Graflar'dan gelen yüzeyler.

$f : E \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$S = \{(x, y, f(x,y)) : (x,y) \in E\}$

Küresel ko-ordinatalar

$S = \{(r, \theta, \phi) : r = 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$

$S = \{(x, y, z) \mid x^2 + y^2 = 1\}$

Silindirik ko-ordinat

$S = \{(r, \theta, z) \mid r = 1,\ 0 \leq \theta < 2\pi,\ -\infty < z < \infty\}$

Parametrik yüzeyler (Tanım)

$E \subset \mathbb{R}^2 \quad \Phi : E \longrightarrow \mathbb{R}^3 \quad C^1$

$S = \{\Phi(x,y) : (x,y) \in E\}.$

Φ : Parametrizasyon denir.

Küre $\quad E = \underset{\theta}{[0, 2\pi]} \times \underset{\phi}{[0, \pi]}$

$\Phi(\theta, \phi) = (\sin\phi \cos\theta,\ \sin\phi \sin\theta,\ \cos\phi)$

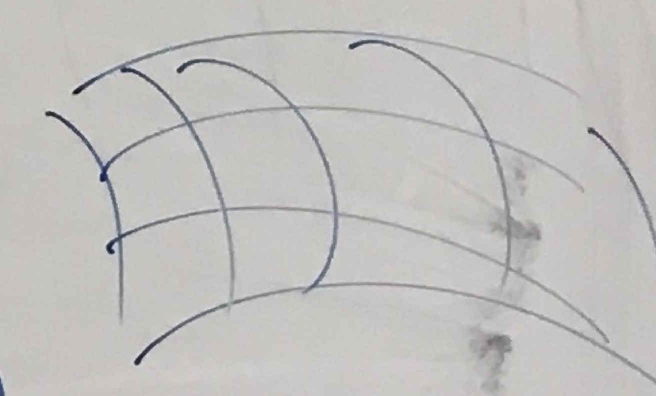
2

2.3.1 Tanım (Yüzey alanı)

$S = \{(x, y, f(x,y)) : (x,y) \in E\}$ bir Graf Yüzey olsun.

Onun Alan nasıl hesaplanır?

$f(x,y)$ bir C^1 fonksiyon olsun.



E üzerinde bir ağ seçelim ($\mathcal{G}$)

İnce. $\quad \mathcal{G} = \{R_1, \cdots, R_p\}$

$$\boxed{\sum_{\substack{R_i \in \mathcal{G} \\ R_i \cap \bar{E} \neq \emptyset}} |R_i \text{ üstündeki } S\text{'nin alanı}| \longrightarrow (*)}$$

$S = \{ \Phi(x,y) \}$

Φ : Parametrizasyon denir.

Küre $E = [0, 2\pi] \times [0, \pi]$

$\Phi(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$f(x,y)$ bir e' fonksiyon olsun.

E üzerinde bir ağ seçelim ($\mathcal{G}$) $\mathcal{G} = \{R_1, \dots, R_p\}$

İnce.

$$\boxed{\sum_{\substack{R_i \in \mathcal{G} \\ R_i \cap \bar{E} \neq \emptyset}} | R_i \text{ üstündeki } S\text{'nin alanı} | \rightarrow (*)}$$

1

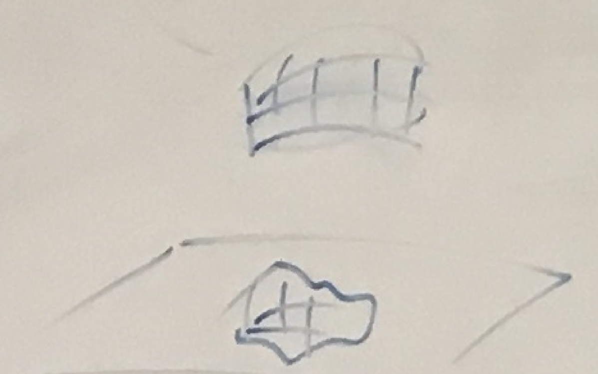
2.3 Yüzeyler.

