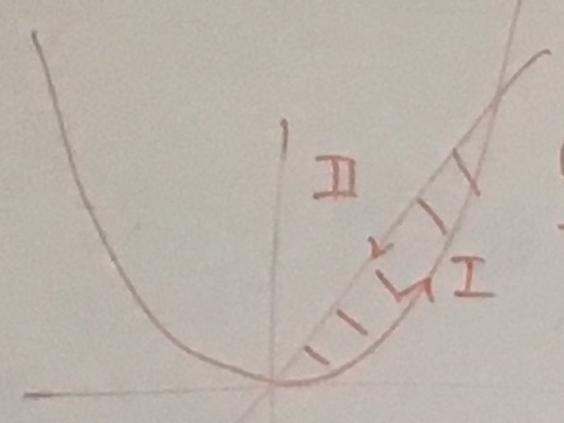


Cevap

$$\int_{\partial E} = \int_{I} y\,dx + x\,dy + \int_{II} y\,dx + x\,dy.$$

$I: \gamma(t) = (t, t^2) : t \in [0,1]$

$\gamma'(t) = (1, 2t)$

$$\int_{I} y\,dx + x\,dy = \int_0^1 (t^2 \cdot 1 + t \cdot 2t)\,dt = 1$$

$$= \int_{0} 2(t-1)\,dt$$

$$= (t-1)^2 \Big|_{0} = -1.$$

$$\Rightarrow \int_{\partial E} y\,dx + x\,dy = 1 - 1 = 0.$$

2.5 Üç Temel Teoremler

① Green Teoremi

$E \subset \mathbb{R}^2$ Jordan bölge.

∂E : C^1 eğri

O zaman,

$$\boxed{\int_{\partial E} M\,dx + N\,dy = \iint_E \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

+ve orientasyon

$\partial E : \{ \vec{\gamma}(t) : 0 \le t \le T \}.$ $\qquad \gamma(t) = (x(t), y(t))$

→ Bölge solda kalıyor

$$\int_{\partial E} (M\,dx + N\,dy) = \int_{\partial E} (M, N) \cdot dT \qquad ①$$

$$= \int_0^T \left[M(\gamma(t))\, x'(t) + N(\gamma(t))\, y'(t) \right] dt$$

$$\int_{\partial E} M\,dx + N\,dy = \iint_E \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

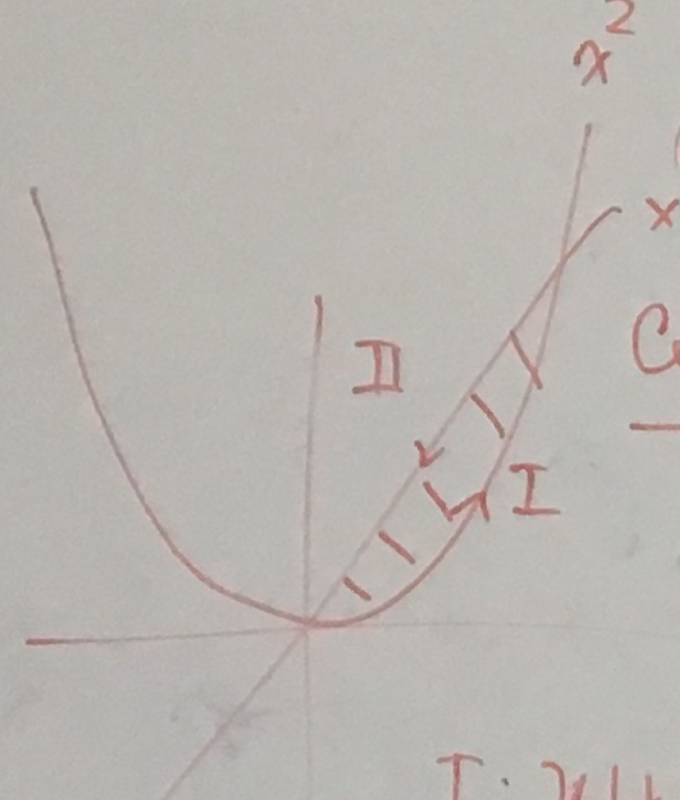
+ve orientasyon

$\partial E : \{ \vec{r}(t) : 0 \le t \le T \}$. $\quad r(t) = (x(t), y(t))$

→ Bölge solda kalıyor

Örnek 1.1 E: $y = x$ ve $y = x^2$ ile sınırlı bölge.

