Strongly Rayleigh measures and the Kadison-Singer Problem

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1. Negative dependance
   - Attempts and examples
   - Geometric approach to Negative dependance
   - Strongly Rayleigh measures

2. The Restricted Invertibility principle and Kadison-Singer
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   - Root shift estimates
   - Analytic Lieb-Sokal
   - Immanantal polynomials
Main actors in story

Richard Kadison
Isadore Singer
Joel Anderson
Charles Akemann
Jean Bourgain
Adam Marcus
Daniel Spielman
Nikhil Srivastava
Nik Weaver
Julius Borcea
Petter Branden
Shayan Oveis-Gharan
Robin Pemantle
Pete Casazza
Janet Tremain
Lior Tzafriri
Mohan Ravichandran, MSGSU, Istanbul
Thomas Liggett
Nima Anari
In this talk we only work with binary (\(\{0, 1\}\) valued) random variables.
\(\mu \in \mathcal{P}(2^{[n]})\), i.e PM on subsets of \([n] = \{1, \cdots, n\}\).

1. **Positive dependance** well understood.

   \[ \text{PLC}(+ \text{ lattice condition}) \quad \mu(S)\mu(T) \leq \mu(S \cup T)\mu(S \cap T), \quad \forall S, T \subset [n]. \]
   \[ \text{PA}(\text{positive association}) \quad \mathbb{E}(f)\mathbb{E}(g) \leq \mathbb{E}(fg), \quad \forall f, g : 2^{[n]} \to \mathbb{R}, \uparrow. \]

   FKG(Fortuin-Kasteleyn-Ginibre) theorem, 1971: \(\text{PLC} \implies \text{PA}\), local-global.

2. **Negative dependance**: Analogous notions for repelling random variables:

   \[ \text{NLC}(- \text{ lattice condition}) \quad \mu(S)\mu(T) \geq \mu(S \cup T)\mu(S \cap T), \quad \forall S, T \subset [n]. \]
   \[ \text{NA}(\text{negative association}) \quad \mathbb{E}(f)\mathbb{E}(g) \leq \mathbb{E}(fg), \quad \forall f, g : 2^{[n]} \to \mathbb{R}, \uparrow \quad \text{supp}(f) \cap \text{supp}(g) = \emptyset. \]

   However, \(\text{NLC} \nRightarrow \text{NA}\)

Popularized by Robin Pemantle (2000 - ...)
Various definitions

Given \( \mu \in P_n := P(2^{[n]}) \), we consider the multi-affine (generating) polynomial (of \( \mu \)),

\[
P_\mu = \sum_{S \subseteq [n]} \mu(S)z^S.
\]

\( X_1, \ldots, X_n \) : co-ordinate random variables, \( X_i(S) = 1 \) if \( i \in S \), else 0.

1. \( \mu \in P_n \) is **pairwise negative correlated** (p-NC) if

\[
\mathbb{E}(X_i) \mathbb{E}(X_j) \geq \mathbb{E}(X_iX_j), \ i \neq j \in [n] \iff \partial_i P_\mu(1) \partial_j P_\mu(1) \geq \partial_{ij} P_\mu(1).
\]

2. \( \mu \in P_n \) satisfies the **strong hereditary negative lattice condition** (h-NLC+) if,

\[
\mu(S) \mu(T) \geq \mu(S \cup T) \mu(S \cap T), \quad \forall S, T \subseteq [n].
\]

and the same holds for

1. **Projections** : Projection onto \( 2^X \) where \( X \subseteq [n] \), \( \tilde{\mu}(S) \sim \sum_{T = S \cup X^c} \mu(T) \) for every \( X \subseteq S \).
2. **Application of external fields** : Given \( (a_1, \ldots, a_n) \in \mathbb{R}_+^n \), \( \tilde{\mu}(S) \sim \mu(S) \prod_{i \in S} a_i \).

