$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

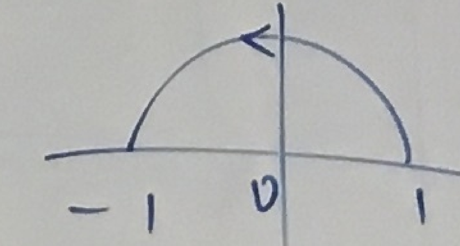
Örnek $f(x,y) = x^2 \cos(y)$

O zaman. $\nabla f = \left(2x\cos(y), -x^2\sin(y)\right)$

$\gamma(a)$

$$\int_\Gamma \nabla f \cdot dT$$

Örnek $\Gamma$ :




$\Gamma = \{(\cos t, \sin t) : t \in [0,\pi]\}$

$f(x,y) = x^2 y$

$\nabla f(x,y) = (2xy, x^2)$

$\gamma(a) = \gamma(t_0)$

1

## 2.5 Analizin Temel Teoremleri.

**Hatırlatma** $f : [a,b] \longrightarrow \mathbb{R}$

$f$ Integrellenebilir olsun

$$F(x) := \int_a^x f(y)\,dy$$

Eğer $F$ türevlenebilir ise,

$$F'(x) = f(x) \quad \forall x \in [a,b]$$

**Hatırlatma** $f$ sürekli ise $F$ otomatik olarak türevlenebilirdir.

$f : [a,b] \longrightarrow \mathbb{R}$ sürekli olsun.

$$F(x) := \int_a^x f(y)\,dy.$$

O zaman,

$$F'(x) = f(x) \quad \forall x \in [a,b]$$

f Integrallenebilir olsun

$$F(x) := \int_a^x f(y)\,dy$$

Eğer F türevlenebilir ise,

$$F'(x) = f(x) \quad \forall x \in$$

$$F(x) := \int_a^x f(y)\,dy.$$

O zaman,

$$F'(x) = f(x) \quad \forall x \in [a,b)$$

$\gamma(a) = \gamma(t_0)$

**Tanım 2.5.1** $f : \mathbb{R}^n \longrightarrow \mathbb{R}.$

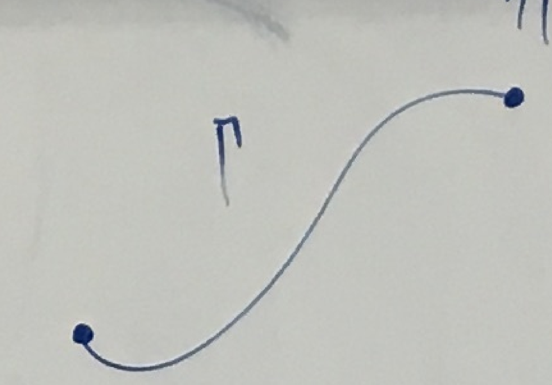
(Gradient)

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

**Örnek** $f(x,y) = x^2 \cos(y)$

O zaman.

$$\nabla f = \left(2x\cos(y), -x^2 \sin(y)\right)$$

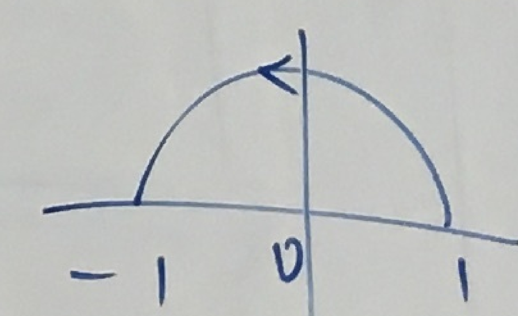



$\gamma : [a,b] \longrightarrow \mathbb{R}^n \quad \Gamma = \{\gamma(t) : t \in [a,b]\}$ (2)

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ Sürekli olsun.

