Geometry as made rigorous by Euclid and Descartes

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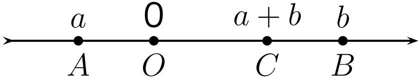
0 Introduction

According to one **textbook** of the subject,

> **analytic geometry** is based on the idea that a one-to-one correspondence can be established between the set of points of a straight line and the set of all real numbers.

- A straight line is an **ordered abelian group** in a geometrically natural way.



$$AO = CB$$

- This ordered group is **isomorphic** to $(\mathbb{R}, +, <)$.

The isomorphism from $(\mathbb{R}, +, <)$ to a straight line induces a **multiplication** on that straight line.

This multiplication has a **geometric meaning.**

This, if anything, is the "Fundamental Principle of Analytic Geometry."

Descartes establishes it.

Details can be worked out from Book I of Euclid's *Elements.*

1 Origins of geometry

Geometry comes from γεωμετρία, formed of γῆ (*land*) and μέτρον (*measure*).

According to Herodotus (b. *c.* 484 B.C.E.), in Egypt, land was taxed in proportion to size. If the Nile's annual flooding robbed you of land, the king sent **surveyors** to measure the loss.

> From this, to my thinking, the Greeks learned the **art of measuring land** (γεωμετρίη); the sunclock and the sundial, and the twelve divisions of the day, came to Hellas not from Egypt but from Babylonia. [2.109]

According to Aristotle (b. 384 B.C.E.),

> as more and more *skills* (τέχναι) were discovered, some relating to the *necessities* (ἀναγκαῖα) and some to the pastimes of life, the inventors of the latter were always considered wiser than those of the former, because their *sciences* (ἐπιστήμαι) did not aim at utility. Hence when all the discoveries of this kind were fully developed, the sciences concerning neither *pleasure* (ἡδονή) nor necessities were invented, and first in those places where men **had leisure** (σχολάζω).
>
> Thus **mathematics** (μαθηματικαί) originated in Egypt (Αἴγυπτος), because there the *priestly class* (ἱερέων ἔθνος) was allowed leisure. [*Metaphysics* I.i.16]

2 Euclid's geometry

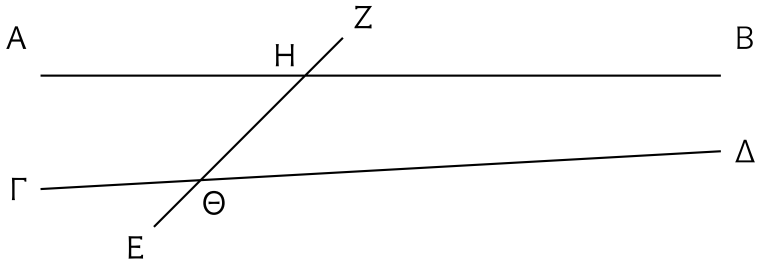
The *Elements* (Στοιχεῖα) of Euclid (fl. 300 B.C.E.) begins with five **Postulates** (Αἰτήματα "Demands").

By the first four, we have three tools of a builder:

- a **ruler** or **chalk line,** (1) to draw a straight line from one point to another, or (2) to extend a given straight line;
- a **compass,** (3) to draw a circle with a given center, passing through a given point;
- a **set square,** whose mere existence ensures (4) that all right angles are equal to one another.

The **Fifth Postulate** is that, if

$$\angle \mathsf{BH\Theta} + \angle \mathsf{H\Theta\Delta} < \mathsf{2} \text{ right angles},$$



then AB and $\mathsf{\Gamma\Delta}$, extended, **meet.**

- This is **unambiguous** by the 4th postulate.
- It tells us what the 2nd postulate can **achieve.**

After the Postulates come the Axioms or **Common Notions** (Κοιναὶ ἔννοιαι):

1. **Equals** to the same are equal to one another.
2. If equals be **added** to equals, the wholes are equal.
3. If equals be **subtracted** from equals, the remainders are equal.
4. Things **congruent** with one another are equal to one another.
5. The whole is **greater** than the part.