3. \( \mu \in P_n \) is **strongly conditionally negatively associated** (CNA+) if it is conditionally negatively associated and the same holds upon applying projections and external fields.
Examples

1. **Determinantal measures:** \( \mu \in \mathcal{P}_n | \exists \text{ PSD } A \in M_n(\mathbb{R}) \text{ such that,} \)

\[
\mu(\{T \subset [n]|S \subset T\}) = \sum_{S \subset T} \mu(T) = \det[A(S)], \quad \forall S \subset [n].
\]

(Lyons, 2003) : \( \mu \) determinantal measure in \( \mathcal{P}_n \). If the associated PSD matrix is a contraction, then it is CNA+.

2. **Symmetric exclusion processes** Take \( n \) points on \( \{0, 1\}^k \). These points jump to neighbours with fixed probabilities but jumps to occupied spots are forbidden.

Goal : Come up with a notion of negative dependence that is preserved under these transitions.

None of p-NC, h-NLC+ or CNA+ are.
Real Stable polynomials

Polynomial $p(z_1, \cdots, p_m)$ called stable if

$$p(z_1, \cdots, z_n) \neq 0, \quad \forall (z_1, \cdots, z_n) \mid \text{Im}(z_k) > 0 \quad \forall k \in [m].$$

**Real Stable:** Stable + real coefficients.

1. Univariate real stability = Real rootedness.
2. $p = \text{det}[A + z_1B_1 + \cdots + z_nB_n]$, where $A$ is symmetric and the $B_i$ are PSD.
3. Real stability preservers,
   1. $p \rightarrow \partial_i p$.
   2. (Julius Borcea, Petter Branden, 2006) Lieb-Sokal lemma: $p \rightarrow q(\partial_1, \cdots, \partial_n)p$.
   3. $p(z_1, \cdots, z_n) \rightarrow p(t, z_1, \cdots, z_n)$ for $t \in \mathbb{R}$.
4. Convexity: (Adam Marcus, Daniel Spielman, Nikhil Srivastava 2013, Terence Tao 2013, Alexander Scott, Alan Sokal, 2010) $a \in \mathbb{R}$ called above the roots of $p$ or $a \in Ab_p$ if $p(a + z) > 0 \quad \forall z \in \mathbb{R}_n^+$. Then,

   Complete monotonicity \quad $(-1)^k \partial_j^k \left( \frac{\partial_i p}{p} \right)(a) \geq 0 \quad \forall a \in Ab_p, \; i, j \in [n], \; k \geq 0$.

**Strongly Rayleigh measures**

**Definition (Julius Borcea, Petter Branden, Thomas Liggett 2009)**

$\mu \in P_n$ is called *Strongly Rayleigh* if $P_\mu := \sum_{S \subseteq [n]} \mu(S)z^S$ is real stable.

Product measures, Determinantal measures, uniform spanning tree measures are SR. Preserved under symmetric exclusion processes. Implies p-NC, h-NLC+ and CNA+.

$X$ random vector taking values in $\mathbb{Z}_+^n$. Define,

$$P_X = \sum \mathbb{P}(X = a)z^a.$$  

$M(X)$: Maximum degree of the variables $z_i$.

**Theorem (Multivariate CLT, Ghosh, Pemantle, Liggett, 2016)**

$X_n$ sequence of random vectors with sequence $s_n$ such that there is a matrix $A$ satisfying,

$$\frac{\text{Cov}(X_n)}{s_n^2} \rightarrow A, \quad \frac{M(X_n)^{1/3}}{s_n} \rightarrow 0.$$

Then,

$$\frac{X - \mathbb{E}(X)}{s_n} \rightarrow N(0, A).$$
**Bourgain-Tzafriri’s RIP**

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map.

The singular values of $T$, denoted by $s_1(T) \geq \cdots \geq s_m(T)$, are the square roots of the eigenvalues of $T^*T$. They are also the square roots of the diagonal entries of the positive semi-definite matrix $T^*T$.

