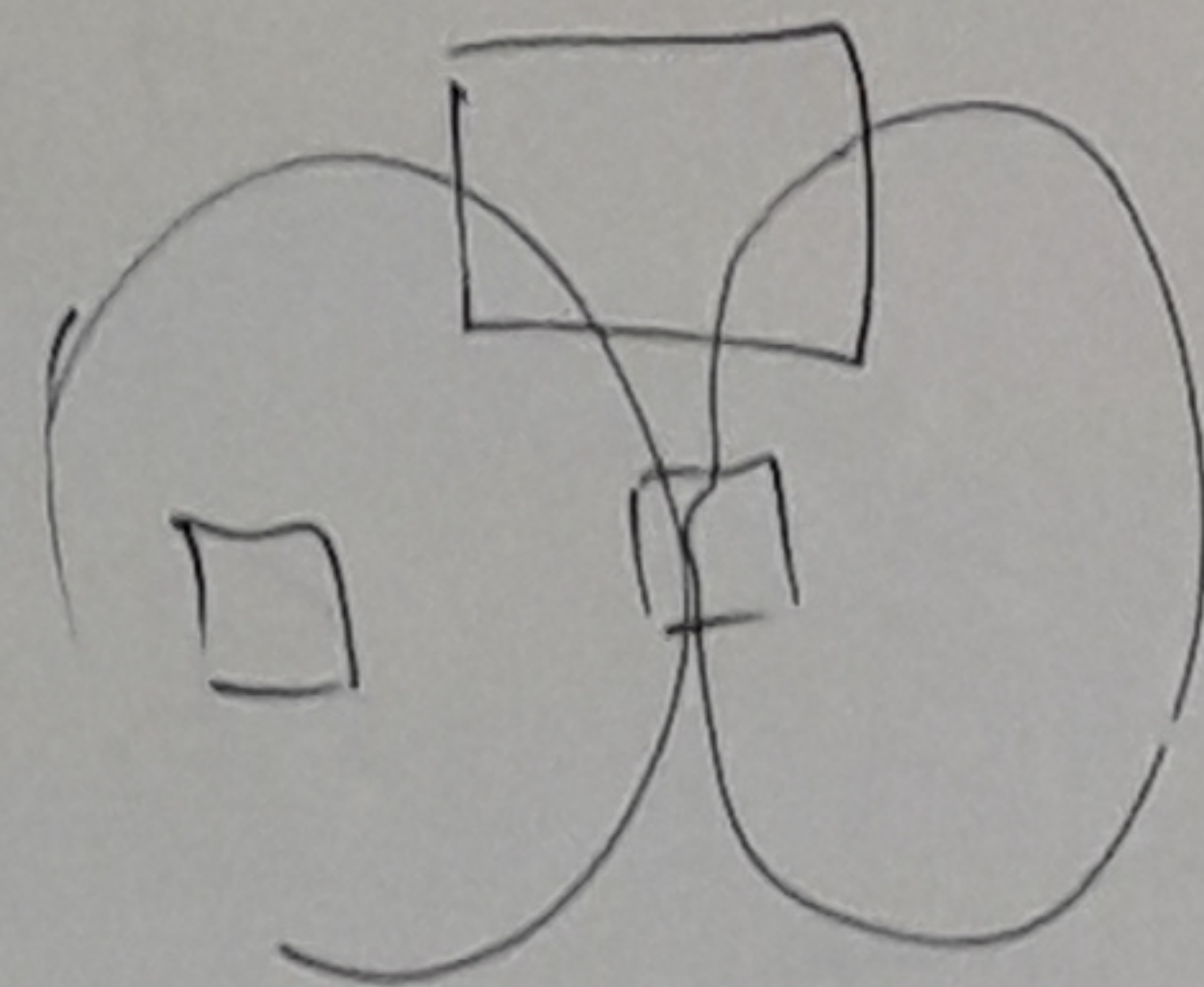
$$\leq M \cdot \frac{\rho}{6M} + U_{E_1}(f, \varphi) + U_{E_2}(f, \varphi)$$

