Definable soluble and nilpotent envelopes "around" subgroups in simple theory
1. Motivation

**Theorem.** $G$ is an infinite group with small theory. Then $G$ has an infinite abelian subgroup.

**Conjecture (Smidt).** Every infinite group has an infinite abelian subgroup.

False. (1968 Adian Novikov).

**Definition.** A **Tarski Monster** is an infinite (countable) group such that every proper subgroup is either $\{1\}$ or cyclic of order a prime $p$.

**Fact (Ol’shanskii 1979).** For every prime $p > 10^{75}$, there are $2^\aleph_0$ non-isomorphic Tarski monsters.

**Corollary.** Any Tarski Monster has $2^\aleph_0$ countable models sharing its theory, up to isomorphisms.
1. Motivation

**Theorem.** $G$ is an infinite group with small theory. Then $G$ has an infinite abelian subgroup.

**Corollary (Wagner).** $G$ is a group with small and stable theory. Then $G$ has a definable infinite abelian subgroup.

**Proof.** A infinite abelian. Take $Z(C(A))$.

**Question.** When can one find definable abelian groups around abelian subgroups?

**Question.** When can one find definable nilpotent/soluble groups around nilpotent/soluble subgroups?
2. What is known

**Remark.** If $G$ has **dcc on centralisers**, and $A \leq G$ is abelian $H$, then $Z(C(A))$ is a definable abelian envelope of $H$.

**Fact (Poizat).** If $G$ is **stable** and $H \leq G$ is $n$-nilpotent/$n$-soluble, $H$ has a definable $n$-nilpotent/$n$-soluble envelope.

**Fact (Shelah).** If $G$ has **NIP** and $A \leq G$ is abelian, $A$ has a definable abelian envelope.

**Fact (Aldama).** If $G$ has **NIP** and $H \leq G$ is $n$-nilpotent/normal $n$-soluble, $H$ has a definable envelope with same property.

**Fact (Altınel, Baginsky).** If $G$ has **dcc on centralisers** and $H \leq G$ is $n$-nilpotent, $H$ has a definable $n$-nilpotent envelope.
3. Question

What happens if $G$ has merely a simple theory? Can one find a definable abelian/nilpotent/soluble envelope of an abelian/nilpotent/soluble $H \leq G$?

The answer is no. But:

**Proposition.** If $G$ is simple and $A \leq G$ is abelian, then $A$ has a definable envelope which is abelian-by-finite.

**Theorem A.** If $G$ is simple and $N \leq G$ is $n$-nilpotent, there is a definable $2n$-nilpotent group finitely many translates of which cover $N$.

**Theorem B.** If $G$ is simple and $S \leq G$ is $n$-soluble, there is a definable $2n$-soluble group finitely many translates of which cover $S$. 
4. Stable and simple definitions and properties

Definition. $X$ is a definable subset of $G$, $\phi(x, y)$ a formula. The $\phi$-Cantor-Bendixson rank of $X$:

- $\text{CB}(X, \phi) \geq 0$ if $X \neq \emptyset$,
- $\text{CB}(X, \phi) \geq n + 1$ if there are infinitely many $2$-disjoint $\phi$-sets $X_1, X_2, \ldots$ with $\text{CB}(X_i \cap X, \phi) \geq n$.

Definition. $G$ is stable if $\text{CB}(G, \phi)$ is finite for every formula $\phi$.

Definition. $X$ is a definable subset of $G$, $\phi(x, y)$ a formula, $k$ a natural number. The $D(\ldots, \phi, k)$-Cantor rank of $X$:

- $D(X, \phi, k) \geq 0$ if $X \neq \emptyset$,
- $D(X, \phi, k) \geq n + 1$ if there are infinitely $k$-disjoint sets defined by $\phi(x, a_1), \phi(x, a_2), \ldots$ with $D(X_i \cap X, \phi, k) \geq n$.

Definition. $G$ has a simple theory if $D(G, \phi, k)$ is finite for every formula $\phi$ and natural number $k$. 
Remark. \( D(X, \phi, k) \leq CB(X, \phi) \): stability implies simplicity.

Fact (Baldwin Saxl’s chain condition). \( G \) is a group with stable theory, \( \phi(x, y) \) a formula. There is some \( n \) such that every descending chain of subgroups defined by \( \phi \)-formulae has no more than \( n \) elements.

