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Örnekler 1 $\sin(x^2+y^2)$'nin $B(0,2)$ üzerinde ortalama değer nedir?

Cevap

$$\text{Ortalama} = \frac{\displaystyle\int_{B(0,2)} \sin(x^2+y^2)\,dx\,dy}{|B(0,2)|}$$

$$= \int_{\substack{r\in[0,2]\\ \theta\in[0,2\pi]}} \sin(r^2)\, r\,dr\,d\theta \Big/ 4\pi$$

Polar Ko-ordinatlar

$E \subset \mathbb{R}^2$ Tordan Bölge

$$\int_E f(x,y)\,dx\,dy = \int_F f(r,\theta)\, r\,dr\,d\theta$$

$$F = \{(r,\theta) \mid (r\cos\theta, r\sin\theta) \in E\}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \sin(r^2)\, r\,dr\,d\theta \Big/ 4\pi$$

$$= \int_{\theta=0}^{2\pi} \left. \frac{-\cos(r^2)}{2} \right|_{r=0}^{2} d\theta \Big/ 4\pi$$

$r \in [0,2]$
$\theta \in [0,2\pi]$ $\quad / 4\pi$

$$= \int_{\theta=0}^{2\pi} -\frac{\cos(r^2)}{2}\Big|_{r=0} d\theta \Big/ 4\pi$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{\cos 0}{2} - \frac{\cos(4)}{2}\right) d\theta \Big/ 4\pi$$

$$= 2\pi\left(\frac{1-\cos(4)}{2}\right) \Big/ 4\pi$$

$$= \pi(1-\cos(4)) / 4\pi$$

$\phi = (\phi_1, \phi_2, \phi_3)$

$$\phi' = \begin{bmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_1}{\partial z} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_2}{\partial z} \\ \frac{\partial \phi_3}{\partial x} & \frac{\partial \phi_3}{\partial y} & \frac{\partial \phi_3}{\partial y} \end{bmatrix}$$

Silindirik ko-ordinatlar.

Kolmogorov ②

$$(r, \theta, z) \xrightarrow{\phi} (r\cos\theta, r\sin\theta, z)$$

$? \; |\det \phi'| = |\Delta_\phi|$

$$\int_E f(x,y,z)\, dx\, dy\, dz = \int_F f(r,\theta,z)\, \boxed{r}\, dr\, d\theta\, dz$$

$$F = \{(r,\theta,z) \mid (r\cos\theta, r\sin\theta, z) \in E\}$$

Örnekler 2

$$\int (x-y) z \, dx\, dy\, dz$$

$$E = \begin{cases} x^2+y^2+z^2 \le 4 \\ z \geq \sqrt{x^2+y^2} \\ x \geq 0 \end{cases}$$

$$\phi'(r,\theta,\phi) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

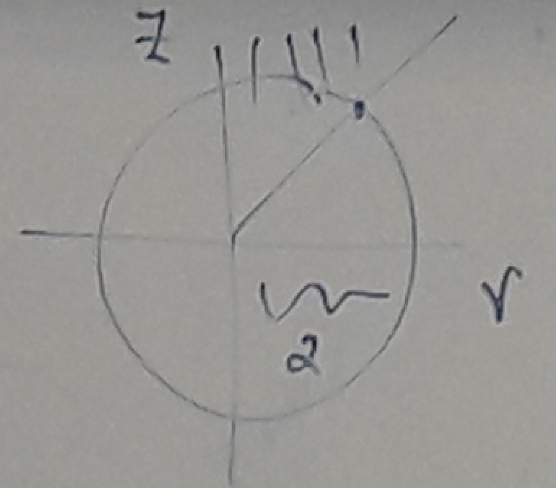
$$\Delta_\phi(r,\theta,\phi) = r$$

$$- r^2 \sin\phi \cos^2\phi \sin^2\theta - r^2 \sin\phi \cos^2\phi \cos^2\theta$$
$$= -r^2 \sin^3\phi - r^2 \sin\phi \cos^2\phi$$
$$= -r^2 \sin\phi$$

E : $r^2 + z^2 \le 4$

$z \ge r$

(silindirik ko-ordinatlar) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



(*) (Jacobian)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{\sqrt{2}} \int_{z=r}^{\sqrt{4-r^2}} r(\cos\theta - \sin\theta)\, z\, r\, dz\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_{r=0}^{\sqrt{2}} \left.\frac{z^2}{2}\right|_{z=r}^{\sqrt{4-r^2}} (\cos\theta - \sin\theta)\, r^2\, dr\, d\theta$$

③

$$\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{\sqrt{2}} (2-r^2)\, r^2 (\cos\theta - \sin\theta)\, dr\, d\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \left. \frac{2r^3}{3} - \frac{r^5}{5} \right|_{r=0}^{\sqrt{2}} (\cos\theta - \sin\theta)\, d\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \frac{8\sqrt{2}}{15} (\cos\theta - \sin\theta)\, d\theta$$

$$= \frac{16\sqrt{2}}{15} \quad \square$$

$$\int_{-\pi/2} \int_{r=0} \int_{z=r} \frac{z}{2} (\cos\theta - \sin\theta)\, r^2\, dr\, d\theta$$

$$= \int_{\theta=-\pi/2} \frac{8\sqrt{2}}{\sqrt{5}} (\cos\theta - \sin\theta)\, d\theta$$

$$= \frac{16\sqrt{2}}{\ldots} \quad \square$$

Örnek 3: $B(0,1) \subset \mathbb{R}^3$ üzerinde $(xyz)^2$ 'nin ortalama değeri nedir?

