Piet Mondrian, *Tableau No. IV; Lozenge Composition with Red, Gray, Blue, Yellow, and Black*

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A derivation of a field is an operation $D$ on the field satisfying

\[ D(x + y) = Dx + Dy, \quad D(x \cdot y) = Dx \cdot y + x \cdot Dy. \]

**Example.** “Taking the derivative,”

\[ f \mapsto f', \]

on $\mathbb{R}(x)$ or the field of meromorphic functions on $\mathbb{C}$.

The derivations of a field $K$ compose a **vector space** over $K$,

\[ \text{Der}(K), \]

where the vector-space operations are given by

\[ (D_0 + D_1)x = D_0x + D_1x, \quad (a \cdot D)x = a \cdot (Dx). \]

Then $\text{Der}(K)$ also has a **multiplication**, given by

\[ [D_0, D_1] = D_0 \circ D_1 - D_1 \circ D_0; \]

this is the **Lie bracket** operation, which I may denote by

\[ b. \]
In this context, a **multiplication** is an operation \( \cdot \) on an abelian group that distributes over addition:

\[
x \cdot (y + z) = x \cdot y + x \cdot z, \quad (x + y) \cdot z = x \cdot z + y \cdot z.
\]

A **ring** in the most general sense is an abelian group with a multiplication.

**Examples.**

1. \( \mathbb{Z} \) and \( \mathbb{Q} \);

2. the **Cayley–Dickson** algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{S}, \ldots \);

3. the ring \( \text{M}_n(R) \) of \( n \times n \) matrices over a ring \( R \);

4. \( (\mathbb{R}^3, \times) \);

5. \( (\text{Der}(K), b) \).
A **group operation** is another kind of multiplication.
The **permutations** of a set $A$ compose a group,

$$(\text{Sym}(A), \circ),$$

the operation being **composition**.
If there is a homomorphism from a group $(G, \cdot)$ to $(\text{Sym}(A), \circ)$,
then $(G, \cdot)$ acts on $A$.
The action is **faithful** if the homomorphism is one-to-one.

**Theorem** (Cayley). A group acts faithfully on its underlying set.
Indeed, if $(G, \cdot)$ is a group, and $g, x \in G$, define

$$\lambda_g(x) = g \cdot x.$$  

Then

$$g \mapsto \lambda_g: (G, \cdot) \to (\text{Sym}(G), \circ).$$
Now let $V$ be an *abelian* group.
The **endomorphisms** of $V$ compose an abelian group,

$$\text{End}(V).$$

**Examples.** $\phi \mapsto \phi(1) : \text{End}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \quad \text{End}(\mathbb{Z} \oplus \mathbb{Z}) \cong M_2(\mathbb{Z}).$

Then $(\text{End}(V), \circ)$ is an **associative ring**: a ring $(R, \cdot)$ satisfying

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

If there is a homomorphism from a field $K$ to $(\text{End}(V), \circ)$, then $V$ is a **vector space** over $K$.
We may say then $K$ **acts** on $V$.

**Example.** $K$ acts on $\text{Der}(K)$.

But also $(\text{Der}(K), b)$ may be said to **act** on $K$.
So $K$ and $(\text{Der}(K), b)$ are **interacting rings**.
The multiplications of $V$ compose an abelian group, 

$$\text{Mult}(V).$$

This has an involutory automorphism, $m \mapsto \cdot m$, where 

$$\cdot m(x, y) = m(y, x).$$

**Example.** $m \mapsto m(1, 1)$: \(\text{Mult}(\mathbb{Z}) \cong \mathbb{Z}\), but $\cdot m = m$.

**Examples.** In place of $V$, consider $\text{End}(V)$:

1. $(\text{End}(V), \circ)$ is an associative ring, as above.
2. $(\text{End}(V), \circ - \cdot)$ is a **Lie ring**, namely, a ring $(R, \cdot)$ in which

\[
(x \cdot y) \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z), \quad x \cdot x = 0.
\]

In particular, $(\text{Der}(K), \mathfrak{b})$ is a Lie ring.

3. $(\text{End}(V), \circ + \cdot)$ is a **Jordan ring**, in which

\[
(x \cdot y) \cdot (x \cdot x) = x \cdot (y \cdot (x \cdot x)), \quad x \cdot y = y \cdot x.
\]
If \((R, \cdot)\) is a ring, \(p, q \in \mathbb{Z}\), and
\[ x \mapsto \lambda_x: (R, \cdot) \rightarrow (\text{End}(R), p \circ - q \circ)\]
(where again \(\lambda_x(y) = x \cdot y\)), let \((R, \cdot)\) be called a \((p, q)\)-ring.