Fact (Wagner’s chain condition). \( G \) is a group with simple theory, \( \phi(x, y) \) a formula. There is some \( n \) such that every descending chain of subgroups defined by \( \phi \)-formulae has no more than \( n \) elements, up to finite index.
## 4. Stable and simple definitions and properties

<table>
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<th>In a stable theory</th>
<th>Analogue in a simple theory</th>
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<td>Uniform dcc</td>
<td>Uniform dcc up to finite index</td>
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<td>abelian groups</td>
<td>FC-groups (eg finite, abelian)</td>
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<tr>
<td>$C_G(H)$</td>
<td>$FC_G(H) = { g \in G : g^H \text{ is finite}}$ (Haimo, 1953)</td>
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<tr>
<td>$Z(H)$</td>
<td>$FC(G) = FC_G(G)$</td>
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<tr>
<td>$Z_{n+1}(G)$</td>
<td>$FC_{n+1}(G) \ (FC_{n+1}(G)/FC_n(G) = FC(G/FC_n(G))$</td>
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<tr>
<td>$n$-nilpotent</td>
<td>$n$-FC-nilpotent ($FC_n(G) = G$, Haimo) (eg finite, nilpotent)</td>
</tr>
<tr>
<td>$n$-soluble</td>
<td>$n$-FC-soluble (Duguid, McLain, 1956)</td>
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**Proposition.** $G$ is a saturated group with simple theory, and $H$ is a definable subgroup. Then $FC_G(H)$ is definable.
5. Main results

**Theorem.** Let $G$ be a group with simple theory and $N$ a subgroup of $G$. If $N$ is $FC$-nilpotent of class $n$, then it is contained in a definable $FC$-nilpotent group of class $n$.

**Theorem.** Let $G$ be a group with simple theory, and let $S$ be a subgroup of $G$. If $S$ is $FC$-soluble of class $n$, then it is contained in a definable $FC$-soluble group of class $n$ the members of whose $FC$ series are definable subgroups.

**Fact (Wagner).** In a group with simple theory, an $FC$-nilpotent definable subgroup is virtually-$m$-nilpotent, with $m \leq 2n$.

**Proposition.** In a group with simple theory, an $FC$-soluble definable subgroup is virtually-$m$-soluble, with $m \leq 2n$. 
5. Main results

**Corollary.** If $G$ is simple and $N$ is $n$-nilpotent, there is a definable $2n$-nilpotent group finitely many translates of which cover $N$.

**Corollary.** If $G$ is simple and $S$ is $n$-soluble, there is a definable $2n$-soluble group finitely many translates of which cover $S$.

**Corollary.** In a group with simple theory, let $N$ be a normal nilpotent subgroup of class $n$. There is a definable normal nilpotent group of class at most $3n$ containing $N$.

**Corollary.** In a group with simple theory, let $S$ be a normal soluble subgroup of class $n$. There is a definable normal soluble group of class at most $3n$ containing $S$. 
6. Next questions: nilpotent and soluble radical

In a group $G$, the **Fitting subgroup** $\text{Fit}(G)$ is the subgroup generated by all normal nilpotent subgroups of $G$. The **soluble radical** $R(G)$ is generated by all normal solvable subgroups of $G$.

**Remark (Ould Houcine).**

1. $\text{Fit}(G)$ is definable if and only if it is nilpotent.
2. $R(G)$ is definable if and only if it is solvable.

**Question.** In a group with simple theory, are $R(G)$ and $\text{Fit}(G)$ definable?

**Fact (Wagner).** *If $G$ is stable, $\text{Fit}(G)$ is definable.*

**Remark.** Known for algebraic groups, groups of finite RM (Nesin).

**Fact (Baudish).** *If $G$ is superstable, $R(G)$ is definable.*

**Remark.** Known for groups of finite RM (Belegradek), and groups of finite U-rank (Baldwin-Pillay).
Fact (Elwes, Jaligot, Macpherson, Ryten). $G$ is a supersimple group of finite $SU$-rank such that $T^{eq}$ eliminates $\exists^\infty$. Then $R(G)$ is definable and soluble.

Question (Elwes, Jaligot, Macpherson, Ryten). $G$ is a supersimple group of finite $SU$-rank such that $T^{eq}$ eliminates $\exists^\infty$. Is $Fit(G)$ definable and nilpotent?

Proposition. Yes, and one does not need to assume that $T^{eq}$ eliminates $\exists^\infty$.

Proposition. $G$ is a supersimple group of finite $SU$-rank. Then $R(G)$ is definable and soluble.