Aggrav

$$= \frac{\rho}{6} + U_{E_1}(f, \varphi) + U_{E_2}(f, \varphi)$$

$$L_{E_1 \cup E_2}(f, \varphi) \geq \frac{-\varepsilon}{6} + L_{E_1}(f, \varphi) + L_{E_2}(f, \varphi)$$

(4)



$$\begin{aligned} \rightarrow U_{E_1 \cup E_2}(f, \varphi) - L_{E_1 \cup E_2}(f, \varphi) &\leq \\ \frac{\rho}{6} + [U_{E_1}(f, \varphi) - L_{E_1}(f, \varphi)] &+ [U_{E_2}(f, \varphi) - L_{E_2}(f, \varphi)] \end{aligned}$$

$$\leq \varepsilon$$

□

$$= \int_0^1 \left[\int_{y=0}^{x=y} e^{x^2} dy \right] dx$$

Theorem 2) E : Jordan Bölgesi
 $f \geq 0$ İntegrelenebilir olsun.

0 zaman

$$\int_E f \geq 0.$$

6) E : Jordan Bölgesi,
 f, g İntegrelenebilir.
 Eğer $f \leq g$ ise,

$$\int_E f \leq \int_E g$$

Not

Burada İstatalamak çok kolay
 E : Jordan f, g İntegrelenebilir.
 $f+g$ 'nin İntegrelenebilir olduğunu
 göstermiştik. Ayrıca

$$\int_E f+g = \int_E f + \int_E g$$

Theorem E : Jordan f İntegrelenebilir.
 0 zaman $|f|$ İntegrelenebilir ve

$$\int_E f \leq \int_E |f|$$

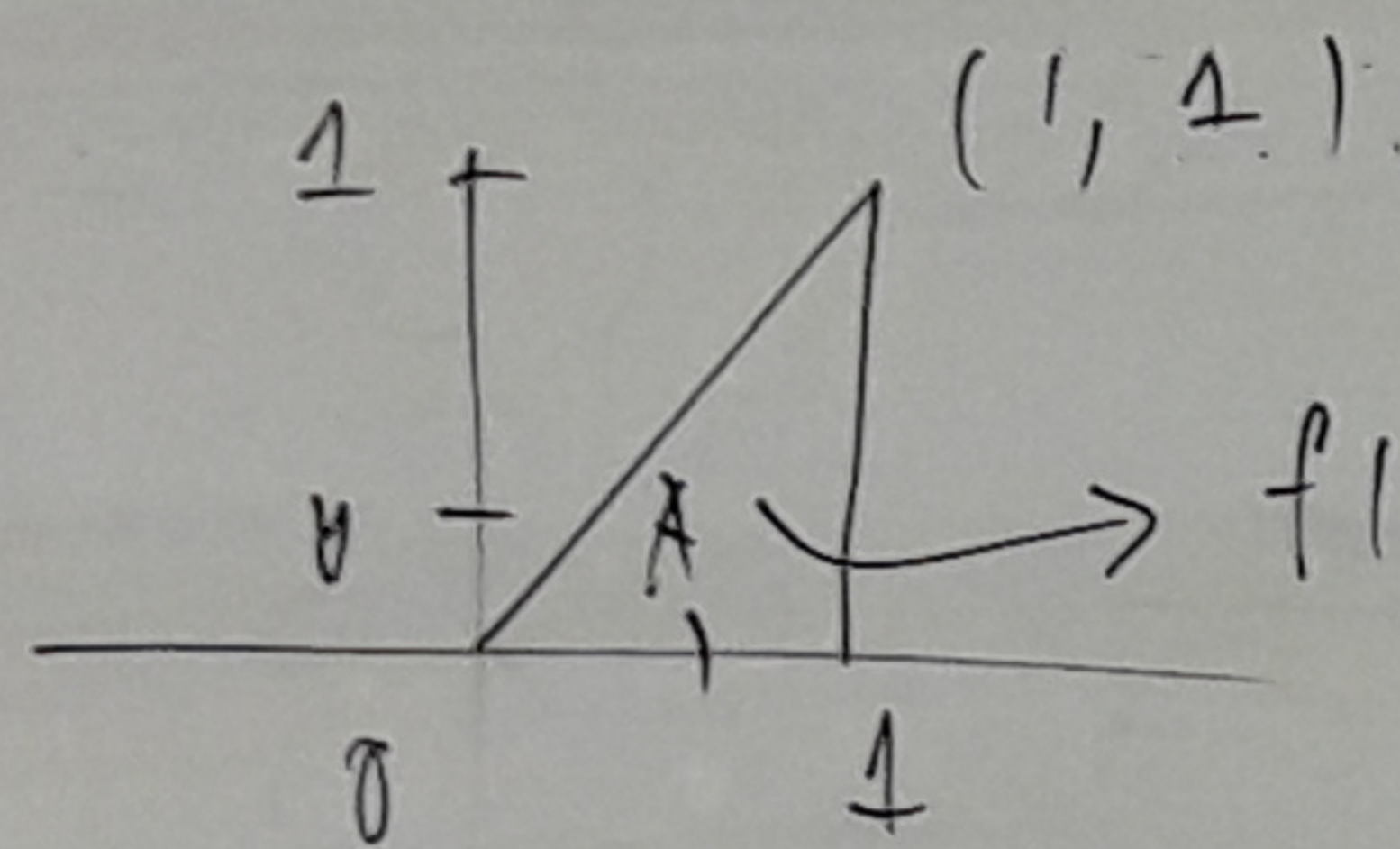
5)