Question: Find a large subspace on which $T$ is well invertible.

The singular rank of $T$, denoted as $\text{srank}(T)$, is defined as:

$$\text{srank}(T) = \frac{\|T\|_2^2}{\|T\|_2^2} = \frac{\text{Trace}(T^* T)}{\|T\|^2} = \frac{\sum s_k(T)^2}{s_1(T)^2}.$$
Remark (Singular vector basis)

\{v_1, \cdots, v_m\} basis of singular vectors for \( T \), i.e. eigenbasis for \( T^* T \).
\( V = \text{span}\{v_1, \cdots, v_k\} \), where \( k = c \text{srank}(T) \) for some \( c < 1 \). Then,

\[
\min_{V} s_{\min} T \geq \sqrt{(1-c) \sqrt{\frac{\text{srank}(T)}{m}}}
\]

Similar statement holds for any basis! One version,

Theorem (The restricted invertibility principle, B-T, Spielman-Srivastava)

\{v_1, \cdots, v_m\} orthonormal basis. Then, for any \( c < 1 \), there exists \( \sigma \subset [m] \) of size \( k = c \text{srank}(T) \),

\[
\min_{P_{\sigma} \mathbb{R}^m} s_{\min} T \geq \frac{1}{5} \sqrt{(1-c) \sqrt{\frac{\text{srank}(T)}{m}}}
\]

Theorem (R, 2016)

Let \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \) be a linear operator. Then, for any \( 0 \leq \delta \leq 1 \), there is a subset \( \sigma \) of size \( |\sigma| = \delta \frac{\|T\|_2^4}{\|T\|_4^4} \) and such that, letting \( c = \frac{|\sigma|}{m} \), we have,

\[
s_{\min}(T |_{P_{\sigma} \mathbb{R}^m}) \geq \sqrt{\frac{\text{srank}(T)}{m}} \left[ \sqrt{1-c} - \sqrt{\delta-c} \right].
\]
Theorem (Joel Anderson’s Paving problem, Adam Marcus, Daniel Spielman, Nikhil Srivastava 13)

There are universal constants $\epsilon < 1$ and $r \in \mathbb{N}$ so that for any zero diagonal contraction $A \in M_n(\mathbb{R})^{sa}$, there are diagonal projections $Q_1, \ldots, Q_r$ with $Q_1 + \cdots + Q_r = I$,

$$\lambda_1(Q_iAQ_i) < \epsilon, \quad 1 \leq i \leq r.$$ 


Restricted Invertibility in analogous form,

Theorem (Restricted Invertibility, R 2016)

For any trace zero contraction $A \in M_n(\mathbb{R})^{sa}$ and any $c \leq \frac{1}{2}$ there is a principal submatrix $A(S)$ of size $cn$ such that

$$\lambda_1[A(S)] \leq 2\sqrt{c - c^2}.$$
Casazza, Speegle, Tremain, Weber 2006: Equivalent to fundamental problems in Geometric Functional Analysis, Convex geometry, Signal processing, Harmonic analysis, Frame theory (Feichtinger conjecture), Coding theory, ...

\[ \mathcal{I} \subset \mathbb{Z}, \quad S(I) := \text{span}(\{e^{int} : n \in S\}) \]

**Theorem (Weyl)**

Given any \([a, b] \subset [0, 1], \epsilon > 0\) there is a partition \(X_1 \cup \cdots \cup X_n = \mathbb{Z}\) such that \(\forall f \in S(X_j), 1 \leq j \leq n,\)

\[(1 - \epsilon)\|f\|_2^2 \leq \frac{\|f \chi[a, b]\|_2^2}{b - a} \leq (1 + \epsilon)\|f\|_2^2.\]

Does the same hold for any measurable set \(E\)? Equivalent to Kadison-Singer.
\[ \mu \in \mathcal{P}_n \text{ Strongly Rayleigh, } A \in M_n(\mathbb{R})^{sa} \text{ self adjoint.} \]

Sample principal submatrices of \( A \), picking \( A_S \) with probability \( \mu(S) \).