O zaman $\displaystyle\int_{\partial E} y\,dx + x\,dy$ hesaplayınız.



Cevap

$$\int_{\partial E} = \int_{I} y\,dx + x\,dy + \int_{II} y\,dx + x\,dy$$

$I : r(t) = (t, t^2) : t \in [0,1]$

$r'(t) = (1, 2t)$

$$\int_I y\,dx + x\,dy = \int_0^1 (t^2 \cdot 1 + t \cdot 2t)\,dt = 1$$

$II : r(t) = (1-t, 1-t) \quad 0 \le t \le 1$

$r'(t) = (-1, -1)$

$$\int_{II} y\,dx + x\,dy = \int_0^1 \left[(1-t)(-1) + (1-t)(-1)\right] dt$$

$$= \int_0^1 2(t-1)\,dt$$

$$= (t-1)^2 \Big|_0^1 = -1.$$

$$\Rightarrow \int_{\partial E} y\,dx + x\,dy = 1 - 1 = 0.$$

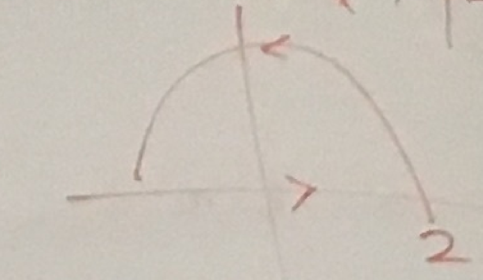
$$= (-1)\int_{x=0} \int_{y=0} \left(\frac{1}{x+1} - e^y\right) dy\,dx$$

$$= (-1)\int_{x=0}^{2} \left[\frac{y}{x+1} - e^y\right]_0^3 dx$$

$$= (-1)\int_{x=0}^{2} \left(\frac{3}{x+1} - (e^3-1)\right) dx$$

Örnek 1.3 $\displaystyle\int_{\partial\left[\begin{smallmatrix}x^2+y^2\le 4\\ y\ge 0\end{smallmatrix}\right]} \underbrace{\sin\sqrt{x^3-x^2}}_{M}\,dx + \underbrace{xy}_{N}\,dy$

+ve orientasyon



Cevap (Green Thm)

$$\iint_{\substack{x^2+y^2\le 4\\ y\ge 0}} y\,dx\,dy = \int_{\theta=0}^{\pi}\int_{r=0}^{2} r\sin\theta\; r\,dr\,d\theta = 16/3.$$

(3)

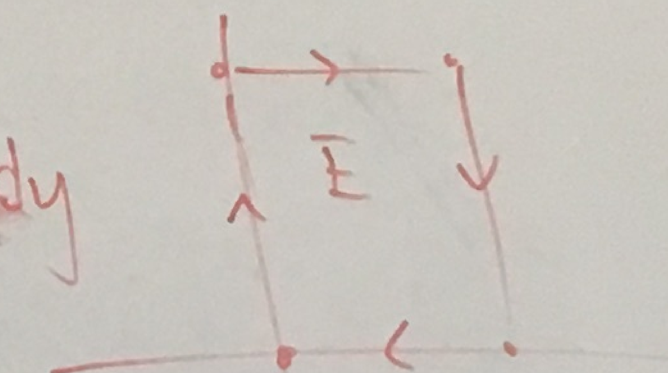
Green Teorem Kullansaydık.