$$\int_\Gamma \nabla f \cdot dT$$

**Örnek** $\Gamma$ :

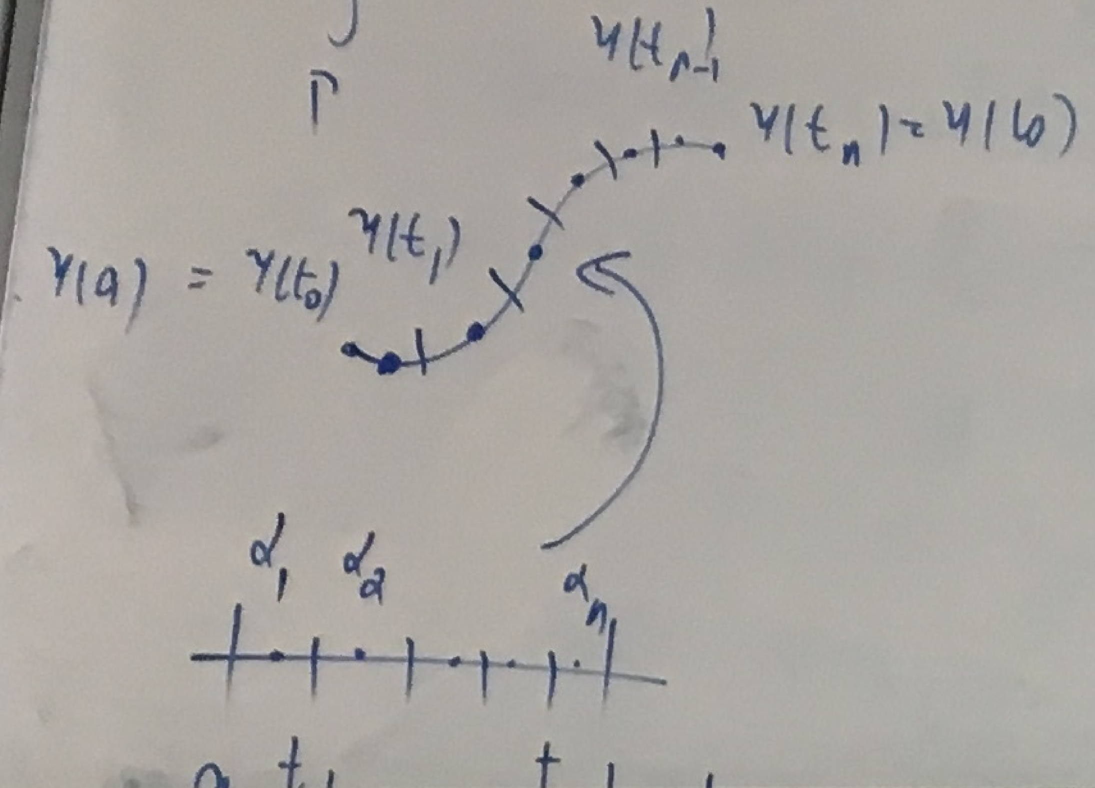



$\Gamma = \{(\cos t, \sin t) : t \in [0,\pi]\}$

$$f(x,y) = x^2 y$$

$$\nabla f(x,y) = (2xy, x^2)$$

$$\int_\Gamma \nabla f \cdot dT = \lim \sum_{i=1}^{n} \nabla f(\gamma(\alpha_i)) \cdot \left(\gamma(t_i) - \gamma(t_{i-1})\right)$$




Ortalama değer Thm

$$f(\gamma(t_i)) - f(\gamma(t_{i-1})) = \nabla f(\gamma(\beta_i)) \cdot \left(\gamma(t_i) - \gamma(t_{i-1})\right)$$

$$\beta_i \in [t_{i-1}, t_i]$$

$a, b \in \mathbb{R}$

$u$ : birim vektör.

O zaman $\exists\, c \in L(a,b)$, (a,b arasındaki doğru)

$$[f(b) - f(a)] \cdot u = [f'(c)(b-a)] \cdot u$$

Eğer $f : \mathbb{R}^n \to \mathbb{R}$ ise,

$$f(b) - f(a) = \nabla f(c) \cdot (b-a)$$

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$$\int_\Gamma \nabla f \cdot dT := \int_0^\pi \nabla f(\gamma(t)) \cdot \gamma'(t)\, dt.$$

$$\gamma(t) = (\cos t, \sin t)$$

$$\gamma'(t) = (-\sin t, \cos t).$$

$$\int_0^\pi \nabla f(\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^\pi (2\cos t \sin t, \sin^2 t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^\pi (-2\cos t \sin^2 t + \sin^2 t \cos t)\, dt$$

$$= \int_0^\pi -\sin^2 t \cos t\, dt = -\sin^3 t \Big|_0^\pi$$

$$= 0.$$

$g(t) = f(a + t(b-a))$ $\qquad g(0) = f(a)$

$\exists\, \alpha \in (0,1)$ $\qquad g(1) = f(b)$

$g(1) - g(0) = g'(\alpha) \cdot 1$ $\qquad g'(t) = \sum \frac{\partial f}{\partial x_i}(a + t(b-a))(b_i - a_i)$