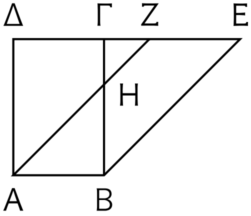
After the Common Notions come the **48 propositions** of Book I of the *Elements*, and then the remaining 12 books.

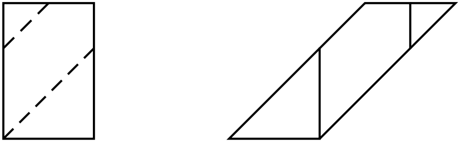
3 Equality and proportion

Equality in Euclid is:

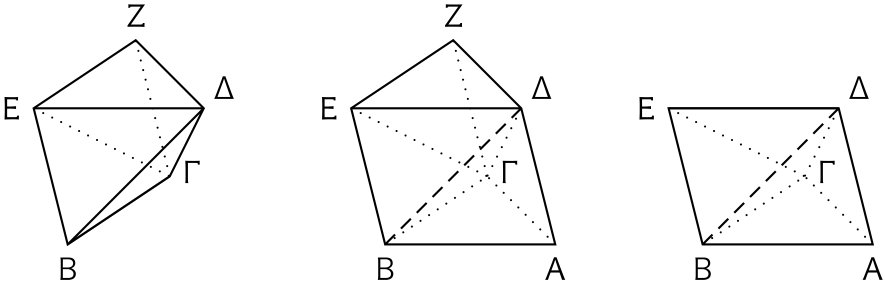
- not identity, by
  - the definitions of the circle and the right angle,
  - the 4th Postulate;
- symmetric (implicitly);
- transitive (Common Notion 1);
- implied by congruence (C.N. 4);
- implied by congruence of respective parts (C.N. 2);
- not universal (C.N. 5).

Equality is congruence of parts only in **Proposition I.35:** *Parallelograms on the same base and in the same parallels are equal.*





Equality is *not* congruence of parts in **Proposition XII.7:** *A triangular prism is divided into three equal triangular pyramids.*



This uses **Proposition XII.5:** *Triangular pyramids of the same height have to one another the* ***same ratio*** *as their bases.*

By **Book V,** a magnitude A has to B the **same ratio** (αὐτός λόγος) that C has to D if, for all positive integers k and n,

$$kA > nB \iff kC > nD.$$

Then the four magnitudes are **proportional** (ἀνάλογος), and today we write $A : B : : C : D$. The pair

$$\left(\left\{\frac{n}{k} : kA > nB\right\}, \left\{\frac{n}{k} : kA \leqslant nB\right\}\right)$$

is a **Dedekind cut.** Thus, for Dedekind (b. 1831), a ratio is a positive real number.

The theory of proportion is said to be due to **Eudoxus of Knidos** (b. 408 B.C.E.), a student of Plato.

By **Propositions V.9** and **16,**

$$\text{if } A : B :: C : C, \text{ then } A = B.$$

Proof. We use the so-called **Axiom of Archimedes** (b. 287 B.C.E.), found in Euclid's definition of **having a ratio** (λόγον ἔχω). Suppose

$$A > B.$$

Then for some n, we have $n(A - B) > B$. Consequently

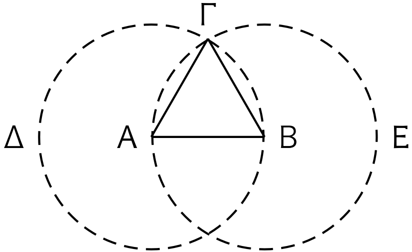
$$nA > (n + 1)B, \qquad nC < (n + 1)C,$$

and therefore

$$A : B > C : C. \qquad \square$$

4 Some propositions

Proposition I.1 of the *Elements* is the *problem* of constructing, on a given bounded straight line, an **equilateral triangle.**

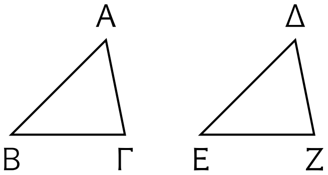


Does this need an **axiom of continuity?**

Side-Angle-Side is an axiom for David Hilbert (b. 1862).