\( A_S \) : Principal submatrix of \( A \) with rows and columns from \( S \) removed.

Sublime idea of MSS : Take expectation not of largest eigenvalue, but of the characteristic polynomial!

**Theorem (MSS, Nima Anari and Oveis-Gharan 2014, R 2016)**

\[
\mathbb{E} \chi[A_S] = \sum_{S \subseteq [n]} \mu(S) \chi[A_S],
\]

is real rooted and further,

\[
\mathbb{P} [\lambda_1 \chi[A_S] \leq \lambda_1 \mathbb{E} \chi[A_S]] > 0.
\]

Further,

\[
\mathbb{E} \chi[A_S] = P_{\mu}(\partial_1, \ldots, \partial_n) \det[Z - A] |_{Z=xl}.
\]
Restricted invertibility: Uniform measure on \( n - k \) element subsets of \([n]\),

\[
P_{\mu} = \binom{n}{k}^{-1} \sum_{|S|=n-k} z^S = \binom{n}{k}^{-1}(\partial_1 + \cdots + \partial_n)^k z_1 \cdots z_n.
\]

Kadison-Singer: Pick subsets of \([n] \times [n]\) of the form \( T \times T^c \).

\[
P_{\mu_2} = 2^{-n} \left( \prod_{i=1}^{n} (\partial z_i + \partial y_i) \right) (z_1 \cdots z_n)(y_1 \cdots y_n).
\]
**Cauchy-Poincare, R.C. Thompson and MSS’ Markov principle**

**Theorem (Cauchy-Poincare)**

\[ A \in M_n(\mathbb{R})^{sa}. \text{ Then, the eigenvalues of } \chi[A] \text{ and } \chi[A_i] \text{ interlace.} \]

**Lemma (Markov principle)**

\[ p_1, \cdots, p_n \text{ be same degree monic real rooted with common interlacer. Then, } \forall k \exists i, \]

\[ \lambda_k(p_i) \leq \lambda_k(p_1 + \cdots + p_n). \]

**Lemma (Obreshkoff)**

\[ \{p_i\}_{i=1}^n \text{ degree } k \text{ monic real rooted. Common interlacer iff every convex combination real rooted.} \]


*Let* \( A \in M_n(\mathbb{R}) \) *be hermitian. Then, \( \exists i \in [n] \) *such that,*

\[ \lambda_1(\chi[A_i]) \leq \lambda_1 \left( \sum \chi[A_i] \right) = \lambda_1(\chi'[A]). \]

*For any* \( k \in [n] \), *there is a size* \( k \) *subset* \( S \subset [n] \) *such that,*

\[ \lambda_1(\chi[A_S]) \leq \lambda_1 \left( \sum_{|S|=k} \chi[A_S] \right) = \lambda_1(\chi^{(k)}[A]). \]
Set $Z = \text{diag}(z_1, \cdots, z_n)$ diagonal matrix of variables.

**Lemma**

If $A \in M_n(\mathbb{R})$ and $S \subset [n]$. Then,

$$\det[AS] = \frac{\partial^S}{\partial^S z} \det[Z + A] \big|_{Z=0}, \quad \chi[AS] = \frac{\partial^S}{\partial^S z} \det[Z - A] \big|_{Z=xI}.$$

$\mu \in \mathcal{P}_n$ Strongly Rayleigh. $A \in M_n(\mathbb{R})^{sa}$ real symmetric.

Create tree. $n + 1$ levels.

Nodes at level $k$ indexed by subsets of $[k - 1]$. Mark node at level $k$ by $\sum_{S \supset T} \mu(S)\chi[AS]$.

