Notes on compactness

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September 8, 2014

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I made these notes while preparing my talk "Compactness" at the Caucasian Mathematics Conference, Tbilisi, Georgia, September 5 & 6, 2014. I did not actually print out the notes for use during the talk; but I spoke of some points from memory. I am likely to use these notes in preparing for the tutorial at the School mentioned below. Documents that I have been able to consult directly (if only in electronic form) are in the References at the end; footnotes contain other documents.

1. The Compactness Theorem is that if every finite subset of a set of sentences has a model, then the whole set has a model.

2. I shall give a tutorial on compactness at the

5th World Congress and School on Universal Logic
June 20–30, 2015, Istanbul
http://www.uni-log.org/

3. John Dawson [1] reports that Vaught,[1] as well as van Heijenoort and Dreben,[2] finds the Compactness Theorem [for countable sets of sentences] to be implicit in Skolem's 1922 paper [9] (Dawson writes 1923). Looking at the paper, I find this plausible. (It would be desirable to work out the details...)

4. But the Compactness Theorem (for countable sets) was not made explicit until Gödel's 1930 article [2] based on his doctoral dissertation.

5. Mal′cev stated the Compactness Theorem (calling it, according to the translator, the "general local theorem"), in 1941,[3] but for a proof referred to his 1936 paper,[4] which had been found dubious in a 1937 review by Rosser.[5]

6. But Mal′cev obtained interesting algebraic results, as:

> A locally soluble group of class k is soluble of class k.
>
> "Locally special groups are direct products of their Sylow subgroups."

In general, the review by I. Kaplansky of the 1941 paper says,

> Let $E_1, \ldots, E_k$ be "elementary" properties of a group. Say that a group G has type $[E_1, \ldots, E_k]$ if it possesses a normal series $G \supset G_1 \supset \cdots \supset G_k$ such that each G_i/G_{i+1} has property E_{i+1}. Theorem: G is of type $[E_1, \ldots, E_k]$ if and only if this is true for each finitely generated subgroup. An interesting application is the case where $k = 2$ and E_2 is "abelian", E_1 is "order $\leqslant m$

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[1] Vaught, R. L. Model theory before 1945. *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pp. 153–172. Amer. Math. Soc., Providence, R.I., 1974.

[2] In the introduction to the *Collected Works, vol. 1* of Gödel, 1986.

[3] Mal′cev, A. On a general method for obtaining local theorems in group theory. (Russian.) *Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Fak. 1* (1941), no. 1, 3–9.

[4] Maltsev, A . I. 1936 'Untersuchungen aus dem Gebiete der mathematischen Logik', *Matematicheskii Sbornik, n.s.,* **1**, 323-336.

[5] Mal′cev's 1936 paper is earlier than the earliest of his on MathSciNet; so I have not seen Rosser's review.

for a certain fixed m". In this way we can pass from Jordan's theorem that any finite group of complex n-by-n matrices has an abelian subgroup of index $\leqslant$ a number m depending only on n, to Schur's theorem that the same is true for any (possibly infinite) torsion group of matrices. Further applications are given to a theorem of Černikov [ibid. **8(50)** (1940), 377–394; MR0003422 (2,217a)] on Sylow series and a theorem attributed to Baer on lattice isomorphisms of groups.

7. As the Prime Ideal Theorem for *countable* rings requires no choice principle (*i.e.* follows from ZF alone) so too the Compactness Theorem for *countable* signatures. This is observed, in effect, by Skolem [9], who needs to prove the Löwenheim–Skolem Theorem (and in effect proves countable Compactness) without using AC, so that the Skolem Paradox results more plausibly.

8. Four bits of evidence that the contents of my talk are not well known or well appreciated:

a) Some model-theoretic sources suggest that the Compactness Theorem is *immediately* equivalent to the compactness of Stone spaces. (The two theorems are equivalent, but not immediately equivalent.) My examples are:

   i. Keisler [4] lets S be the set of all Th($\mathfrak{A}$) ("in a first order language L"). Given a sentence φ [*sic*[6]] of L, he defines

$$[\varphi] = \{p \in S \colon \varphi \in p\},$$

   and he notes that such sets are a basis of closed sets for a topology on S. He calls S thus topologized the **Stone space** of L. He notes that, by the compactness theorem [*sic*[7]], every set of basic closed sets with the finite intersection property has nonempty intersection.

[6] I think an arbitrary *sentence* ought to be denoted by σ, while φ can be an arbitrary *formula*.

[7] Keisler does not capitalize the name of the theorem.

> In other words, the compactness theorem states that the Stone space S of L is compact.

