REPRESENTATION THEOREMS FOR RINGS

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Abstract. Among rings, the associative rings, along with the rings that respect the Jacobi identity, as well as rings with trivial multiplication, are unique in possessing a certain kind of representation theorem. Such rings can be obtained from an arbitrary abelian group E by taking the abelian group of endomorphisms of E and introducing, as a multiplication, the appropriate combination of composition and reverse composition. Other rings, for example Jordan rings, can be obtained this way as well; but only for an associative ring, or a ring satisfying the Jacobi identity, will the multiplication always be respected by the canonical homomorphism from the underlying abelian group to the endomorphism group of that group.

1. Introduction

Associative rings, along with rings that respect the Jacobi identity, as well as rings with trivial multiplication, are unique in possessing a certain kind of representation theorem. This observation is a consequence of Theorem 2 below.

The observation arises from considering a field K together with the set $\operatorname{Der}(K)$ of its derivations. Then $\operatorname{Der}(K)$ has the structure of both a vector-space over K and a Lie ring. Let V be a subspace and sub-ring of $\operatorname{Der}(K)$, and let k be the constant field of V. Then V is what is termed by Rinehart [5] a (k, K)-*Lie algebra.* Other terms include *pseudo-algèbre de Lie* [1] and *Lie d-ring* [4], as one may learn from Stasheff [6, p. 228], in whose terminology (V, K) is a *Lie-Rinehart pair over* k. This term applies more generally to the situation where k is just a (commutative associative) ring and K is a commutative algebra over k. In any case, reference to k may distract one from seeing the symmetry or dualism present in the pair (V, K). But it is just this dualism that I want now to bring out.

In the pair (V, K), first of all, K and V are abelian groups, each acting faithfully as a group of endomorphisms of the other. That is, we have, say,

$$(x, \boldsymbol{v}) \mapsto x * \boldsymbol{v} \colon K \times V \to V,$$
$$(\boldsymbol{v}, x) \mapsto \boldsymbol{v}\, D\, x \colon V \times K \to K,$$

both maps being bi-additive:

$$(x + y) * \boldsymbol{v} = x * \boldsymbol{v} + y * \boldsymbol{v}, \qquad x * (\boldsymbol{u} + \boldsymbol{v}) = x * \boldsymbol{u} + x * \boldsymbol{v}, \tag{1a}$$
$$(\boldsymbol{u} + \boldsymbol{v})\, D\, z = \boldsymbol{u}\, D\, z + \boldsymbol{v}\, D\, z, \qquad \boldsymbol{v}\, D(x + y) = \boldsymbol{v}\, D\, x + \boldsymbol{v}\, D\, y. \tag{1b}$$

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Faithfulness of the actions is given by the formulas

$$\exists \boldsymbol{v}\ (x * \boldsymbol{v} = 0 \Rightarrow x = 0), \tag{2a}$$

$$\exists x\ (\boldsymbol{v}\, D\, x = 0 \Rightarrow \boldsymbol{v} = 0). \tag{2b}$$

Moreover, K has the multiplication $(x, y) \mapsto xy$; and V has the multiplication $(\boldsymbol{u}, \boldsymbol{v}) \mapsto [\boldsymbol{u}, \boldsymbol{v}]$, which make the abelian groups a commutative associative ring and a Lie ring respectively. This means just that the identities

$$xy - yx = 0, \qquad (xy)z = x(yz) \tag{3a}$$

hold in K; and in V,

$$[\boldsymbol{v}, \boldsymbol{v}] = 0, \qquad [[\boldsymbol{u}, \boldsymbol{v}], \boldsymbol{w}] = [\boldsymbol{u}, [\boldsymbol{v}, \boldsymbol{w}]] - [\boldsymbol{v}, [\boldsymbol{u}, \boldsymbol{w}]], \tag{3b}$$

where the latter equation in (3b) is the Jacobi identity. Here

$$[\boldsymbol{u}, \boldsymbol{v}] = \boldsymbol{u} \circ \boldsymbol{v} - \boldsymbol{v} \circ \boldsymbol{u}, \tag{4}$$

although Kaplansky [2, p. 2] also uses the notation $\boldsymbol{uv}$, 'to emphasize all possible analogies with rings other than Lie rings'—which is just what I want to do.