Children of node indexed by $S \subset [k] : n - k$ nodes indexed by $S \cup i$, for $i \notin S$.

Leaf nodes : $\chi[AS]$ for $S \subset [n]$.

Top node : $\sum_{S \subset [n]} \mu(S)\chi[AS]$. 
Theorem (R, 2016)

Let $A \in M_n(\mathbb{R})$ be real symmetric. Then, the sum of the characteristic polynomials of all the $2^n$ pavings of $A$ is real rooted and satisfies,

$$\sum_{S \sqcup T = [n]} \chi[A_S \oplus A_T] = \left[ \prod_{m=1}^{\lfloor n/2 \rfloor} (\partial z_m + \partial y_m) \right] \det[Z - A] \det[Y - A] \mid Z = Y = xI .$$

Further, there is a paving $(S, T) \in \mathcal{P}_2$ such that

$$\lambda_1 \chi[A_S \oplus A_T] \leq \lambda_1 \left[ \sum_{S \sqcup T = [n]} \chi[A_S \oplus A_T] \right] .$$

Lemma (R, 2016)

$$\left[ \prod_{m=1}^{\lfloor n/2 \rfloor} (\partial z_m + \partial y_m) \right] \det[Z - A] \det[Y - A] \mid Z = Y = xI = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det[Z - A]^2 \mid Z = xI .$$
Definition (Mixed determinant)

\[ A, B \in M_n(\mathbb{R}), \]
\[ D(A, B) := \sum_{S \coprod T = [n]} \det[A(S)] \det[B(T)]. \]

Definition

Given a matrix \( A \in M_n(\mathbb{R}) \), define

\[ \det_r(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i \sigma(i)} (-1)^{\text{sgn}(\sigma)} r_c(\sigma), \quad \chi_r[A] := \det_r(xI - A). \]

where \( c(\sigma) \) denotes the number of cycles in \( \sigma \).

Lemma (R, 2016)

\[ E_{\mathcal{P}_2([n])} \chi[A, x] = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det[Z - A]^2 \big|_{Z = xI = \chi_2[A]} = D(xI - A, xI - A). \]
Conjecture

$A \in M_n(\mathbb{R})^+, \text{ positive contraction, diagonal entries of } A \text{ all be at most } \alpha \leq \frac{1}{2}. \text{ Then,}$

$$\max \text{root } \chi_2[A] \leq \frac{1}{2} + \sqrt{\alpha(1-\alpha)} = \frac{1}{4} \left( \sqrt{2\alpha} + \sqrt{2(1-\alpha)} \right)^2.$$

$$MSS : \frac{1}{2} + \sqrt{2\alpha} + \alpha, \quad BCMS : \frac{1}{2} + \sqrt{2\alpha(1-2\alpha)}.$$

Theorem (R 2016, 2paving)

$A \in M_n(\mathbb{R})^+, \text{ positive contraction, diagonal entries of } A \text{ all be at most } \alpha \leq \frac{1}{4}. \text{ Then,}$

$$\max \text{root } \chi_2[A] \leq \frac{1}{4} \left( \sqrt{\alpha} + \sqrt{3(1-\alpha)} \right)^2.$$

Theorem (R 2016, paving diagonal 1/2 projections)

$A \in M_n(\mathbb{R})^+, \text{ positive contraction, diagonal entries of } A \text{ all be at most } \alpha \leq \frac{1}{2}. \text{ Then,}$

$$\max \text{root } \chi_4[A] \leq \frac{(3 + \sqrt{7})^2}{32} \approx 0.996.$$
$p$ : Real rooted degree $n$ polynomial. For $b \geq \lambda_1(p)$ and $\varphi > 0$, define

$$\Phi_p(b) := \frac{p'}{p} = \sum \frac{1}{b - \lambda_i}, \quad \text{smax}_\varphi(p) := \Phi^{-1}(\varphi) = \lambda_1(p' - \varphi p).$$

Note : For any $\varphi > 0$, we have : $\lambda_1(p) < \text{smax}_\varphi(p)$.