$$\int_E f \leq \int_E g$$

$$\int_E f \leq \int_E |f|$$

1.3 Konu

Aradışık İntegraller
(Iterated Integrals)



$$f(x,y) = e^{x^2}$$

$$\int_A e^{x^2} = \int_0^1 \left[\int_{x=y}^1 e^{x^2} dx \right] dy$$

$$= \int_0^1 \left[\int_{y=0}^x e^{x^2} dy \right] dx$$

$$= \int_0^1 e^{x^2} \cdot x dx = \left. \frac{e^{x^2}}{2} \right|_0^1 = \frac{e-1}{2} \quad (6)$$

Integrellenebilir olsun

$$G(x) := \int_a^b f(x, y) dy$$

$$\rightarrow (L) \int_R f \leq (L) \int_a^b G(x) dx \leq (U) \int_a^b G(x) dx \leq (U) \int_R f$$

uygu'ama

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

İstbat

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx \right] dy = \left[\int_{-\infty}^{\infty} e^{-x^2} dx \right] \left[\int_{-\infty}^{\infty} e^{-y^2} dy \right]$$

$$= \int_{\mathbb{R}^2} e^{-x^2-y^2} dx dy = I^2$$

$$\lim_{M \rightarrow \infty} \int_{-M}^M e^{-x^2} dx$$

Polar koordinat

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta$$

$\theta=0$ $r=0$

$$= 2\pi \left[-\frac{e^{-r^2}}{2} \right]_0^{\infty}$$

$$= \pi$$

$$I^2 = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$

$$= \int_{\mathbb{R}^2} e^{-x-y^2} dx dy$$

$$= I^2$$

$$= \pi$$

$$I^2 = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$

Theorem 1.3.1

$R = [a, b] \times [c, d]$ bir dik-dörtgen

olsun. $f: R \rightarrow \mathbb{R}$ simetrik

olsun ve

her x için $y \mapsto f(x, y)$

İntegrelenebilir olsun

her y için $x \mapsto f(x, y)$

İntegrelenebilir olsun.