$\Rightarrow f(b) - f(a) = \nabla f(c) \cdot (b-a).$ $\qquad = \nabla f(a + t(b-a)) \cdot (b-a)$

$c = a + \alpha(b-a)$

$\gamma'(t) = (-\sin t, \cos t)$.

$$\int_0^\pi \nabla f(\cos t, \sin t)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_0^\pi (2\cos t \sin t, \sin^2 t)\cdot(-\sin t, \cos t)\,dt$$

$= 0$

$g(t) = f(a + t(b-a))$ $\quad g(0) = f(a)$, $g(1) = f(\ldots)$

$\exists\, \alpha \in (0,1)$

$g(1) - g(0) = g'(\alpha)\cdot 1$ $\qquad g'(t) = \sum_i \frac{\partial f}{\partial x_i}(a + t(b-a))\,(b_i - a_i)$

$\Rightarrow f(b) - f(a) = \nabla f(c)\cdot(b-a)$. $\qquad = \nabla f(a + t(b-a))\cdot(b-a)$

$c = a + \alpha(b-a)$

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**2.5.2 Not**

$\Gamma = \{\gamma(t) : t \in [a,b]\} \subset \mathbb{R}^n$

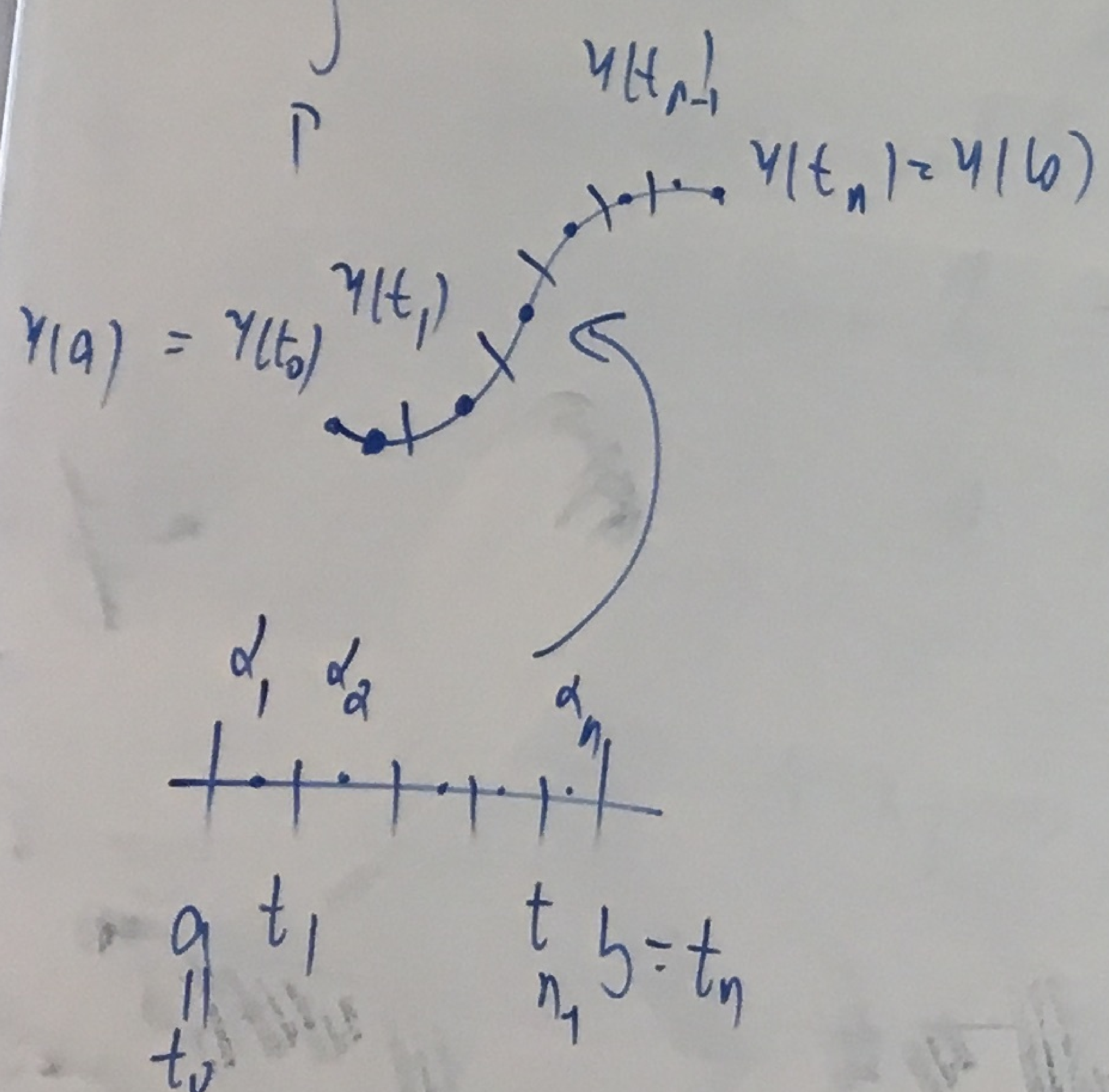
$f : \mathbb{R}^n \to \mathbb{R}$. $C^1$ fonksiyon.