For Euclid it is **Proposition I.4,** the first proper *theorem.* Suppose

$$\mathsf{AB} = \mathsf{\Delta E},$$
$$\mathsf{A\Gamma} = \mathsf{\Delta Z},$$
$$\angle \mathsf{BA\Gamma} = \angle \mathsf{E\Delta Z}.$$



Then, by the meaning of equality:

1. AB can be applied exactly to $\mathsf{\Delta E}$.
2. At the same time, $\angle\mathsf{BA\Gamma}$ can be applied to $\angle\mathsf{E\Delta Z}$.
3. Then $\mathsf{A\Gamma}$ will be applied exactly to $\mathsf{\Delta Z}$.
4. Consequently $\mathsf{B\Gamma}$ will be applied exactly to EZ.

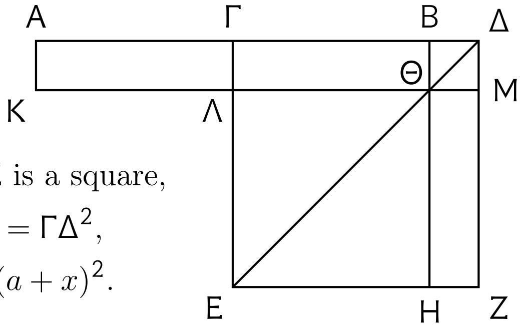
Proposition I.5 is that the base angles of an isosceles triangle are equal.

Immanuel Kant (b. 1724) alludes to it in the *Critique of Pure Reason*:

> **Mathematics** has from the earliest times... travelled the secure path of a science. Yet it must not be thought that it was as easy for it as for logic... to find that royal path... its transformation is to be ascribed to a **revolution,** brought about by the happy inspiration of a single man... a new light broke upon the first person who demonstrated [Proposition I.5] (whether he was called "Thales" or had some other name). [B x–xi]

Euclid's **Proposition II.6** is a *synthesis*:

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.



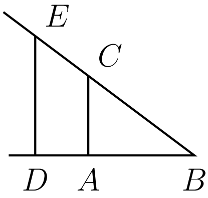
$\because \quad \mathsf{A\Gamma} = \mathsf{\Gamma B}$ and $\mathsf{\Gamma\Delta ZE}$ is a square,

$$\therefore \quad \mathsf{A\Delta} \cdot \mathsf{\Delta B} + \mathsf{\Gamma B}^2 = \mathsf{\Gamma\Delta}^2,$$

$$(2a + x) \cdot x + a^2 = (a + x)^2.$$

5 Descartes's geometry

Euclid's products are **areas.** As Descartes (b. 1596) observes, they can be **lengths,** if a unit length is chosen.



If AB is the unit, and $DE \parallel AC$, then

$$BE = BD \cdot BC.$$

Thus any number of lengths can be **multiplied.**

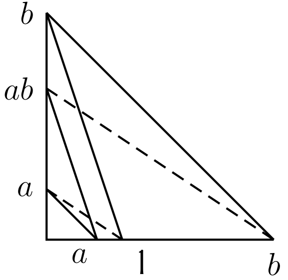
Descartes quotes Pappus (fl. 320 C.E.) as noting that any number of **ratios** can be multiplied:

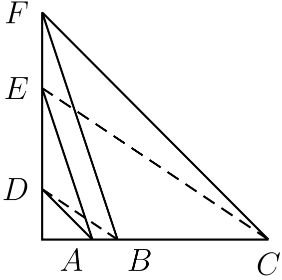
$$A : B \mathbin{\&} B : C \mathbin{\&} \dots \mathbin{\&} Y : Z :: A : Z.$$

As Hilbert shows, multiplication is commutative by a version of **Pappus's Hexagon Theorem.**

Let $AD \parallel CF$ and $AE \parallel BF$. Then

$$ab = ba \iff BD \parallel CE,$$
$$BD \parallel CE.$$

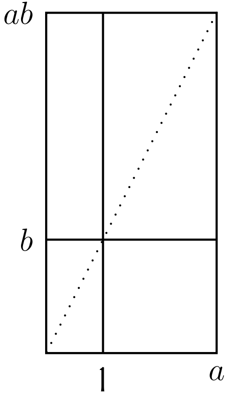




Alternatively, we can establish Hilbert's **algebra of segments** on the basis of Book I of the *Elements* alone.