0 zaman

$$(L) \int_R f \leq (L) \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

$$\leq (U) \int_a^b \left[\int_c^d f(x, y) dy \right] dx \leq (U) \int_R f$$

Not $G(x) := \int_{y=c}^d f(x, y) dy$

$$\Rightarrow (L) \int_R f \leq (L) \int_a^b G(x) dx \leq (U) \int_a^b G(x) dx \leq (U) \int_R f$$

(8)

$$\int_0^1 1 = \frac{ac}{2} + \frac{a(a-c)}{2} = \frac{a^2}{2}$$

$$\boxed{|R| = da}$$

$$= \int_{x=0}^1 \frac{dx}{c} = \frac{dc}{2}$$

Aynı

$$F(y) = \int_a^b f(x, y) dx$$

0. 2. an

$$(L) \int_R f \leq (L) \int_C F(y) dy \leq (U) \int_C F(y) dy \leq (U) \int_R f$$

Teorem 1.3.2 / Fubini Teoremi

$R = [a, b] \times [c, d]$ ve $f: R \rightarrow \mathbb{R}$ İntegrelenbilir

olsun. Ayrıca,

- Her x için $y \rightarrow f(x, y)$ İntegrelenbilir olur
- Her y için $x \rightarrow f(x, y)$ İntegrelenbilir olur.

0. 2. an, d

$$a) F(x) = \int_c^d f(x, y) dy$$

$[a, b]$ üzerinde İntegrelenbilir

$$b) G(y) = \int_a^b f(x, y) dx$$

$[c, d]$ üzerinde İntegrelenbilir.