$$\int_{\substack{\partial E\\ +}} \underbrace{y}_{M}\,dx + \underbrace{x}_{N}\,dy = \iint_E \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_E 0\,dx\,dy = 0.$$

Örnek $F = (e^y, \ln(x+1))$

Bunu $(0,0)$ $(2,0)$ $(0,3)$ $(2,3)$ köşeli dikdörtgen'in sınırı (−ve orientasyon) üzerinde integre ediniz.



$$\int_{\substack{\partial E\\ -ve}} F\cdot dT = \int_{\substack{\partial E\\ -ve}} e^y\,dx + \ln(x+1)\,dy$$

Green

$$= (-1)\iint_E \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

E $\quad \partial E$ $\quad \partial E$

$-ve$ $\quad -ve$

Green

$$= (-1) \iint_E \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

(4)

$$= (-1) \iint_{[0,2]\times[0,3]} \left(\frac{1}{x+1} - e^y \right) dx\,dy$$

$$= (-1) \int_{x=0}^{2} \int_{y=0}^{3} \left(\frac{1}{x+1} - e^y \right) dy\,dx$$

$$= (-1) \int_{x=0}^{2} \left. \frac{y}{x+1} - e^y \right|_0^3 dx$$

$$= (-1) \int_{x=0}^{2} \left(\frac{3}{x+1} - (e^3 - 1) \right) dx$$

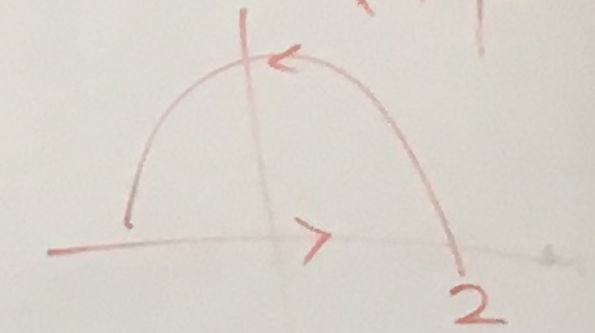
$$= (-1) \left[3\ln(x+1) - (e^3-1)x \right]_{x=0}^{2}$$

$$= (-1)\left(3\ln 3 - 2e^3 + 2\right)$$

örnek 1.3 $\displaystyle\int_{\partial\left[\begin{array}{l} x^2+y^2 \le 4 \\ y \ge 0 \end{array}\right]} \underbrace{\sin\sqrt{x^3 - x^2}}_{M}\, dx + \underbrace{xy}_{N}\, dy$

$+ve$ orientasyon

$x^2+y^2=4$



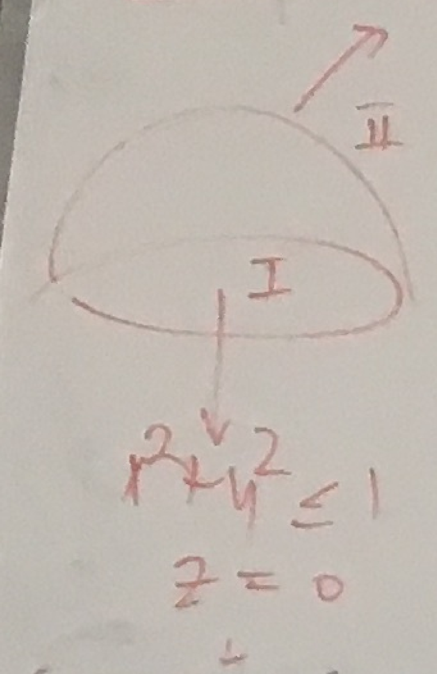
Cevap (Green Thm)

$$\iint_{\substack{x^2+y^2 \le 4 \\ y \ge 0}} y\,dx\,dy = \int_{\theta=0}^{\pi} \int_{r=0}^{2} r\sin\theta \, r\,dr\,d\theta$$

$$= 16/3.$$

S

Cevap (Direkt hesaplama)