The abelian group $\mathrm{End}(E)$ of endomorphisms of an abelian group E becomes
(1) an associative ring, when equipped with composition;
(2) a Lie ring, when equipped with the Lie bracket operation, $(\boldsymbol{u}, \boldsymbol{v}) \mapsto [\boldsymbol{u}, \boldsymbol{v}]$.

The embeddings $x \mapsto (\boldsymbol{v} \mapsto x * \boldsymbol{v})$ of K in $\mathrm{End}(V)$ and $\boldsymbol{v} \mapsto (x \mapsto \boldsymbol{v}\, D\, x)$ of V in $\mathrm{End}(K)$ are in fact homomorphisms of associative rings and Lie rings respectively:

$$(xy) * \boldsymbol{v} = x * (y * \boldsymbol{v}), \tag{5a}$$

$$(\boldsymbol{uv})\, D\, x = \boldsymbol{u}\, D(\boldsymbol{v}\, D\, x) - \boldsymbol{v}\, D(\boldsymbol{u}\, D\, x). \tag{5b}$$

The two actions interact according to:

$$(x * \boldsymbol{v})\, D\, y = x(\boldsymbol{v}\, D\, y), \tag{6a}$$

$$(\boldsymbol{u}\, D\, x) * \boldsymbol{v} = [\boldsymbol{u}, x * \boldsymbol{v}] - x * [\boldsymbol{u}, \boldsymbol{v}]. \tag{6b}$$

Indeed, (6a) is because V is a space *of derivations.* Moreover, (6b) is a result of (4) and the Leibniz rule of differentiation. The latter is (7a) below and is dual to a rearrangement of (6b):

$$\boldsymbol{v}\, D(xy) = x(\boldsymbol{v}\, D\, y) + (\boldsymbol{v}\, D\, x)y, \tag{7a}$$

$$x * [\boldsymbol{u}, \boldsymbol{v}] = [\boldsymbol{u}, x * \boldsymbol{v}] - (\boldsymbol{u}\, D\, x) * \boldsymbol{v}. \tag{7b}$$

The foregoing pairs of formulas can be taken as the definition of a **Lie–Rinehart pair.** Model-theoretically speaking, the class of Lie–Rinehart pairs is elementary, with $\forall\exists$ axioms.

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Theorem 1. *In the foregoing definition, the identities* (3) *(except $xy - yx = 0$) and* (7) *are redundant. In fact, the identities* (6b)*,* (7a)*, and* (7b) *are equivalent.*

Proof. The redundancy of (3) follows from (5) and (2). Indeed, as noted above, by (5), we have a *homomorphism* from K to an associative ring and from V to a Lie ring; by (2), those homomorphisms are embeddings. As for the second claim, we have already noted the equivalence of (6b) and (7b). Moreover,

$$\begin{aligned}
(x\,D(yz)) * w &= x((yz) * w) - (yz) * (xw) && \text{[by (6b)]}\\
&= x(y * (z * w)) - y * (z * (xw)) && \text{[by (5a)]}\\
&= (x\,D\,y) * (z * w) + y * (x(z * w)) \\
&\qquad - y * (x(z * w) - (x\,D\,z) * w) && \text{[by (6b) or (7b)]}\\
&= (x\,D\,y) * (z * w) + y * ((x\,D\,z) * w) && \text{[by (1a)]}\\
&= ((x\,D\,y)z) * w + (y(x\,D\,z)) * w, && \text{[by (5a)]}
\end{aligned}$$

whence (7a), by (1a) and (2a). So (7) follows from (6b) (and everything else). Conversely,