$$\int_\Gamma \nabla f \cdot d\Gamma = \lim \sum_{i=1}^{n} \nabla f(\gamma(\alpha_i))\cdot\big(\gamma(t_i) - \gamma(t_{i-1})\big)$$

Ortalama değer Thm

$$f(\gamma(t_i)) - f(\gamma(t_{i-1})) = \nabla f(\gamma(\beta_i))\cdot\big(\gamma(t_i) - \gamma(t_{i-1})\big)$$

$\beta_i \in [t_{i-1}, t_i]$




**Ortalama Değer Thm**

$f : \mathbb{R}^n \to \mathbb{R}^n$

$a, b \in \mathbb{R}^n$

$u$ : birim vektör.

O zaman $\exists\, c \in L(a,b)$, (a,b arasındaki doğru)

$$[f(b) - f(a)]\cdot u = [f'(c)(b-a)]\cdot u$$

Eğer $f : \mathbb{R}^n \to \mathbb{R}$ ise,

$$f(b) - f(a) = \nabla f(c)\cdot(b-a)$$

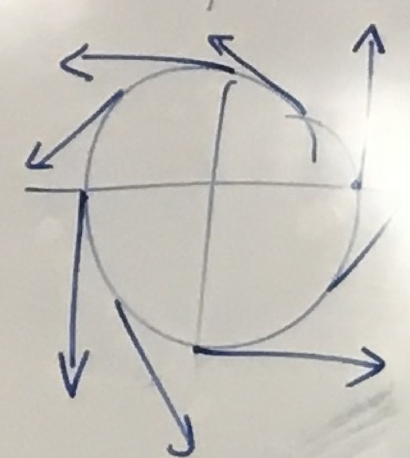
Örnek (önemli) 2.5.4

$F : \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$.

$\int_{S^1} F \cdot dT$

+ve orientasyon



$\gamma(t) = (\cos t, \sin t)$ (6)

$$\int_\Gamma F \cdot dT = \int_{t=0}^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_{t=0}^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi. \quad (\neq 0)$$

(5)

$$\int_\Gamma \nabla f \cdot dT = \lim \sum \nabla f(\gamma(\alpha_i)) \cdot (\gamma(t_i) - \gamma(t_{i-1}))$$

$$f(b) - f(a) = \sum f(\gamma(t_i)) - f(\gamma(t_{i-1}))$$

$$= \sum \nabla f(\gamma(\beta_i)) \cdot (\gamma(t_i) - \gamma(t_{i-1}))$$

$a = t_0 \leq \ldots \leq t_n = b$.