I Paramatrizasyon $(x,y,0) : x^2+y^2\le 1$

$$\int_{I} (xy, xz, -yz)\cdot dS$$

$\Phi(x,y)=(x,y,0)$

$\Phi_x=(1,0,0)$ $\quad \Phi_x\times\Phi_y=(0,0,1)$ (Ters yön)

$\Phi_y=(0,1,0)$

5

2. Stokes Theorem.

$S\subset\mathbb{R}^3$ bir C^1 yüzey olsun

ve ∂S bir C^1 eğri olsun.

$F=(M,N,P)$ bir vektör alan olsun.

O zaman, uyumlu orientasyon ile,

$$\iint_S F\cdot dS = \int_{\partial S} (\nabla\times F)\cdot dT$$

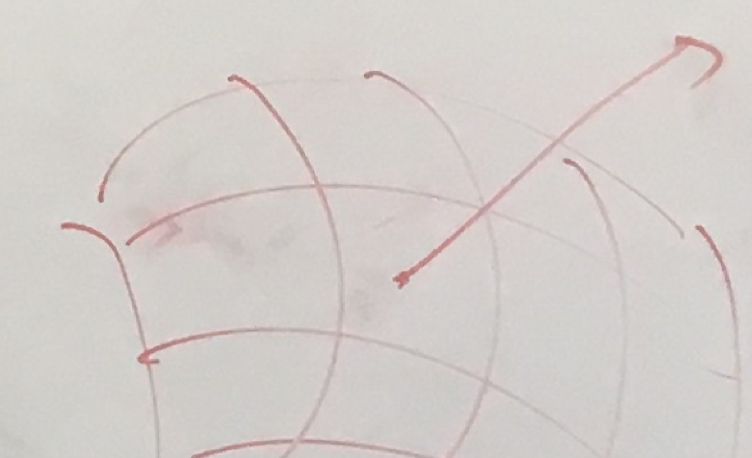
Flux Integral — Eğri Integral.

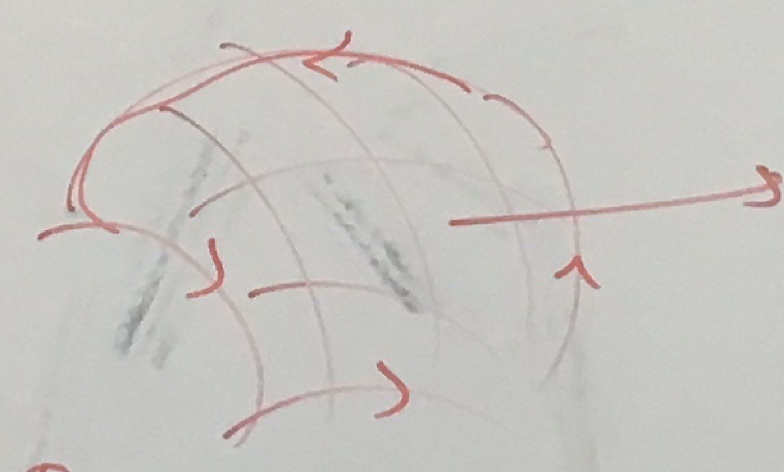
Flux Integral

$S=\{\Phi(x,y) : (x,y)\in E\subset\mathbb{R}^2\}$

$$\iint_S F\cdot ds = \iint_E F(\Phi(x,y))\cdot(\Phi_x\times\Phi_y)\,dx\,dy$$

Uyumlu Orientasyon





Yüzey için orientasyon: Her noktada (sürekli şekilde) birim dik vektör belirlenmiş,