$S \subset \mathbb{R}^3$ $\quad \mathbb{R}^3$ içinde yerel olarak iki boyutlu alan.

· Graflar'dan gelen yüzeyler.

$E \subset \mathbb{R}^2 \quad f: E \longrightarrow \mathbb{R}$

$S = \{ (x, y, f(x,y)) : (x,y) \in E \}$



· Parametrik yüzeyler.

$S = \{ (x,y,z) \mid x^2 + y^2 + z^2 = 1 \}$

Küresel ko-ordinatlar.

$S = \{ (r, \theta, \phi) : r = 1, \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi \}$

$S = \{ (x,y,z) \mid x^2 + y^2 = 1 \}$

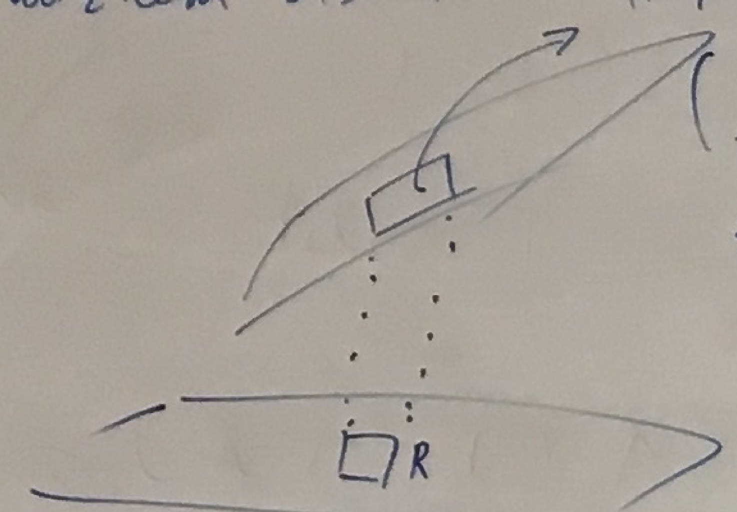
Silindrik ko-ordinat

$S = \{ (r, \theta, z) \mid r = 1, \ 0 \le \theta < 2\pi, \ -\infty < z < \infty \}$

R_i küçük ise, (*) bir düzlem parçası ile çok yakın.

$z = ax+by+C$ bir düzlem olsun. Hacim ne kadar?

($R \subset \mathbb{R}^2$ Düzlem içinde R üzerindeki parça)



$K = \{(x,y,z) \mid (x,y)\in R \to$ var dan ger. $z = ax+by+c\}$

Örnek M: $x+y+z=2$

$R = [0,1]\times[0,1] \subset \mathbb{R}^2$

M içinde ve R üstünde:

$(0,0) \to (0,0,2)$

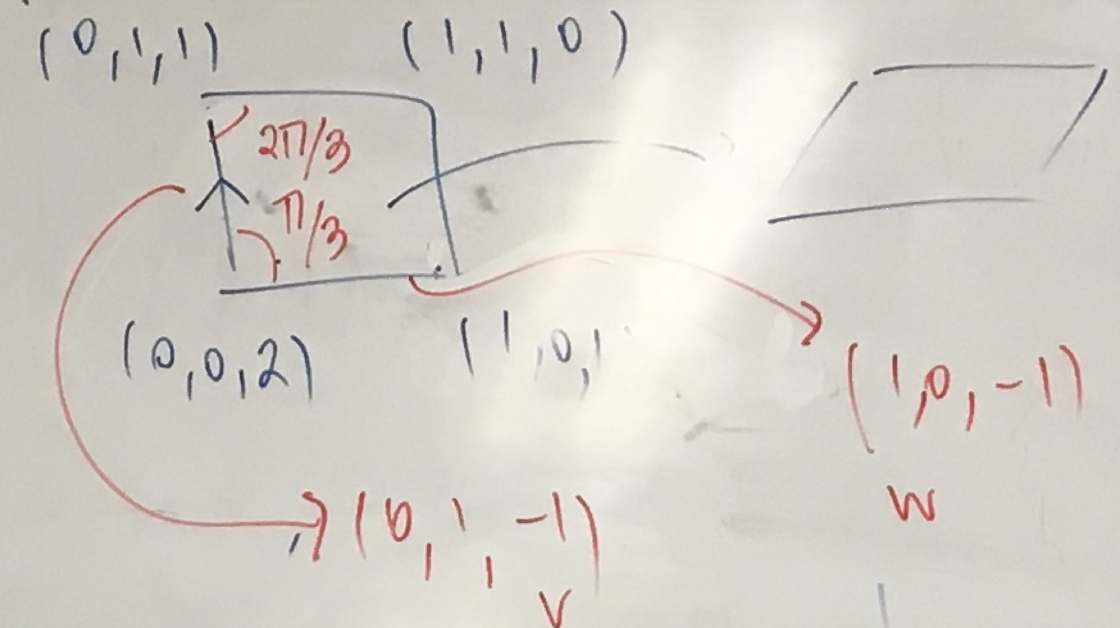
$(1,0) \to (1,0,1)$

$(0,1) \to (0,1,1)$

$(1,1) \to (1,1,0)$

$\cos\theta = \dfrac{v\cdot \vec{w}}{\|v\|\cdot\|w\|} = \dfrac{1}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$

$\Rightarrow \theta = \dfrac{\pi}{3}$



M: $z = Ax+By+C$

$\gamma(t): 0\le t\le 1 \subset \mathbb{R}^2$ bir doğru olsun.

$\gamma(t) = (a+tc,\ b+td)$, $0\le t\le 1$

(a,b) $(a+c,\ b+d)$

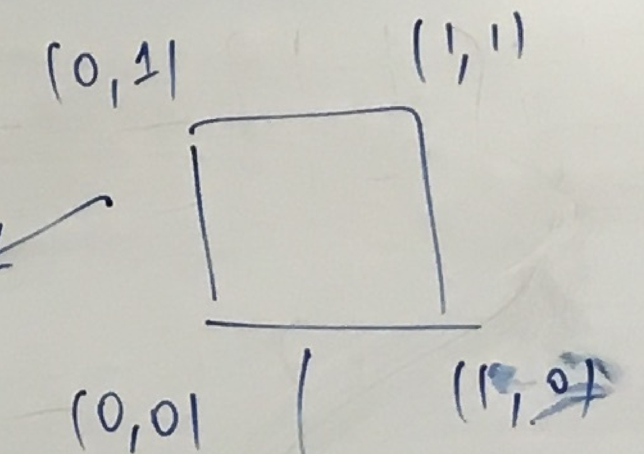
M'e taşıdığı zaman, bu

$(a+tc,\ b+td,\ A(a+tc)+B(b+td)+C)$

Yön: $(c,\ d,\ Ac+Bd)$

$R \subset \mathbb{R}^2$ $[0,1]\times[0,1]$

$z = Ax+By+C$ (4)



$(0,1,B)$

$(1,0,A)$

Hacim $\|(1,0,A)\times(0,1,B)\|$

$g = (x,\ y,\ f(x,y))$, $f(x,y) = z$

$\left\|\dfrac{\partial g}{\partial x}\times\dfrac{\partial g}{\partial y}\right\|$