Küresel Ko-ordinatlar ④

$r \geq 0$

$0 \leq \theta < 2\pi$

$0 \leq \phi \leq \pi$

$$(r, \theta, \phi) \xrightarrow{\Phi} (r\sin\phi\cos\theta,\ r\sin\phi\sin\theta,\ r\cos\phi)$$

r: 0'dan uzaklık

$$\Phi'(r,\theta,\phi) = \begin{bmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}$$

$$\Delta_\Phi = \det \Phi' = -r^2\sin^3\phi\cos^2\theta - r^2\sin^3\phi\sin^2\theta - r^2\sin\phi\cos^2\phi\sin^2\theta - r\sin\phi\cos^2\phi\cos^2\theta$$

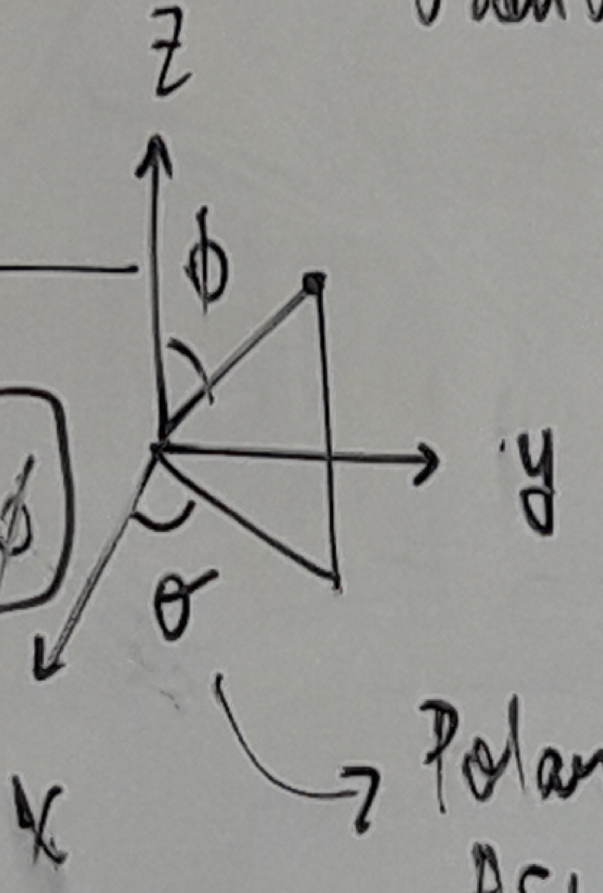
$$= -r^2\sin^3\phi - r^2\sin\phi\cos^2\phi$$

$$= -r^2\sin\phi$$

Cevap: **Küresel Ko-ordinat formula**

$$\boxed{\int_E f(x,y,z)\,dx\,dy\,dz = \int_F f(r,\theta,\phi)\, \underbrace{r^2\sin\phi}_{\text{Jacobian}}\, dr\, d\theta\, d\phi}$$

Azimuth Açı



Polar Açı

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Evalp $\displaystyle\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} r^6\left[\cos^2\phi\,\sin^4\phi\right]\left[\sin^2\theta\,\cos^2\theta\right] r^2\sin\phi\,dr\,d\theta\,d\phi$

$$\frac{\sin^2 2\theta}{4} = \frac{1-\cos 4\theta}{8}$$

$$\left[\cos^2\phi\,(1-\cos^2\phi)^2\right]\sin\phi \qquad y=\cos\phi$$

$$= \left[\int_{r=0}^{1} r^8\,dr\right]\left[\int_{\theta=0}^{2\pi}(\sin^2\theta\cos^2\theta)\,d\theta\right]\left[\int_{\phi=0}^{\pi}\cos^2\phi\,\sin^5\phi\,d\phi\right]$$

$$\int_{-1}^{1}\left[y^2(1-y^2)^2\right]dy$$

$$= \left.\frac{y^3}{3}-\frac{2y^5}{5}+\frac{y^7}{7}\right|_{-1}^{1}$$

$$= \frac{2}{3}-\frac{4}{5}+\frac{2}{7}$$

$$= \frac{70-84+30}{105}$$

$$= \frac{1}{9}\cdot\frac{2\pi}{8}\cdot\frac{16}{105} \quad \square$$

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rmk for 4

$B^n = \{(x_1,\dots,x_n)\in R^n \mid \sum x_i^2 \le 1\}.$

$\mathrm{Vol}(B_2^n) = \int^{1} B_2^{n-1}\left(0,\sqrt{1-x_1^2}\right)dx_1$

$A \subset \mathbb{R}^n$

$|\alpha A| = |\alpha|^n |A|$

$= \frac{1}{9} \cdot \frac{2\pi}{8} \cdot \frac{16}{105}$ □

$= \frac{y}{3} - \frac{2y}{5} + \frac{y}{7}$

$= \frac{2}{3} - \frac{4}{5} + \frac{2}{7}$ -1

$= \frac{70 - 84 + 30}{105}$

örnekler 4 ⑥

$B_2^n = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 \le 1 \}.$

$|B_2^n| = ?$

1. $\sim$ const ~~Kadir II~~
2. $\sim 2^n$ ~~Kadir I~~
3. $\sim n^p$ $p \geq 1$
4. Azalıyor Kadir III ✓

$\sim \frac{2^n}{n!}$

$\mathrm{Vol}(B_2^n) = \int_{x_1=-1}^{1} B_2^{n-1}\left(0, \sqrt{1-x_1^2}\right) dx_1$

$x_2^2 + \dots + x_n^2 \le 1 - x_1^2$

$= \int_{x_1=-1}^{1} |B_2^{n-1}| \cdot (1-x_1^2)^{n/2} \, dx_1$

$|B_2^n| = |B_2^{n-1}| \int_{x_1=-1}^{1} (1-x_1^2)^{n/2} \, dx_1$

$A \subset \mathbb{R}^n$

$|\alpha A| = |\alpha|^n |A|$

Ödev