Multiplication as in the diagram is commutive, given that:

- the rectangles about the diagonal are equal **(I.43)**,
- all rectangles of equal dimensions are congruent **(I.8, 33)**.



For associativity, we use I.43 and its converse:

By definition of ab, cb, and $a(cb)$,

$$A + B = E + F + H + K,$$
$$C = G,$$
$$A = D + E + G + H.$$

Also $a(cb) = c(ab)$ if and only if

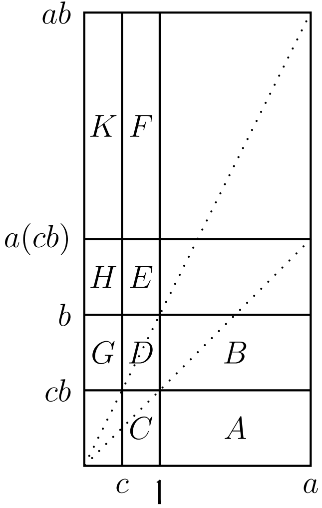
$$C + D + E = K.$$

We compute

$$D + C + B = F + K.$$

We finish by noting

$$B = E + F.$$

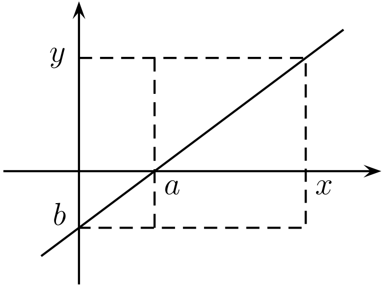


6 Conclusion

We thus **interpret** an ordered field in the Euclidean plane.

The **positive elements** of this ordered field are congruence-classes of line-segments.

We impose a **rectangular coordinate system** as usual.



Straight lines are now given by **linear equations:**

$$a \cdot (y - b) = -b \cdot x,$$
$$bx + ay = ab.$$

Conversely, let an **ordered field** K be given.

In $K \times K$, obtain the **Cauchy–Schwartz Inequality,** and then the **Triangle Inequality.** Define

- **line segments:** $\boldsymbol{ab}$ is the set
$$\{\boldsymbol{x} \colon |\boldsymbol{b}-\boldsymbol{a}| = |\boldsymbol{b}-\boldsymbol{x}| + |\boldsymbol{x}-\boldsymbol{a}|\};$$
- their **congruence:** $\boldsymbol{ab} \cong \boldsymbol{cd}$ means
$$|\boldsymbol{b}-\boldsymbol{a}| = |\boldsymbol{d}-\boldsymbol{c}|;$$
- **angle congruence:** $\angle\boldsymbol{bac} \cong \angle\boldsymbol{edf}$ means
$$\frac{(\boldsymbol{c}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a})}{|\boldsymbol{c}-\boldsymbol{a}|\cdot|\boldsymbol{b}-\boldsymbol{a}|} = \frac{(\boldsymbol{f}-\boldsymbol{d})\cdot(\boldsymbol{e}-\boldsymbol{d})}{|\boldsymbol{f}-\boldsymbol{d}|\cdot|\boldsymbol{e}-\boldsymbol{d}|}.$$

Alternatively, define figures to be **congruent** when they can be transformed into one another by a composition of a **translation**

$$\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{a}$$

and a **rotation**

$$\boldsymbol{x} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \boldsymbol{x},$$

where $a^2 + b^2 = 1$.

K should be **Euclidean** or at least **Pythagorean.**

One just ought to be clear what one is doing.

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