**Proposition (Marcus, 2014)**

Let $p$ be real rooted and $\varphi > 0$. Then,

$$\text{smax}_\varphi(p') \leq \text{smax}_\varphi(p) - \frac{1}{\varphi}, \quad \Rightarrow \quad \text{smax}_\varphi(p^{(k)}) \leq \text{smax}_\varphi(p) - \frac{k}{\varphi}.$$

Follows from concavity of $\frac{1}{\Phi_p}$ above $\lambda_1(p)$. 
A ∈ Mₙ(ℝ)ˢᵃ, set

\[ p₀ = \det[Z - A]², \quad p₁ = \frac{∂}{∂z₁} \det[Z - A]², \quad \ldots, \quad pₙ = \frac{∂ⁿ}{∂z₁ \cdots ∂zₙ} \det[Z - A]². \]

Real stable polynomial \( p(z₁, \ldots, zₙ) \), say \( z ∈ \mathbb{R}^n \) is in \( Abₚ \) if \( p(z + t) \neq 0 \) for any \( t ∈ \mathbb{R}^n. \)

(Upper) potential of \( p \) in direction \( j \),

\[ \Phi_j^p(z) = \frac{\partial_j p}{p}(z). \]

Basic fact, for any \( z ∈ Abₚ \) and \( i, j ∈ [n] \),

\[ \Phi_j^p > 0, \quad \partial_i \Phi_j^p < 0 \text{ (Monotonicity),} \quad \partial_i^² \Phi_j^p > 0 \text{ (Convexity).} \]

**Lemma (MSS, R)**

\[ \Phi_j^p(z + δe_i) ≤ \Phi_j^p(z), \quad \delta = \frac{1}{1 - \Phi_j^p}, \quad i, i ∈ [n]. \]

Suppose \( p \) is of degree at most 2 in \( z_i \),

\[ \Phi_j^p(z - δe_i) ≤ \Phi_j^p(z), \quad \delta = \frac{1}{2\Phi_j^p}, \quad j ∈ [n]. \]
Theorem (R, 2016)

\( p(z_1, \cdots, z_n) \) real stable and of degree at most 2 in each of the variables. (For instance, \( p = \det [Z - A]^2 \)). Let \( q = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} p \). Then, for any \( z \in Ab_p \),

\[
\Phi^j_q(z - \delta) \leq \Phi^j_p(z), \quad j \in [n] \quad \text{where} \quad \delta = \min_{j \in [n]} \frac{1}{2\Phi^j_p(z)}.
\]

Lemma (R, 2016)

Suppose \( p = \det [Z - A]^2 \) where \( A \) is a positive contraction and \( z = zI \) where \( z > \lambda_1(A) \), then,

\[
\Phi^j_p(zI) \leq \frac{\delta}{z - 1} + \frac{1 - \delta}{z}, \quad \delta = \max(A_{ii}).
\]

Theorem (R, 2016)

\( A \in M_n(\mathbb{R})^+ \), positive contraction, diagonal entries of \( A \) all be at most \( \alpha \leq \frac{1}{4} \). Then,

\[
\maxroot \chi_2[A] \leq \inf_{z \geq 1} z - \frac{1}{2} \left( \frac{\alpha}{z - 1} + \frac{1 - \alpha}{z} \right)^{-1} = \frac{1}{4} \left( \sqrt{\alpha} + \sqrt{3(1 - \alpha)} \right)^2.
\]
Remark

Suppose we could shift the barrier to the left by \( \frac{1}{\Phi_j(p)(z)} \) instead of \( \frac{1}{2\Phi_j(p)(z)} \), we would have the conjectured optimal estimate of maxroot \( \chi_2[A] \leq \frac{1}{2} + \sqrt{\alpha(1 - \alpha)} \).