**Theorem.**

1. All associative rings are \((1, 0)\)-rings.
2. All Lie rings are \((1, 1)\)-rings.

**Corollary.** If
\[ (p, q) \in \{(0, 0), (1, 0), (1, 1)\}, \]
then \((\text{End}(V), p \circ - q \circ)\) is a \((p, q)\)-ring.

**Theorem** (P). The converse holds.

**Proof.** \(x \mapsto \lambda_x: (\text{End}(V), p \circ - q \circ) \rightarrow (\text{End(End}(V)), p \circ - q \circ)\)

\[ \iff \lambda_{x \cdot y} = \lambda_x \cdot \lambda_y \]

\[ \iff \lambda_{px \circ y - qy \circ x}(z) = (p \lambda_x \circ \lambda_y - q \lambda_y \circ \lambda_x)(z) \]

\[ \iff p(px \circ y - qy \circ x) \circ z - qz \circ (px \circ y - qy \circ x) = \ldots \]
A **differential field** is a pair

\[(K, V),\]

where

1. \(K\) is a field,
2. \(V\) is both a subspace and a sub-ring of \(\text{Der}(K)\).

**Theorem.** If \((K, V)\) is a differential field, and \(\dim_K(V) = m\), then \(V\) has a basis

\[\{\partial_0, \ldots, \partial_{m-1}\},\]

where in each case

\[[\partial_i, \partial_j] = 0.\]

The *structures* \((K, \partial_0, \ldots, \partial_{m-1})\) have a **theory**, which I denote by

\[\text{DF}^m.\]

**Example.** \((\mathbb{C}(x_0, \ldots, x_{m-1}), \partial/\partial x_0, \ldots, \partial/\partial x_{m-1}) \models \text{DF}^m.\)
Let $\mathfrak{A}$ be a structure with underlying set $A$. (So $\mathfrak{A}$ might be a group, a differential field, an ordered set, . . . ) By introducing names for all elements of $A$, we get the structure $\mathfrak{A}_A$.

The diagram of $\mathfrak{A}$ is the quantifier-free theory of $\mathfrak{A}_A$.

**Example.** The diagram of the field $\mathbb{F}_2$ is axiomatized by

\[
\begin{align*}
0 + 0 &= 0, & 1 + 0 &= 1, & 0 + 1 &= 1, & 1 + 1 &= 0, \\
0 \cdot 0 &= 0, & 1 \cdot 0 &= 0, & 0 \cdot 1 &= 0, & 1 \cdot 1 &= 1, \\
0 &\neq 1.
\end{align*}
\]

This does *not* entail field-theory.

For example, it does not entail

\[\forall x \forall y \ x \cdot y = y \cdot x.\]

Neither does field-theory entail $1 + 1 = 0$.  


Let ACF be the theory of **algebraically closed fields**, such as \( \mathbb{C} \). That is, ACF has the field axioms, along with, for each positive integer \( n \), the axiom

\[
\forall u_0 \ldots \forall u_{n-1} \exists x \ u_0 + u_1 \cdot x + \cdots + u_{n-1} \cdot x^{n-1} + x^n = 0.
\]

**Theorem.** If \( K \) is a field, then the theory

\[
\text{ACF} \cup \text{diag}(K)
\]

is **complete** (it entails either \( \sigma \) or \( \neg \sigma \) for each \( \sigma \ldots \)).

**Proof.** Use the \( \text{Łoś–Vaught Test} \).
(This relies on \( \text{Gödel's Completeness Theorem} \).)

1. The theory \( \text{ACF} \cup \text{diag}(K) \) has no finite models.

2. by Steinitz, all algebraically closed fields that include \( K \), but are of cardinality \( (|K| + \aleph_0)^\dagger \), are isomorphic over \( K \).

(\( \text{Gödel’s Incompleteness Theorem: a particular theory} \)
—namely \( \text{Th}(\mathbb{N}, +, \cdot, <) \)—has no complete axiomatization.)
Definition (A. Robinson). A theory $T$ is **model complete** if, for all models $\mathfrak{A}$ of $T$, the theory

$$T \cup \text{diag}(\mathfrak{A})$$

is complete, that is,

$$T \cup \text{diag}(\mathfrak{A}) \vdash \text{Th}(\mathfrak{A}_A).$$

Examples (A. Robinson).
1. Torsion-free divisible abelian groups (*i.e.* vector spaces over $\mathbb{Q}$),
2. algebraically closed fields, such as $\mathbb{C}$ (by the last slide),
3. real-closed fields, such as $\mathbb{R}$.