$(a+tc,\ b+td,\ A(a+tc)+B(b+td)+C)$

Yön : $(c,\ d,\ Ac+Bd)$

$R \subset \mathbb{R}^2$. $[0,1]\times[0,1]$

$(1,0,A)$

Hacım $\|(1,0,A)\times(0,1,B)\| = ?$

$g = (x, y, f(x,y))$

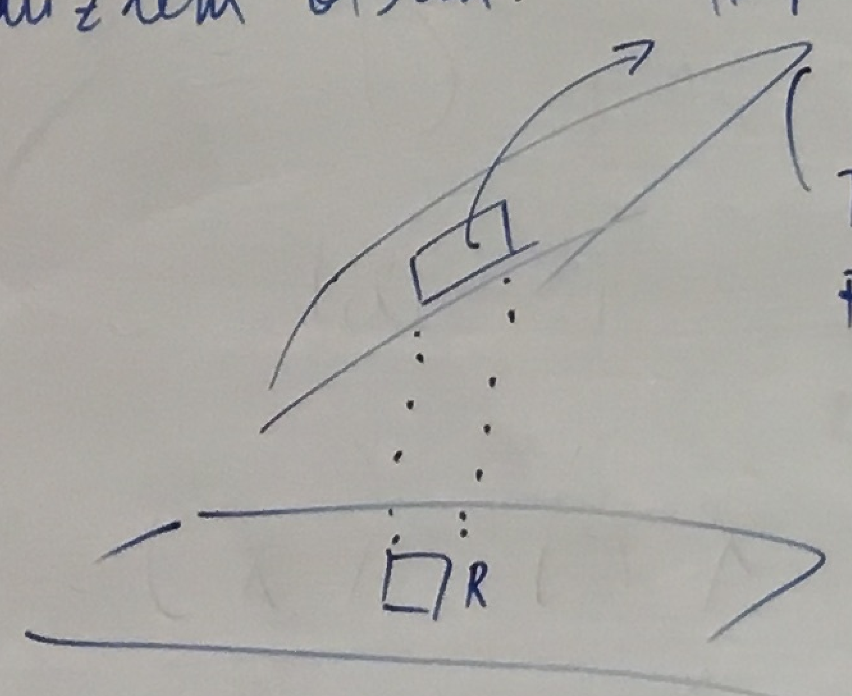
$\left\| \frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} \right\|$

③

R_i küçük ise, (k) bir düzlem parçası ile çok yakın.

$z = ax+by+C$ bir düzlem olsun. Hacim ne kadar?

(R ⊂ $\mathbb{R}^2$ Düzlem içinde R üzerindeki parça)



$K = \{(x,y,z) \mid (x,y)\in R \to$ Dik dörtgen. $z = ax+by+c\}$.

Örnek P: $x+y+z=2$.

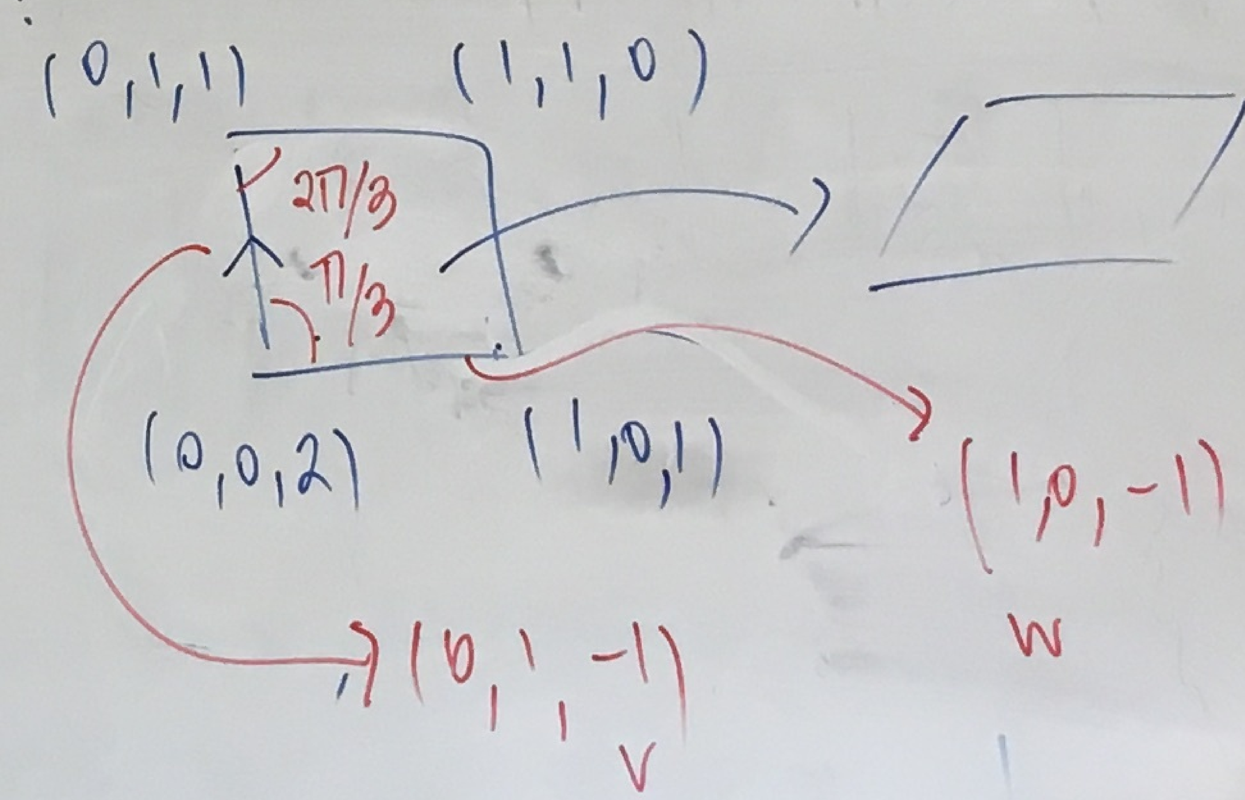
$R = [0,1]\times[0,1] \subset \mathbb{R}^2$