$\alpha_i, \beta_i \in [t_{i-1}, t_i]$

$t_{i-1} \quad \alpha_i \; \beta_i \quad t_i$

$f$ $C^1$ olduğu için $\nabla f$ sürekli

$\alpha_i, \beta_i \in [t_{i-1}, t_i]$, $t_i - t_{i-1}$ küçük ise,

$\nabla f(\gamma(\beta_i)) \approx \nabla f(\gamma(\alpha_i))$

Sonuç,

Thm 2.5.3 $\Gamma \subset \mathbb{R}^n$, $C^1$ eğri ($C^1$ parametrizasyon'a sahip).

$f : \mathbb{R}^n \to \mathbb{R}$ $C^1$ olsun.

O zaman,

$$\boxed{\int_\Gamma \nabla f \cdot dT = f(b) - f(a)}$$

$\frac{GM}{r}$

$f(b) - f(a) = \sum$

$= \sum \nabla f(\gamma(\beta_i)) \cdot (\gamma(t_i) - \gamma(t_{i-1}))$

$a = t_0 \leq \ldots \leq t_n = b.$

$\alpha_i, \beta_i \in [t_{i-1}, t_i]$

$f$ $C^1$ olduğu için $\nabla f$ sürekli ...

$\alpha_i, \beta_i \in [t_{i-1}, t_i]$, $t_i - t_{i-1}$ küçük ise,

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ $C^1$ olsun.

O zaman,

$$\boxed{\int_\Gamma \nabla f \cdot dT = f(b) - f(a)}$$

$\frac{GM}{r}$

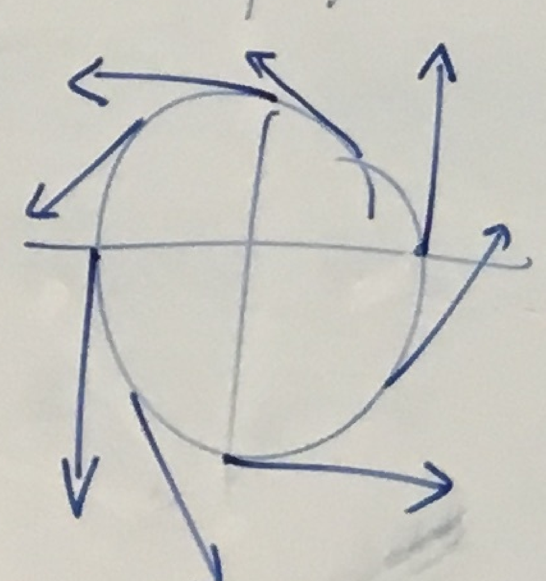
Örnek (önemli) 2.5.4

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$F(x,y) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$

$\int_{S^1} F \cdot dT$

+ve orientasyon



$F(\cos t, \sin t) = (-\sin t, \cos t).$

$\Gamma(t) = \{ \gamma(t) : t \in [0, 2\pi] \}$

$\gamma(t) = (\cos t, \sin t)$

$$\int_\Gamma F \cdot dT = \int_{t=0}^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_{t=0}^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_{t=0}^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi. \quad (\neq 0)$$

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$F = \nabla f$ sağlayan
bir $C^1$ fonksiyon $f$ var.

İspat Salı Günü

7

**Tanım 2.5.5** Korunumlu Vektör Alan.

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ sürekli vektör alan.

Eğer her kapalı $C^1$ eğri $\Gamma$ için

$$\int_\Gamma F \cdot dT = 0 \quad \text{ise}$$

$F$ KORUNUMLU vektör alan denir.

**Not** a) Eğer bir $C^1$ fonksiyon $f$ için

$F = \nabla f$ ise,

$F$ korunumlu bir vektör alandır.

b) $\left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ bir korunumlu vektör alan değildir.

için $\int_{\Gamma} F \cdot dT = 0$ ise

F KORUNUMLU vektor alan denir.

$\left( \frac{\;}{x^2+y^2} , \frac{\;}{x^2+y^2} \right)$

vektor alan değildir.

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Teorem 2.5.6 sürekli ve

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ bir korunumlu

vektor alan olsun. O zaman

$$F = \nabla f \quad \text{sağlayan}$$

bir $C^1$ fonksiyon $f$ var.