This is not literally wrong, but it uses a nonstandard *definition* of a Stone space, and so it obscures the significance of the Compactness Theorem.

ii. Tent and Ziegler [10] say at the head of their Section 2.2, just before proving the Compactness Theorem,

> Its name is motivated by the results in Section 4.2 which associate to each theory a certain compact topological space.

Section 4.2 begins:

> We now endow the set of types of a given theory with a topology. The Compactness Theorem 2.2.1 then translates into the statement that this topology is compact, whence its name.
>
> Fix a theory T. An n-type is a maximal set of formulas $p(x_1, \ldots, x_n)$ consistent with T. We denote by $\mathrm{S}_n(T)$ the set of all n-types of T. We also write $\mathrm{S}(T)$ for $\mathrm{S}_1(T)$...
>
> Remark. The Stone duality theorem asserts that the map
>
> $$X \mapsto \{C \mid C \text{ clopen subset of } X\}$$
>
> yields an equivalence between the category of 0-dimensional compact spaces and the category of Boolean algebras. The inverse map assigns to every Boolean algebra B its Stone space $\mathrm{S}(B)$, the set of all ultrafilters (see Exercise 1.2.4) of B.

Here "consistent" means satisfiable. Exercise 1.2.4 defines filters, ultrafilters, and ultraproducts and asks for a proof of Łoś's

Theorem. Again nothing is literally wrong here. But the notation suggests that type spaces are *by definition* Stone spaces, so that one might think the Compactness Theorem was a special case of the compactness of Stone spaces.

b) The same sources and others prove Compactness using the full force of the Axiom of Choice (Keisler uses transfinite recursion; Tent & Ziegler, Zorn's Lemma). Poizat [6] spends 7 pages discussing AC and mentions that the Prime Ideal Theorem is strictly weaker, but does *not* point out that this is sufficient for the Compactness Theorem; rather, he calls AC indispensible for this and other theorems. (Rothmaler [7] does refer the reader to Hodges [3, §6.2], where matters are made explicit.)

c) The basic Compactness Theorem is not always distinguished from stronger statements. Rubin & Rubin [8] give almost literally the result of Vaught [12] given in the abstract quoted in its entirety below; but they do not give Vaught's parenthesis:

> Consider the following three known theorems concerning the existence of models of sentences (of the first-order predicate logic) : (1) a sentence having a model of power $\mathfrak{b}$ has also a model of every power $\mathfrak{a}$ such that $\aleph_0 \leqq \mathfrak{a} \leqq \mathfrak{b}$; (2) a sentence having a model of power $\aleph_0$ has also a model of every power $\mathfrak{c} \geqq \aleph_0$; (3) if Q is a set of sentences, in which the set of individual constants involved may have an arbitrary power $\mathfrak{d}$, and every finite subset of Q has a model, then Q has a model, whose power is not greater than $\mathfrak{d}+\aleph_0$. Theorem: *Each of* (1), (2), *and* (3) *implies the Axiom of Choice in its general form.* As regards (3), this answers a question, raised by Henkin (Trans. Amer. Math. Soc. vol. 74 (1953) p. 420), as explicitly formulated. (On the other hand, the statement, obtained from (3) by dropping the final phrase ("whose...") has been shown by Henkin and Łoś to be equivalent to the existence of prime ideals in Boolean algebras, and its exact relationship to the Axiom of Choice

> is still not determined.) The Theorem is derived from the known result (of Tarski) that the Axiom of Choice is implied by the statement: if $\aleph_0 \leqq \mathfrak{b}$, then $\mathfrak{b}^2 = \mathfrak{b}$. (Received January 13, 1956.)

Rubin and Rubin list 8 equivalent formulations of the *Boolean* Prime Ideal Theorem; but none of them is the (arbitrary) Prime Ideal Theorem or the Compactness Theorem.

Rubin and Rubin refer to Vaught's three statements as a downward Löwenheim–Skolem theorem, an upward Löwenheim–Skolem theorem, and a compactness theorem, designating them as `LOG 1`, `LOG 2`, and `LOG 3` respectively. Some sort of correction to `LOG 3` as indicated is needed (otherwise Γ could use singular predicates, for example, to say that all models are strictly larger than κ).