ve b d

$$c) \int_A f = \int_a^b \int_c^d f(x, y) dy dx$$

$$= \int_c^d \int_a^b f(x, y) dx dy$$

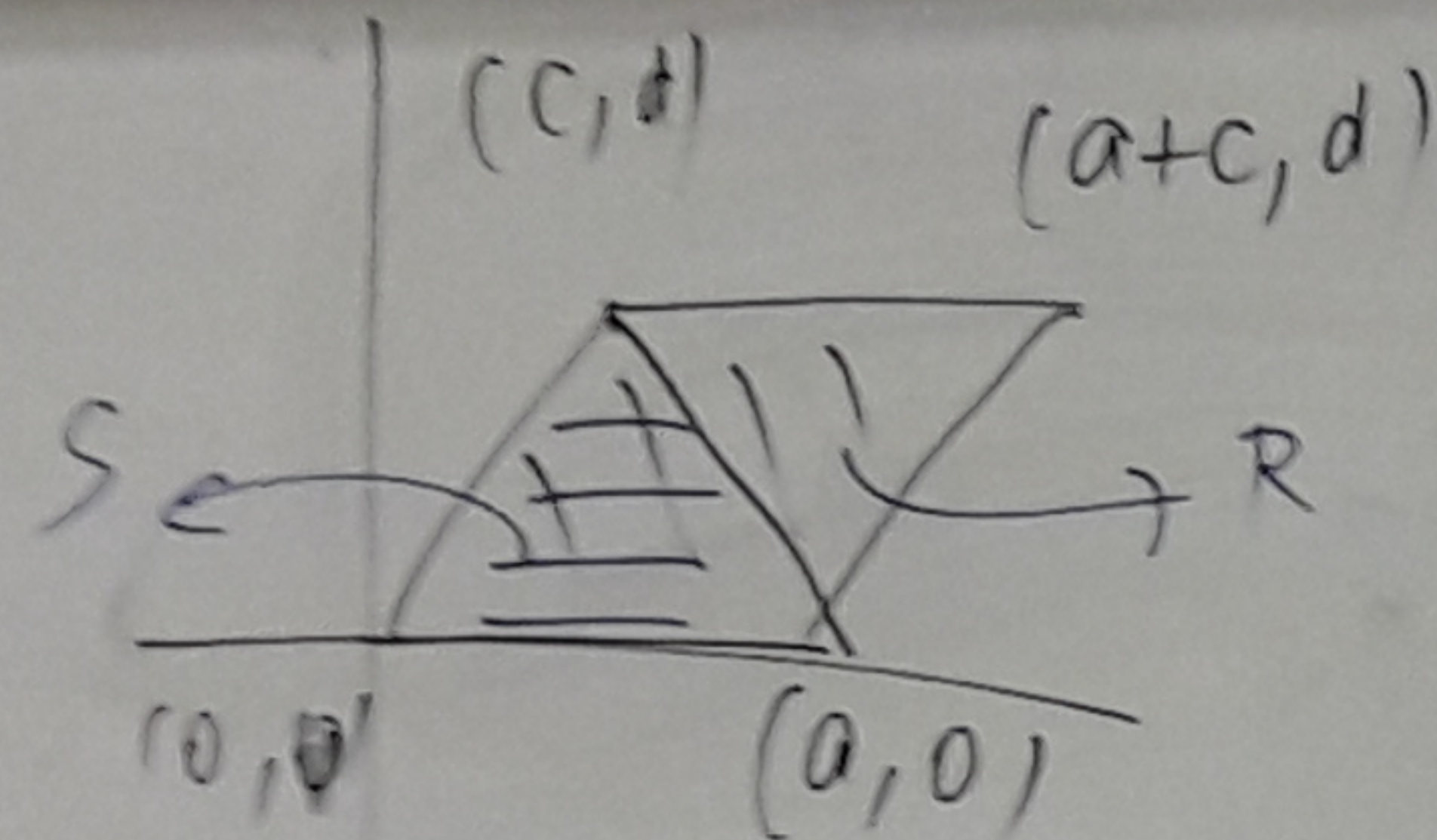
Örnek 1. Ayrica,

- Her x için $y \rightarrow f(x, y)$ İntegrelenbilir olsun.
- Her y için $x \rightarrow f(x, y)$ İntegrelenbilir olsun.

$$dA = \int_a^b \int_c^d f(x, y) dx dy$$

Örnek

(1)