İspat Salı Günü

$$f = (xy,\ yz,\ x^2yz)$$

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & x^2yz \end{vmatrix} = i(x^2z - y) + j(0 - xyz) + k(0 - x) = (x^2z - y,\ -xyz,\ -x)$$

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## GREEN'İN TEOREMİ

$E \subset \mathbb{R}^2$ bölge.

$\partial E$ bir $C^1$ eğri olsun

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ vektör alan.

$F = (M, N)$. O zaman

$$\int_{\partial E} F \cdot dT = \int_E \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

+ ve oriyantasyon

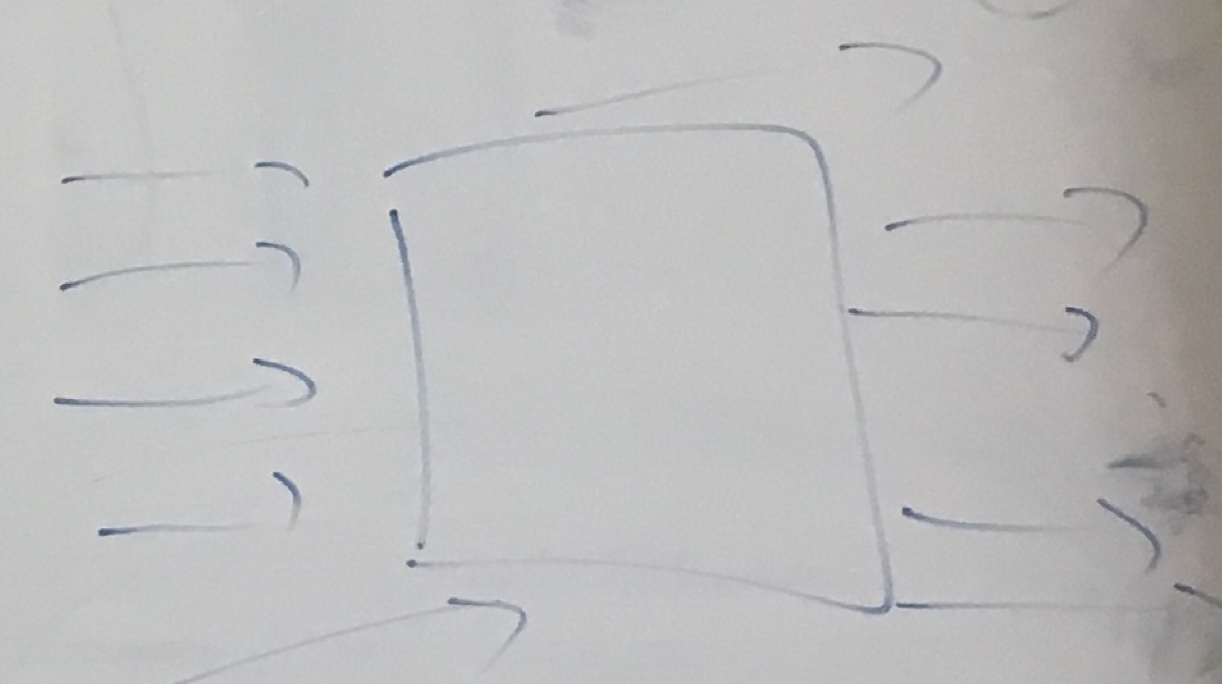
## GAUSS TEOREM

$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

$V \subset \mathbb{R}^3$ Bölge.

$\partial V$ bir $C^1$ yüzey.

O zaman



$$\int_{\partial V} F \cdot dS = \int_V \nabla \cdot F$$

$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ vektör alan.

zaman

$$\oint_{\partial E} F \cdot dT = \int_E \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

+ve orientasyon

zaman

$$\int_{\partial V} F \cdot dS = \int_V \nabla \cdot F$$

Tanımlar 2.5.7

a) $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

GRADIENT

$$\nabla f := \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

CURL $f = (F, G, H)$

$$\nabla \times f := \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F & G & H \end{vmatrix}$$

$f = (xy, yz, x^2yz)$

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & x^2yz \end{vmatrix} = i(x^2z - y) + j(0 - xyz) + k(0 - x) = (x^2z - y, -xyz, -x)$$

(9)

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$f = (F, G, H)$

DIVERGENCE $\nabla \cdot f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\nabla \cdot f = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F, G, H) = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}$$