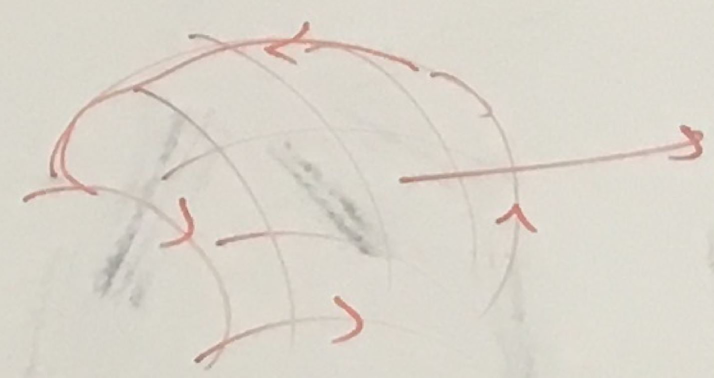
Buna göre: Eğri Ters Saat Yönüyle hareket ediyor olsun.

O zaman, uyumlu orientasyon ile,

$$\iint_S F \cdot dS = \int_{\partial S} (\nabla \times F) \cdot dT$$

Flux Integral — Eğri Integral

Uyumlu Orientasyon



Yüzey için orientasyon: Her noktada (sürekli şekilde) birim dik vektör belirlenmiş,

Örnek 2.1

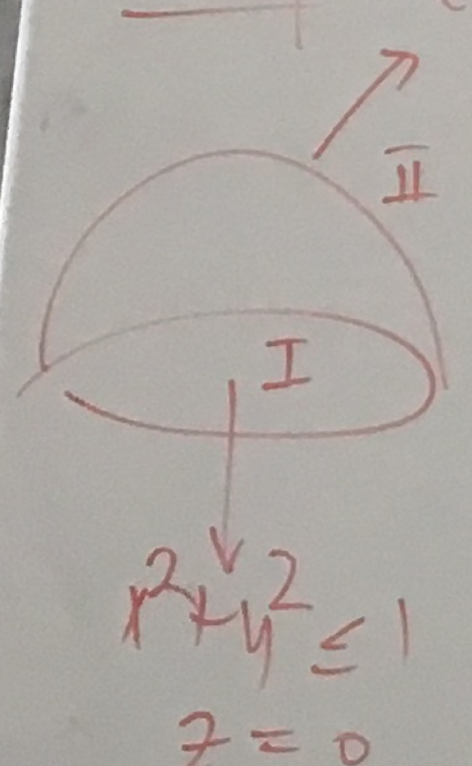
$S : \{(x,y,z) : x^2+y^2+z^2=1\}$, $z \geq 0$

Dışarıya giden vektörler kullanarak orientasyon

$$\int_S (xy, xz, -yz) \cdot dS$$

$$= (-1)\iint_{x^2+y^2 \leq 1} (xy, 0, 0) \cdot (0,0,1)\, dx\, dy = 0$$

Cevap (Direkt hesaplama)



I Parametrizasyon $(x,y,0) : x^2+y^2 \leq 1$

$$\int_I (xy, xz, -yz) \cdot dS$$

$\Phi(x,y) = (x,y,0)$

$\Phi_x = (1,0,0)$ $\Phi_x \times \Phi_y = (0,0,1)$ (Ters yön)

$\Phi_y = (0,1,0)$

$S \qquad \partial S$

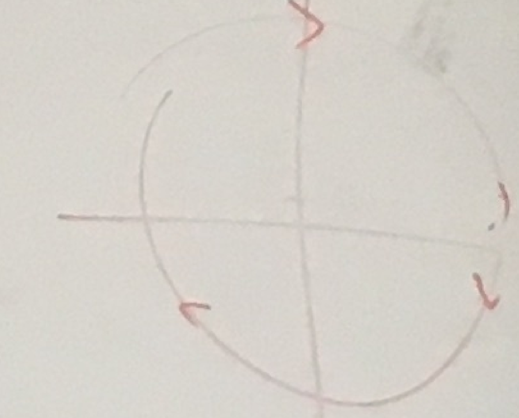
$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & -yz \end{vmatrix} = (-y-x, 0, z-x)$$

$$\int_{\partial S} (\nabla\times F)\cdot dT = \int_{\partial S} (-y-x, 0, z-x)\cdot dT$$

$$= \int_{\theta=0}^{2\pi} (\sin\theta-\cos\theta, 0, -\cos\theta)\cdot(-\sin\theta, -\cos\theta, 0)\,d\theta$$

$$= \int_{\theta=0}^{2\pi}(-\sin^2\theta+\sin\theta\cos\theta)\,d\theta$$

$$= -\pi \quad \square$$



(7)