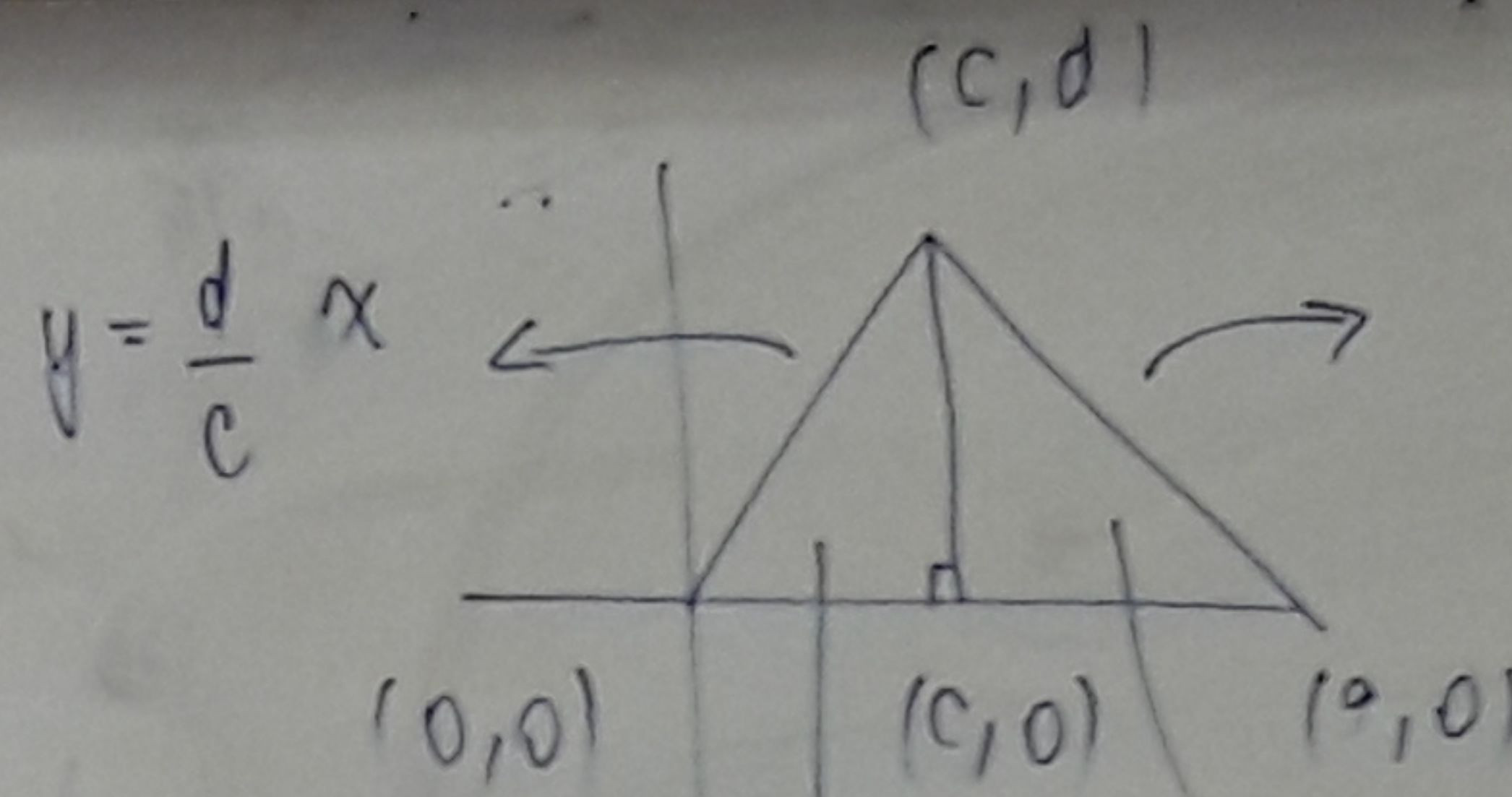


$$R = \{(x, y) \mid (0,0), (a,0), (c,d)\}$$

İçerdiği hacim,

$$\int_S 1 = \frac{dc}{2} + \frac{d(a-c)}{2} = \frac{da}{2}$$

$$\boxed{|R| = da}$$



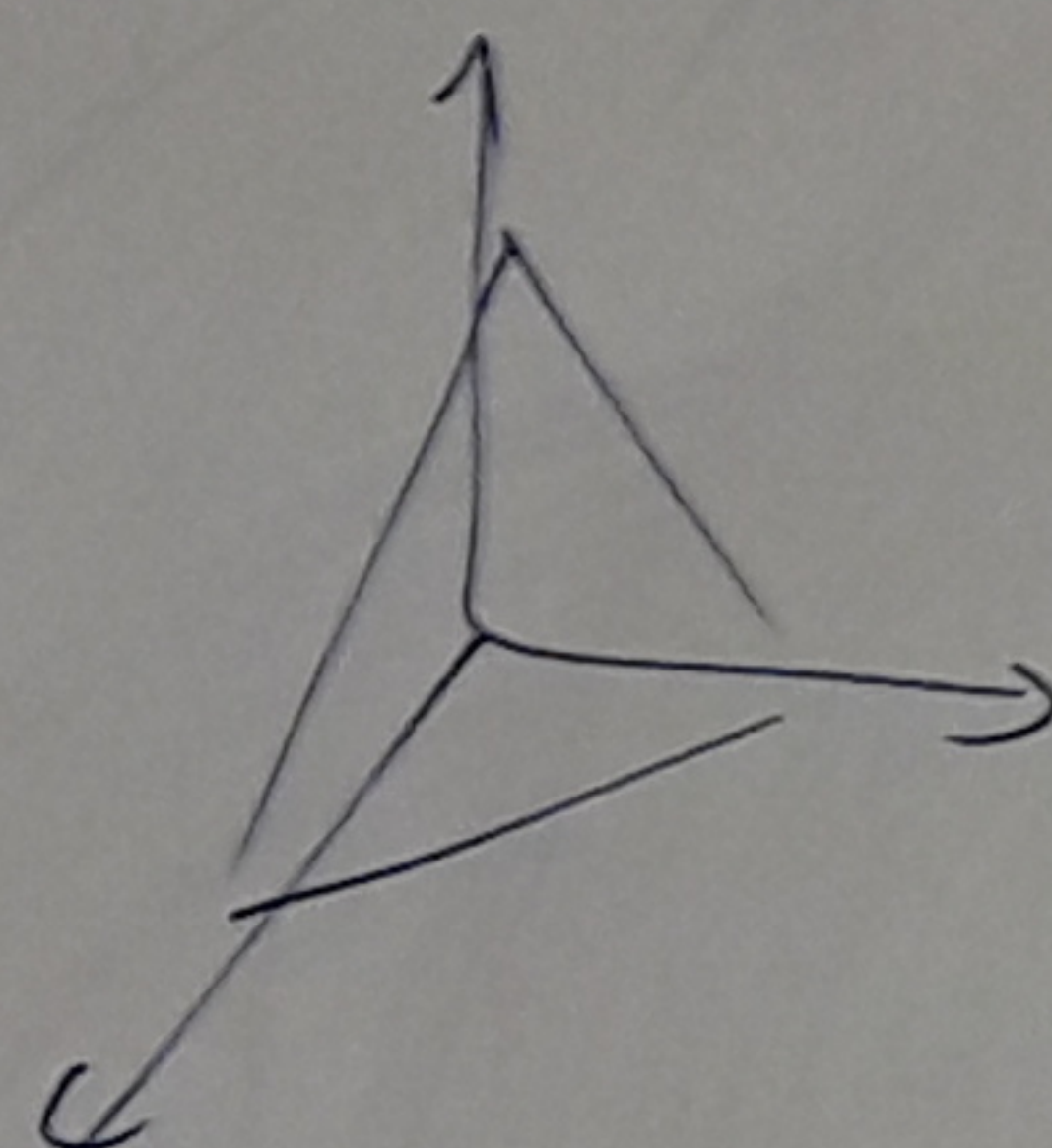
$$\int_{x=0}^c \int_{y=0}^{\frac{d}{c}x} 1 dy dx = \int_0^c \frac{dx}{c} dx = \frac{dc}{2}$$

$$\begin{aligned} \frac{y}{x-a} &= \frac{d}{c-a} \\ y &= \frac{dx}{c-a} - \frac{da}{c-a} \\ \int_{x=c}^a \int_{y=0}^{\frac{dx}{c-a} - \frac{da}{c-a}} 1 dy dx &= \frac{d(a-c)}{2} \end{aligned}$$

$$\eta=0 \quad x=y$$

örnek 2

$$\left\{ (x, y, z) \in \mathbb{R}^3 \mid \begin{array}{l} 0 \leq x, y, z \\ x + y + z \leq 1 \end{array} \right\}$$



$$\int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 1 \, dz \, dy \, dx$$

x sabit
 $y, z \geq 0$
 $y + z \leq 1 - x$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} (1-x-y) \, dy \, dx$$

$$= \int_{x=0}^1 \left[y - xy - \frac{y^2}{2} \right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^1 \left[1-x - x(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \frac{(1-x)^2}{2} dx$$