By the method of constants, `LOG 3` $\Rightarrow$ `LOG 2`. By the original Löwenheim–Skolem Theorem, `LOG 2` $\Rightarrow$ `LOG 1`. Now suppose

$$\aleph_0 \leqslant \kappa, \qquad \mu = 2^{\kappa\cdot\aleph_0}.$$

Then

$$\mu^2 = \left(2^{\kappa\cdot\aleph_0}\right)^2 = 2^{\kappa\cdot\aleph_0\cdot 2} = 2^{\kappa\cdot\aleph_0} = \mu,$$
$$\aleph_0 \leqslant \kappa \leqslant \mu.$$

In a signature $\{F\}$, where F is a binary operation symbol, let σ say F is surjective and injective. Then σ has a model of cardinality μ. If `LOG 1` holds, then σ has a model of cardinality κ, so

$$\kappa^2 = \kappa. \qquad (*)$$

This in turn implies the Well Ordering Theorem. For, let κ^* be the least aleph that is not less than κ:

$$\kappa^* = \{\xi \colon \xi \in \mathbf{ON} \;\;\&\;\; \xi \leqslant \kappa\}.$$

By (∗) (for all κ),

$$\kappa + \kappa^* = (\kappa + \kappa^*)^2 \geqslant \kappa \cdot \kappa^* = \kappa^* \cdot \kappa.$$

This implies (by a result of Tarski) that κ and κ^* are comparable. Therefore $\kappa = \kappa^*$.

Under the supplied correction to `LOG 1`, without assuming the Axiom of Choice (but with the Prime Ideal Theorem), all we know is that the model of Γ is bounded in cardinality by $\bigcup_{n \in \omega} \kappa^n + \aleph_0$. Assuming this bound is equal to $\kappa + \aleph_0$ is already equivalent to AC, as we have just seen.

9. The Maximal Ideal Theorem can be proved with Zorn's Lemma; but the method of constructing the maximal ideal by transfinite recursion is useful to know. The method is available automatically (without AC) in the countable case.

10. It is worthwhile to prove the Prime Ideal Theorem without AC, to emphasize that primeness (unlike maximality) is local, hence amenable to Compactness.

11. According to Dawson, in topology, *bicompact* was first defined ("every open covering possesses a finite subcovering") "in a paper submitted in 1923 but not published until 1929" by Alexandroff and Urysohn.

12. The current *name* of the Compactness Theorem (again according to Dawson) first appeared in Tarski's 1950 address to the ICM in Cambridge, Massachusetts (published in 1952). One of his versions of the theorem is that **Str** is compact; but the formulation (as his Theorem 17) is basically as follows. For every *formula* φ of $\mathscr{S}$, let $\mathbf{Mod}(\varphi)$ be the class of structures $\mathfrak{A}$ of $\mathscr{S}$ such that every tuple of A satisfies φ in $\mathfrak{A}$. Thus $\mathbf{Mod}(\varphi) = \mathbf{Mod}(\forall \boldsymbol{x}\ \varphi)$, if $\varphi \in \mathrm{Fm}_{\mathscr{S}}(\boldsymbol{x})$. If a set of such classes has empty intersection, then so does some finite subset. Tarski's notation is more complicated, involving first the function $\mathfrak{F}$ associated with φ that sends each $\mathfrak{A}$ to the subset of A^ω whose elements satisfy φ in $\mathfrak{A}$. Then

our **Mod**(φ) is his $\mathfrak{EL}(\mathfrak{F})$, which he calls an **arithmetical class.** $\mathfrak{F}$ itself is an **arithmetical function,** and there is a compactness theorem (Theorem 13) for these as well.

13. Dawson notes, "few logic texts bother to explain the topological context of the compactness theorem at all." His example of an exception is Monk [5]. Monk derives the Compactness Theorem from the Completeness Theorem (proved by Henkin's method). He immediately says,

> The compactness theorem lies at the start of model theory, and it will play a very important role in Part IV. For some motivation for the name compactness theorem, see Exercise 11.59.

The exercise, quoted by Dawson, is

> Let $\mathbf{K}$ be a nonempty set of $\mathscr{L}$-structures. For each $\mathbf{L} \subseteq \mathbf{K}$ let $C\mathbf{L} = \{\mathfrak{A} \in \mathbf{K} : \mathfrak{A}$ is a model of every sentence which holds in all members of $\mathbf{L}\}$. Show that with respect to C as a closure operator, $\mathbf{K}$ is a compact topological space.

Dawson shows that this is false as stated, because there is a counterexample in which $\mathbf{K}$ consists of the finite substructures of $\langle\mathbb{N}, <\rangle$: look at the sentences saying there are at least n elements. Dawson *could* just have noted that, while the given closure operator makes **Str** a compact space, not every subclass of this is compact. Dawson does note that presumably Monk intended $\mathbf{K}$ to be "elementary, or perhaps AC_δ"—but here AC_δ means closed, that is, elementary.

References

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