Theorem (A. Robinson). A theory $T$ is model complete if, for all models $\mathfrak{A}$ of $T$,

$$T \cup \text{diag}(\mathfrak{A}) \vdash \text{Th}(\mathfrak{A}_A)_\forall,$$

that is, if $\mathfrak{A} \subseteq \mathfrak{B}$, and $\mathfrak{B} \models T$, then:

*every system over $\mathfrak{A}$ soluble in $\mathfrak{B}$ is soluble in $\mathfrak{A}$.***
Let
\[ \text{DF}^m_0 = \text{DF}^m \cup \{ p \neq 0 : p \text{ prime} \}. \]

**Theorem** (McGrail, 2000). \( \text{DF}^m_0 \) has a **model companion**, \( \text{DCF}^m_0 \): that is,
\[ (\text{DF}^m_0)_\forall = (\text{DCF}^m_0)_\forall \]
and \( \text{DCF}^m_0 \) is model complete.

**Theorem** (Yaffe, 2001). The theory of fields of characteristic 0 with \( m \) derivations \( D_i \), where
\[ [D_i, D_j] = \sum a^k_{i j} D_k, \]
has a model companion.

**Theorem** (P, 2003; Singer, 2007). The latter follows readily from the former.

**Theorem** (P, submitted March, 2008). \( \text{DF}^m \) has a model companion, \( \text{DCF}^m \), given in terms of varieties.
If \((K, V)\) is a differential field, what is the model theory of \(V\)?

Piet Mondrian, *Broadway Boogie Woogie*
Theorem. Let \((V, \cdot)\) be a Lie ring, and
\[ R = (\text{End}(V), \circ). \]
Then \((V, \cdot)\) acts on \(R\) as a Lie ring of derivations.
The action takes \(D\) to the derivation
\[ f \mapsto Df \]
of \(R\), where
\[ Df(x) = D \cdot (f(x)) - f(D \cdot x). \]
That is,
\[ Df = [\lambda_D, f]. \]
In short,
\[ D \mapsto \lambda_{\lambda_D} : (V, \cdot) \to (\text{Der}(R), b). \]
Again \((V, \cdot)\) is a Lie ring, so it acts on \(R\), namely \((\text{End}(V), \circ)\).

Let \(t \in \text{End}(V)\). It may happen that \((\{Dt: D \in V\}, \circ)\)
— is a well-defined sub-ring of \(R\),
— is closed under the action of \((V, \cdot)\), and
— is a field.

Then \(V\) is a vector space over \(K\),
and \((V, \cdot)\) acts on \(K\) as a ring of derivations.

It may happen further that \(V\) acts on \(K\) as a *space* of derivations:
That is, if \(a, f \in K\) and \(D \in V\), it may happen that

\[
a(D)f = a \circ (Df).
\]

Then let \((V, \cdot, t)\) be called a **vector Lie ring**.

**Example.** If \((K, V)\) is a differential field, \(t \in K\), and \(Dt \neq 0\) for some \(D\) in \(V\), then \((V, b, t)\) is a vector Lie ring, and

\[
(\{Dt: D \in V\}, \circ) = K.
\]
**Theorem** (P). The class of \( m \)-dimensional vector Lie rings is elementary, with \( \forall \exists \) axioms. Its theory has a model companion, whose models are those \((V, \cdot, t)\) such that, when we let

\[
K = (\{Dt : D \in V\}, \circ),
\]

then \( V \) has a commuting basis \((\partial_i : i < m)\) over \( K \), and

\[
(K, \partial_0, \ldots, \partial_{m-1}) \models \text{DCF}^m.
\]

Here \( \dim_C(V) = \infty \), where \( C \) is the constant field.

However, for an infinite field \( K \), the theory of Lie algebras over \( K \) apparently has no model-companion (Macintyre, announced 1973). Is there a model-complete theory of infinite-dimensional Lie algebras with no extra structure?
Adolph Gottlieb, *Centrifugal*

We can also consider \((V, K)\) as a two-sorted structure.
A vector space can be understood model-theoretically as a triple

\[(V, K, \ast),\]

where

1. \(V\) is an abelian group;
2. \(K\) is a field;
3. \(\ast\) is the \textbf{action} of \(K\) on \(V\), that is,

\[(x, \mathbf{v}) \mapsto x \ast \mathbf{v}: K \times V \rightarrow V,
\]

and \(x \ast \mathbf{v} = \lambda_x(\mathbf{v})\), where \(x \mapsto \lambda_x: K \rightarrow (\text{End}(V), \circ)\).

Let the theory of vector spaces of dimension \(n\) be

\[T_n,\]

where \(n \in \{1, 2, 3, \ldots, \infty\}\).