P içinde ve R üstünde :

$(0,0) \to (0,0,2)$
$(1,0) \to (1,0,1)$
$(0,1) \to (0,1,1)$
$(1,1) \to (1,1,0)$

$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|v\|\cdot\|w\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$.

$\Rightarrow \theta = \frac{\pi}{3}$.



O zaman

$$|S| = \iint_S 1\, dS = \iint_E \left\| \frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} \right\| dx\,dy$$

Yüzey Integral

$$g(x,y) = (x, y, f(x,y))$$

$$= \iint \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy$$

$$\left\| \frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} \right\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$$

$|S| =$

(6)

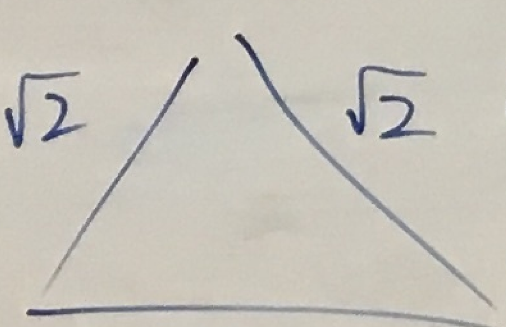
Örnek 2.3.3

$$S = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x \geq 0,\ y \geq 0,\ z \geq 0 \\ x + y + z = 1 \end{array} \right\}$$

Bu $\mathbb{R}^3$ içinde bir üçgen.

köşeler: $(1,0,0)$ $(0,1,0)$ $(0,0,1)$

Direkt Hesaplama



Alan: $\frac{\sqrt{3}}{4} \times 2 = \frac{\sqrt{3}}{2}$.

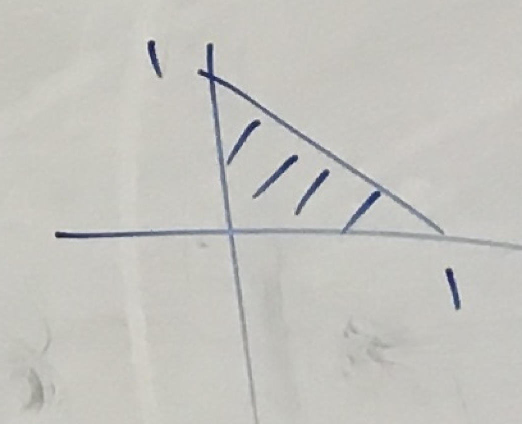
Formula kullanarak,

$$|S| = \iint_S 1\, dS = \iint \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy$$