Alas, not true in general. Also fails for polynomials of the form \( \det[Z - A]^2 \). Similar estimates can be gotten for \( \chi_3[A] \) and \( \chi_4[A] \) through brute force means.

Theorem (R, 2016)

Let \( A \in M_n(\mathbb{R})^+ \), positive contraction, diagonal entries of \( A \) all be at most \( \alpha \). Then,

\[
\max \text{root } \chi_3[A] \leq \frac{1}{9} \left( \sqrt{5(1 - \alpha)} + 2\sqrt{\alpha} \right)^2, \quad \alpha \leq \frac{4}{9}.
\]

\[
\max \text{root } \chi_4[A] \leq \frac{1}{16} \left( \sqrt{7(1 - \alpha)} + 3\sqrt{\alpha} \right)^2, \quad \alpha \leq \frac{9}{16}.
\]
Questions: Analytic Borcea-Branden + Lieb-Sokal + Immanantal polynomials

Root shift estimates

**Question**

$p$ and $q$ real stable polynomials in $n$ variables. Estimates for zero free regions of,

$$q(\partial_1, \cdots, \partial_n)p(z_1, \cdots, z_n).$$

Special case of great interest,

$$e_k(\partial_1, \cdots, \partial_n) \det[Z - A].$$

One variable case: $\varphi > 0$. Define $\text{smax}_{\varphi}(p) = \phi_p^{-1}(\varphi)$. 

**Theorem**

$p$ real rooted. Then,

$$\text{smax}_{\varphi}(\partial p) \leq \text{smax}_{\varphi}(p) - \frac{1}{\varphi}, \quad \text{smax}_{\varphi}[(\partial - \alpha)p] \leq \text{smax}_{\varphi}(p) - \frac{1}{\varphi - \alpha}.$$ 

**Theorem (One variable Analytic Lieb-Sokal)**

$p, q$ real rooted. Then,

$$\text{smax}_{\varphi}(q(\partial)p) \leq \text{smax}_{\varphi}(p) - \Phi_q(\varphi).$$
Multivariable case: $\varphi \in \mathbb{R}_+^n$. 

Let $\text{smax}_\varphi(p) = \{b \in \mathbb{R}^n : \Phi_p(b) = \varphi\}$. 

Given two sets $A, B \in \mathbb{R}^n$, say $A \prec B$ if for all $b \in B$ and $h \in \mathbb{R}_+^n$, $b + h \notin A$. 

**Conjecture**

$p, q$ real stable in $\mathbb{R}[z_1, \cdots, z_n]$ and let $a \in A b_p$. Then,

$$\text{smax}_\varphi(q(\partial)p) \prec \text{smax}_\varphi(p) - \Phi_q(\varphi).$$
Given a class function \( \phi \) on \( S_n \) and a matrix \( A \), the expression 

\[
\det_\phi (A) := \sum_{\sigma \in S_n} \left( \prod_{i \in [n]} a_{i \sigma(i)} \right) \phi(\sigma),
\]

is called an immanant. One may define the expression,

\[
\chi_\phi [A] := \det_\phi [xI - A].
\]

c(\sigma) : number of cycles in \( \sigma \).
When \( \phi(\sigma) = (-1)^{\text{sgn}(\sigma)} \), we get \( \chi[A] \).
When \( \phi(\sigma) = (-1)^{\text{sgn}(\sigma)} c(\sigma) \), we get \( \chi_r[A] \).
\( r \in \mathbb{N} \) : We have that \( \chi_r[A] \) is real rooted for hermitian \( A \).

**Question**

Which immanantal polynomials are real rooted for all hermitian arguments?

**Conjecture**

*Those immanants such that*

\[
\det_\phi (A) = p(\partial_1, \cdots, \partial_n) \det[Z + A]^k \big|_{Z=0}, \quad \text{deg}(p) = (k - 1)n, \quad p \text{ real stable } + \text{ symmetric}.
\]