$$= -\frac{(1-x)^3}{6} \Big|_0^1$$

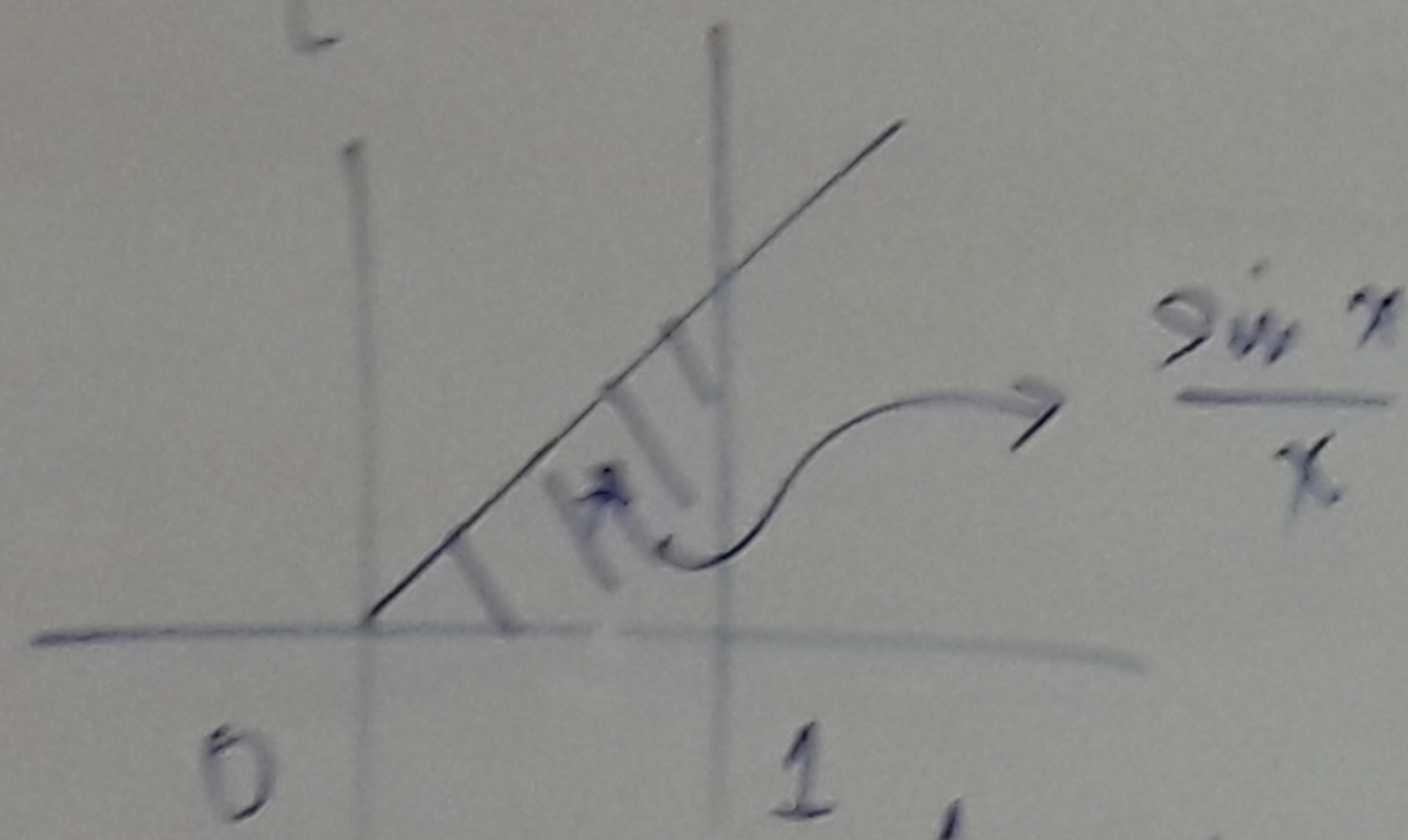
$$= \frac{1}{6}$$

(11)

$$\int_{x=0}^x \int_{y=0}^{1-x} (1-x-y-\frac{1}{2}) dy dx = \int_{x=0}^1 \left(-x - y(1-x) - \frac{1-x}{2} \right) dy$$

or not

$$\int_{y=0}^1 \left[\int_{x=y}^1 \frac{\sin(x)}{x} dx \right] dy$$



$$\int_1^{\frac{1}{2}} \frac{\sin x}{x} = \int_{y=0}^1 \int_{x=y}^1 \frac{\sin x}{x} dx dy$$

two, (if it's a solution, same set).

$$= \int_{x=0}^1 \int_{y=0}^x \frac{\sin x}{x} dy dx$$

$$= \int_{x=0}^1 \frac{\sin x}{x} \left[y \right]_{y=0}^x dx = \int_0^1 \sin x dx$$

$$= 1 - \cos(1)$$