$$S = (x, y, \underbrace{1 - x - y}_{f(x,y)})$$

$x, y \geq 0$

$0 \leq x + y \leq 1$

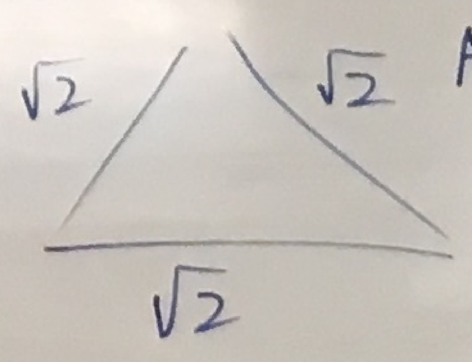


$$= \iint_{\substack{0 \leq x+y \leq 1 \\ x,y \geq 0}} \sqrt{3}\, dx\,dy = \frac{\sqrt{3}}{2}.$$

Bu $\mathbb{R}^3$ içinde bir ...

köşeler: $(1,0,0)$ $(0,1,0)$ $(0,0,1)$

Direkt Hesaplama

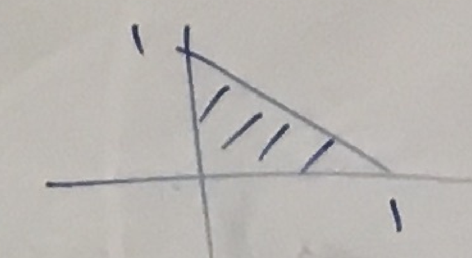


Alan: $\frac{\sqrt{3}}{4} \times 2 = \frac{\sqrt{3}}{2}$.

$S = (x, y, 1-x-y)$

$x, y \geq 0$

$0 \leq x+y \leq 1$



$0 \leq x+y \leq 1$

$x, y \geq 0$

$= \frac{\sqrt{3}}{2}$.

(5)

2.3.2 Formula

$E \subset \mathbb{R}^2 \quad f : E \to \mathbb{R} \quad C^1$

$$S = \{ (x, y, f(x,y)) : (x,y) \in E \}$$

O zaman

$$\boxed{\underbrace{|S| = \iint_S 1 \, dS}_{\text{Yüzey Integral}} = \iint_E \left\| \frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} \right\| dx\,dy \qquad g(x,y) = (x, y, f(x,y)) \qquad = \iint_E \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx\,dy}$$

$$\frac{\partial g}{\partial x} = \left(1, 0, \frac{\partial f}{\partial x}\right)$$

$$\frac{\partial g}{\partial y} = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

$$\left\| \frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} \right\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$$

$S = \{ \Phi(x,y) : (x,y) \in E \}$

O zaman yüzey alan

$$\boxed{|S| = \iint_S 1\, dS = \iint_E \| \Phi_x \times \Phi_y \| \, dx\, dy}$$

$\Phi(\theta, \phi) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$

$\Phi_\theta = (-\sin\phi \sin\theta, \sin\phi \cos\theta, 0)$

$\Phi_\phi = (\cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi)$

$$\Phi_\theta \times \Phi_\phi = \begin{vmatrix} i & j & k \\ -\sin\phi \sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi \sin\theta & -\sin\phi \end{vmatrix}$$

$$= (-\sin^2\phi \cos\theta, -\sin^2\phi \sin\theta, -\sin\phi\cos\phi)$$

$\|\Phi_\theta \times \Phi_\phi\| = |\sin\phi|$

Yüzey alanı

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} |\sin\phi| \, d\theta\, d\phi$$

$$= 2\pi \int_{\phi=0}^{\pi} |\sin\phi| \, d\phi$$

$$= 2\pi \int_{\phi=0}^{\pi} \sin\phi \, d\phi$$

$= 4\pi$.

Formula 2.3.5

(8)

Yüzeyler üzerinde fonksiyonları integre etmek.

$S \subset \mathbb{R}^3 \quad S = \{ \Phi(x,y) : (x,y) \in E \subset \mathbb{R}^2 \}$ e' parametrizasyon

$g : S \longrightarrow \mathbb{R}$ e' fonksiyon.

$$\boxed{\iint_S g\, dS = \iint_E \| \Phi_x \times \Phi_y \| \, g(\Phi(x,y)) \, dx\, dy.}$$

YÜZEY INTEGRAL

$= 2\pi \int |\sin\phi| \, d\phi$

$= 2\pi \int_{\phi=0}^{\pi} \sin\phi \, d\phi$

$= 4\pi.$

$$\iint_S g \, dS = \iint_E \|\Phi_x \times \Phi_y\| \, g(\Phi(x,y)) \, dx \, dy.$$

YÜZEY INTEGRAL

Formula 2.3.2 (Parametrik yüzeyler).