**Theorem** (Kuzichev, 1992). \(T_n\) admits elimination of quantified vector-variables.
A theory is **inductive** if unions of chains of models are models.

**Theorem** (Łoś & Suszko 1957, Chang 1959). A theory $T$ is inductive if and only if

$$T = T_{\forall \exists}.$$  

Hence all model complete theories have $\forall \exists$ axioms.

Of an arbitrary $T$, a model $\mathfrak{A}$ is **existentially closed** if

$$T \cup \text{diag}(\mathfrak{A}) \vdash \text{Th}(\mathfrak{A}_A)_{\forall}.$$  

**Theorem** (Eklof & Sabbagh, 1970). Suppose $T$ is inductive.

1. $T$ has a model companion if and only if the class of its existentially closed models is elementary.

2. In this case, the theory of this class is the model companion.
Again, $T_n$ is the theory of vector spaces of dimension $n$. If $n > 1$, then no completion $T_n^*$ of $T_n$ can be model complete, because it cannot be $\forall \exists$ axiomatizable. For example, let 

$$a_0 = \mathbf{v}_0 = 2, \quad a_{s+1} = \mathbf{v}_{s+1} = \sqrt{a_s},$$

$$K_s = \mathbb{Q}(a_s),$$

$$V_s = \text{span}_{K_s}(\mathbf{v}^s, \ldots, \mathbf{v}^{s+n-1}).$$

Then

$$a_{s+1} * \mathbf{v}_{s+1} = \mathbf{v}^s,$$

so we have a chain

$$(V_0, K_0) \subseteq (V_1, K_1) \subseteq \cdots$$

of models of $T_n$ whose union has dimension 1.

The situation changes if there are predicates for linear dependence.
Let $\text{VS}_n$ (where $n$ is a positive integer) be the theory of vector spaces with a new $n$-ary predicate $P^n$ for linear dependence. So $P^n$ is defined by

$$\exists x^0 \cdots \exists x^{n-1} \left( \sum_{i<n} x^i \cdot v_i = 0 \land \bigvee_{i<n} x^i \neq 0 \right).$$

Let $\text{VS}_\infty$ be the union of the $\text{VS}_n$.

**Theorem** (P).

1. $\text{VS}_n$ has a model companion, the theory of $n$-dimensional spaces over algebraically closed fields.

2. $\text{VS}_\infty$ has a model companion, the theory of infinite-dimensional spaces over algebraically closed fields.
Proof. Given a field-extension $L/K$, where where

$$[L : K] \geq n + 1,$$

we can embed $(K^{n+1}, K)$ in $(L^n, L)$, as models of VS$_n$, under

$$\begin{pmatrix} x^0 \\ \vdots \\ x^{n-1} \\ x^n \end{pmatrix} \mapsto \begin{pmatrix} x^0 \\ \vdots \\ x^{n-1} \end{pmatrix} - x^n \begin{pmatrix} a^0 \\ \vdots \\ a^{n-1} \end{pmatrix},$$

where the $a^i$ are chosen from $L$ so that the tuple

$$(a^0, \ldots, a^{n-1}, 1)$$

is linearly independent over $K$. \qed
Compare:
Let $T$ be the theory of fields with an algebraically closed subfield. The existentially closed models of $T$ have transcendence-degree 1, because of

**Theorem** (A. Robinson). We have an inclusion

$$K(x, y) \subseteq L(y)$$

of pure transcendental extensions, where

$$K(x, y) \cap L = K,$$

provided

$$L = K(\alpha, \beta),$$

where

$$\alpha \notin K(x, y)^{\text{alg}}, \quad \beta = \alpha x + y.$$

(Hence $T$ has no model companion.)
A **Lie–Rinehart pair** is a quadruple \((V, K, D, \ast)\), where

1. \(V\) and \(K\) are abelian groups,
2. \(D\) is an action of \(V\) on \(K\); and \(\ast\), of \(K\) on \(V\); so
   \[
   (u + v) \cdot D x = u \cdot D x + v \cdot D x, \quad (x + y) \ast v = x \ast v + y \ast v,
   \]
   \[
   v \cdot D(x + y) = v \cdot D x + v \cdot D y, \quad x \ast (u + v) = x \ast u + x \ast v;
   \]
3. The actions are faithful:
   \[
   \exists x \ (v \cdot D x = 0 \Rightarrow v = 0), \quad \exists v \ (x \ast v = 0 \Rightarrow x = 0);
   \]
4. if \(u, v \in V\), there is a unique element \([u, v]\) of \(V\) such that
   \[
   [u, v] \cdot D x = u \cdot D(v \cdot D x) - v \cdot D(u \cdot D x),
   \]
   \[
   (u \cdot D x) \ast v = [u, x \ast v] - x \ast [u, v];
   \]
5. if \(x, y \in K\), there is a unique element \(x \cdot y\) of \(K\) such that
   \[
   (x \cdot y) \ast v = x \ast (y \ast v),
   \]
   \[
   (x \ast v) \cdot D y = x \cdot (v \cdot D y).
   \]
Assuming \((V, K, D, \ast)\) is a Lie–Rinehart pair, one shows that \(V\) does act on \(K\) as a Lie ring of derivations:

\[
\nu D(x \cdot y) = (\nu D x) \cdot y + x \cdot (\nu D y).
\]

Let the theory of those Lie–Rinehart pairs \((V, K, D, \ast)\) in which \((K, \cdot)\) is a field be denoted by

\[\text{LR}.\]

In this case, \((K, V)\) is a differential field. The theory \text{LR} is not inductive. However, let the theory of those models \((V, K, D, \ast)\) of \text{LR} in which

\[\dim_K(V) \leq m\]

be denoted by

\[\text{LR}^m.\]

Then \(\text{LR}^m\) is inductive and companionable.
Let \( T \) be the theory of pairs \((V, K)\), where \( K \) is a field, \( \text{char}(K) = 0 \), and \( V \) acts on \( K \) as a space of derivations.  
Let \( \text{DCF}_0^{(m)} \) be the model-companion of the theory of fields of characteristic 0 with \( m \) derivations \textit{with no required interaction}.

**Theorem** (Özcan Kasal). The existentially closed models of \( T \) are just those models \((V, K)\) such that

1. \( \text{tr-deg}(K/\mathbb{Q}) = \infty \);
2. \((K, v_0, \ldots, v_{m-1}) \models \text{DCF}_0^{(m)} \) whenever \((v_0, \ldots, v_{m-1})\) is linearly independent over \( K \);
3. if \((x^0, \ldots, x^{n-1})\) is algebraically independent, and \((y^0, \ldots, y^{n-1})\) is arbitrary, then for some \( \nu \) in \( V \),

\[
\bigwedge_{i<n} \nu D x^i = y^i.
\]

These are not first-order conditions: they require the constant field to be \( \mathbb{Q}^{\text{alg}} \).

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The picture changes when (for each $n$) a predicate $Q_n$ is introduced for the $n$-ary relation on scalars defined by

$$\bigvee_{i<n} \forall \mathbf{v} \left( \bigwedge_{j \neq i} \mathbf{v} D x^j = 0 \Rightarrow \mathbf{v} D x^i = 0 \right).$$

Let the new theory be

$$T',$$

so this entails

$$-Q_n x^0 \cdots x^{n-1} \iff \exists (\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}) \bigwedge_{i<n \atop j<n} \mathbf{v}_i D x^j = \delta^j_i.$$

Say $(a^0, \ldots, a^{n-1})$ from $K$ is $D$-**dependent** if

$$(V, K) \models Q_n a^0 \cdots a^{n-1}.$$  

So algebraic dependence implies $D$-dependence.

Also, $D$-dependence also makes $K$ a pregeometry.
Theorem (Özcan Kasal). The existentially closed models of $T'$ are those $(V, K)$ such that $D\text{-dim}(K) = \infty$ and whenever

1. $U$ is quasi-affine over $\mathbb{Q}(a^0, \ldots, a^{k-1}, \vec{b})$ with a generic point $(x^0, \ldots, x^{\ell+m-1}, \vec{y})$,

   where $\vec{x}$ is algebraically independent over $\mathbb{Q}(\vec{a}, \vec{b})$,

2. $(\vec{v}_0, \ldots, \vec{v}_{k+\ell-1})$ is linearly independent,

3. $(I_k \mid 0) = (v_j \, D \, a^i)_{j<k+\ell}^{i<k}$,

4. $\begin{pmatrix} F & I_{\ell} \\ G & H \end{pmatrix}$ is $(\ell + m) \times (k + \ell)$ with entries from $\mathbb{Q}(\vec{a}, \vec{b})[U]$,

then $U$ contains $(\vec{c}, \vec{d})$ such that

1. $\begin{pmatrix} F & I_{\ell} \end{pmatrix} (\vec{c}, \vec{d}) = (v_j \, D \, c^i)_{j<k+\ell}^{i<\ell+m}$,

2. $D\text{-dim}(c^\ell, \ldots, c^{\ell+m-1}, \vec{d}/\vec{a}, c^0, \ldots, c^{\ell-1}) = 0$. 
Franz Kline, *Palladio*