II $\Phi(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$\Phi_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$

$\Phi_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$

$$\Phi_\phi \times \Phi_\theta = \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

Dışarıya gidiyor

$$\int_{II} \cdot = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} (\sin^2\phi\cos\theta\sin\theta, \sin\phi\cos\phi\cos\theta, -\sin\phi\cos\phi\sin\theta)\cdot \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} (\sin^4\phi\cos^2\theta\sin\theta + \sin^3\phi\cos\phi\sin\theta\cos\theta - \sin^2\phi\cos^2\phi\sin\theta)\,d\theta\,d\phi$$

$\rightarrow$ Yapılabilir

$-\sin\phi \sin\theta \quad \sin\phi\cos\theta \quad 0$

Yapılamaz

$= \sin\phi\,(\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$

Dışarıya gidiyor

Stoks kullanarak.

$$\iint_S \underbrace{(xy, xz, -yz)}_{F} \cdot ds = \int_{\partial S} (\nabla \times F)\cdot dT$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & -yz \end{vmatrix} = (-y-x,\ 0,\ z-x)$$

$$\int_{\partial S} (\nabla \times F)\cdot dT = \int_{\partial S} (-y-x,\ 0,\ z-x)\cdot dT$$

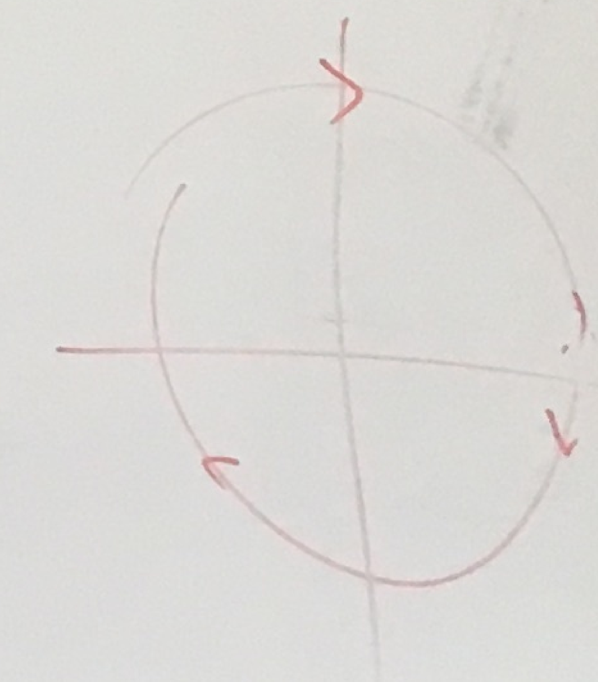
$\partial S : \{ (\cos\theta, -\sin\theta, 0) : 0 \le \theta \le 2\pi \}$ 8

(Saat yönü)

$$= \int_{\theta=0}^{2\pi} (\sin\theta - \cos\theta,\ 0,\ -\cos\theta)\cdot(-\sin\theta,\ -\cos\theta,\ 0)\,d\theta$$

$$= \int_{\theta=0}^{2\pi} (-\sin^2\theta + \sin\theta\cos\theta)\,d\theta$$

$$= -\pi$$



$-4 \le z \le 0$

Dış normal.