$S = \{ \Phi(x,y) : (x,y) \in E \}$

O zaman yüzey alan

$$\boxed{|S| = \iint_S 1 \, dS = \iint_E \|\Phi_x \times \Phi_y\| \, dx \, dy}$$

$\|\Phi_\theta \times \Phi_\phi\| = |\sin\phi|$

Örnek 2.3.4 Kürenin yüzey alan.

(7)

$\Phi(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$\Phi_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$

$\Phi_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$

$$\Phi_\theta \times \Phi_\phi = \begin{vmatrix} i & j & k \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix}$$

$= (-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi)$

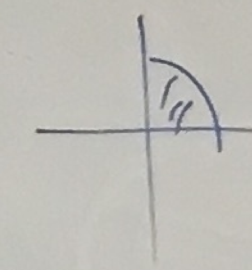
Ağırlık ne kadar?

Cevap $\iint (x^2+y^2)\,\|\Phi_x \times \Phi_y\|\,dx\,dy$

$x^2+y^2 \le 2$

$x,y \geq 0$

$= \iint_{\substack{x^2+y^2\le 2 \\ 0\le x,y}} (x^2+y^2)\sqrt{1+4x^2+4 \ldots}\ \ldots dy$



$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{\sqrt{2}} r^2\sqrt{1+4r^2}\; r\,dr\,d\theta$

$= \frac{\pi}{2}\int_{r=0}^{\sqrt{2}} r^2\sqrt{1+4r^2}\; r\,dr$

$1+4r^2 = y$

$8r\,dr = dy$

$= \frac{\pi}{2}\int_{y=1}^{9}\left(\frac{y-1}{4}\right)\sqrt{y}\;\frac{dy}{8}$

$= \frac{\pi}{32}\left(\frac{y^{5/2}}{5/2} - \frac{y^{3/2}}{3/2}\Bigg|_{y=1}^{9}\right)$

$= \frac{\pi}{32}\left[\frac{243-1}{5/2} - \frac{27-1}{3/2}\right]$

(10)

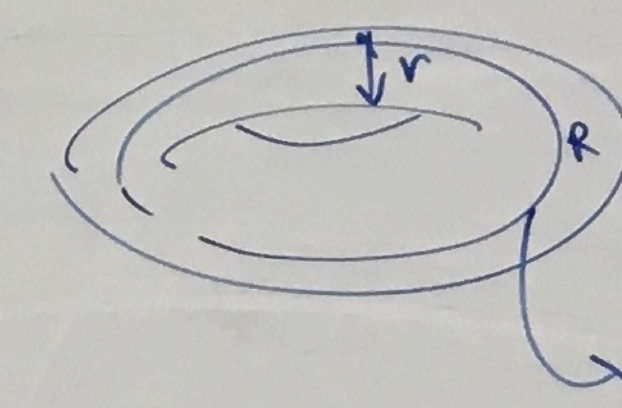
Örnek 2.3.7 (Torus).

$S = \{\Phi(x,y) : x,y \in [0,1]\}$.

$\Phi(\theta,\phi) = R(\cos\phi, \sin\phi, 0) +$

$\quad r(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$

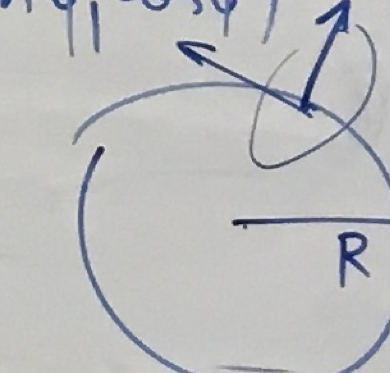
Yüzey alan?