$F(x,y,z) = (x+y^2+\sin z,\ x+y^3+\cos z,\ \cos x+\sin y+z)$

$\iint F\cdot dS$

$F=(M,N,P)$

$\nabla\cdot F = \frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}$ (Divergence

(10)

$S=\partial V: \quad V = x^2+y^2-4 \le z \le 4-x^2-y^2$

$x^2+y^2 \le 4.$

$\nabla\cdot F = \frac{\partial}{\partial x}(x+y^2+\sin z) + \frac{\partial}{\partial y}(x+y^3+\cos z) + \frac{\partial}{\partial z}(\cos x+\sin y+z)$

$= 1+3y^2+1 = 2+3y^2.$

$$\iint_S F\cdot dS = \iiint_V \nabla\cdot F\, dx\,dy\,dz$$

$$= \iint_{x^2+y^2\le 4} \int_{x^2+y^2-4}^{z=4-x^2-y^2} (2+3y^2)\,dz\,dx\,dy$$

$$= \iint_{x^2+y^2\le 4} (2+3y^2)(2x^2+2y^2-8)\,dx\,dy$$

$$= \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (2+3r^2\sin^2\theta)(2r^2-8)\,r\,dr\,d\theta$$

$$= \int_{r=0}^{2}\left[(4r^2-16)\cdot 2\pi + 3\pi r^2(2r^2-8)\right] r\,dr \quad \square$$

$x^2+y^2 \le 4$.

$$\nabla \cdot F = \frac{\partial}{\partial x}(x+y^2+\sin z) + \frac{\partial}{\partial y}(x+y^3+\cos z) + \frac{\partial}{\partial z}(\cos x+\sin y+z)$$

$$= 1+3y^2+1 = 2+3y^2.$$

$$\iint_S F\cdot ds = \iiint_V \nabla\cdot F\,dx\,dy\,dz$$

x^2+y^2-4

$$= \iint_{x^2+y^2\le 4} \int_{z=4-x^2-y^2} (2+3y^2)\,dz\,dx\,dy$$

$$= \iint_{x^2+y^2\le 4} (2+3y^2)(2x^2+2y^2-8)\,dx\,dy$$

$$= \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (2+3r^2\sin^2\theta)(2r^2-8)\,r\,dr\,d\theta$$

$$= \int^{2}\Big[(4r^2-16)\cdot 2\pi + 3\pi r^2(2r^2-8)\Big]r\,dr$$

Örnek 2

$S:\ z=4-x^2-y^2$

$0\le z\le 4$ nın birleşimi.

ve

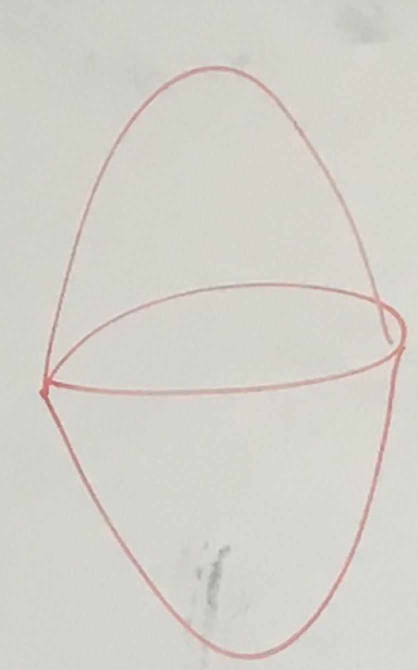
$z = x^2+y^2-4$

$-4\le z\le 0$

Dış normal.

$$F(x,y,z) = (x+y^2+\sin z,\ x+y^3+\cos z,\ \cos x+\sin y+z)$$

$$\iint F\cdot ds$$



Gauss / Divergence Thm (9)

$$\iint_S F\cdot ds = \iiint_V \nabla\cdot F\,dx\,dy\,dz$$

Dış orientasyon.

$S=\partial V$.

$F=(M,N,P)$

$$\nabla\cdot F = \frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}.\ \text{(Divergence)}$$