$(R\cos\phi, R\sin\phi, 0)$

Merkezi çember.

$(-\sin\phi, \cos\phi)$ $(\cos\phi, \sin\phi)$



$(R\cos\phi, R\sin\phi, 0) + r\,v$

$(\alpha\cos\phi, \alpha\sin\phi, \beta)$

$$= \frac{\pi}{32}\left(\frac{y^{5/2}}{5/2} - \frac{y^{3/2}}{3/2} \right)\Bigg|_{y=1}^{9}$$

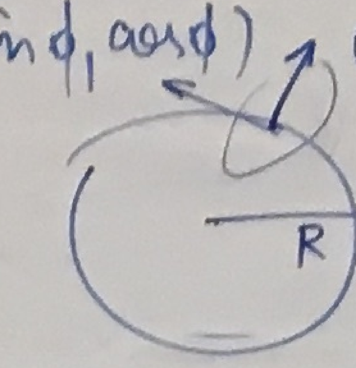
$$= \frac{\pi}{32}\left[\frac{243-1}{5/2} - \frac{27-1}{3/2} \right]_0$$

$$\Phi(\theta,\phi) = R(\cos\phi, \sin\phi, 0) + r(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$$

Yüzey alan?

Merkezi çember:

$(-\sin\phi, \cos\phi)$ $(\cos\phi, \sin\phi)$



$(R\cos\phi, R\sin\phi, 0) + r\,v$

$(a\cos\phi, a\sin\phi, B)$

Örnek 23.6

$$S: \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x \geq 0,\ y \geq 0,\ z \geq 0 \\ z = 2 - x^2 - y^2 \end{array} \right\}.$$

$$\Phi(x,y) = (x, y, 2 - x^2 - y^2)$$
$$x^2 + y^2 \leq 2$$
$$x, y \geq 0.$$

Not $\Phi(x,y) = (x, y, f(x,y))$ ise

$$\|\Phi_x \times \Phi_y\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

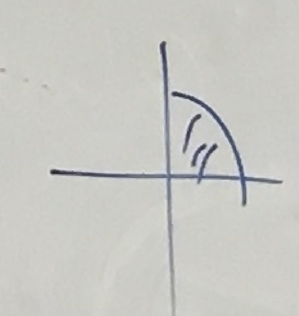
Yoğunluk: $x^2 + y^2$

Ağırlık ne kadar?

Cevap

$$\iint_{\substack{x^2+y^2 \leq 2 \\ x,y \geq 0}} (x^2+y^2)\, \|\Phi_x \times \Phi_y\|\, dx\, dy$$

$$= \iint_{\substack{x^2+y^2 \leq 2 \\ 0 \leq x,y}} (x^2+y^2) \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$



$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{\sqrt{2}} r^2 \sqrt{1+4r^2}\; r\, dr\, d\theta$$

$$= \frac{\pi}{2} \int_{r=0}^{\sqrt{2}} r^2 \sqrt{1+4r^2}\; r\, dr$$

$1 + 4r^2 = y$

$8\, r\, dr = dy$

$$\Phi_\theta = r(-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta) \qquad r < R.$$

$$\Phi_\phi = R(-\sin\phi, \cos\phi, 0) + r(-\sin\phi\cos\theta, \cos\phi\cos\theta, 0)$$

$$= [(-\sin\phi(R + r\cos\theta), \cos\phi(R + r\cos\theta), 0)$$

$$\Phi_\theta \times \Phi_\phi = r(R + r\cos\theta) \begin{vmatrix} i & j & k \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$$

$$= r(R + r\cos\theta)(-\cos\theta\cos\phi, -\cos\theta\sin\phi, -\sin\theta)$$

$$\|\Phi_\theta \times \Phi_\phi\| = r(R + r\cos\theta)$$

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$$|S| = \iint_S 1\, dS$$

$$= \iint r(R + r\cos\theta)\, d\theta\, d\phi$$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq 2\pi$

$$= 2\pi r \quad 2\pi R$$

$$= 4\pi^2 r R$$