$$\begin{aligned}
(x(y * z))\,D\,w &= x\,D((y * z)\,D\,w) - (y * z)\,D(x\,D\,w) && \text{[by (5b)]}\\
&= x\,D(y(z\,D\,w)) - y(z\,D(x\,D\,w)) && \text{[by (6a)]}\\
&= y(x\,D(z\,D\,w)) + (x\,D\,y)(z\,D\,w) - y(z\,D(x\,D\,w)) && \text{[by (7a)]}\\
&= y((xz)\,D\,w) + (x\,D\,y)(z\,D\,w) && \text{[by (5b)]}\\
&= (y * (xz))\,D\,w + ((x\,D\,y) * z)\,D\,w && \text{[by (6a)],}
\end{aligned}$$

whence (6b) and (7b), by (1b) and (2b). □

2. Representation theorems

The original representation theorem is attributed to Cayley: every group embeds in a symmetry group.[1] Indeed, if $(G, \cdot)$ is a group, let λ be the function that takes each element g of G to the permutation $x \mapsto g \cdot x$ of G. Then λ is an embedding of $(G, \cdot)$ in $(\mathrm{Sym}(G), \circ)$.

To Cayley's Theorem, Stone [7] gives what he calls a precise analogue: every Boolean ring embeds in a Boolean ring of *sets,* namely the sets of prime ideals of the ring. The embedding takes x to the set of prime ideals that do *not* contain x.

For abelian groups, the situation is not so neat. There is a *prototypical* abelian group, $\mathbb{Z}$; but the variety of abelian groups as such is multifarious. However, if we fix an abelian group A, then, for every abelian group G, we have the homomorphism $x \mapsto (y \mapsto y(x))$ from G to $\mathrm{Hom}(\mathrm{Hom}(G, A), A)$. Suppose in particular that A is divisible and has elements of all orders: for example, A might be the circle $\mathbb{R}/\mathbb{Z}$, which features in Pontryagin duality for topological groups. Then the homomorphism from G to $\mathrm{Hom}(\mathrm{Hom}(G, A), A)$ is an embedding. Indeed, if $x \in G$, then $\langle x \rangle$ embeds in A, and this embedding extends, by divisibility of A, to an element y of $\mathrm{Hom}(G, A)$, so that $y(x) \neq 0$ if $x \neq 0$. Thus every abelian group embeds in an abelian group of homomorphisms into a particular group.

In the most general sense, a **multiplication** on an abelian group is a bi-additive binary operation. Then a **ring** is just an abelian group with a multiplication. As noted in § 1, if E is an abelian group, then $(\mathrm{End}(E), \circ)$ is an associative ring. Indeed, if symmetry groups are the prototypical groups, then $(\mathrm{End}(E), \circ)$ might be the prototypical associative ring. Suppose a multiplication $\cdot$ is given on E. We have λ as a homomorphism from E to $\mathrm{End}(E)$ by the same formula as in the discussion of Cayley's Theorem above. Then $(E, \cdot)$ is associative if and only if λ is a homomorphism from $(E, \cdot)$ into $(\mathrm{End}(E), \circ)$,

[1] See [3] for discussion of the attribution of Cayley's Theorem.

since each of these conditions is just that the second equation (3a) is an identity on E. Note however that, even if it is a homomorphim, λ need not be an *embedding* of $(E, \cdot)$ in $(\mathrm{End}(E), \circ)$.

Lie rings have a corresponding representation, the so-called adjoint representation. The Jacobi identity in (3b) means simply that λ is a homomorphism from $(E, \cdot)$ to $(\mathrm{End}(E), \mathsf{b})$, where b is the bracket. If λ is indeed a homomorphism here, then we have also $x^2y = 0$ on E, but perhaps not $x^2 = 0$, the other identity in (3b), unless λ is an embedding.

3. More rings

The possible multiplications on an abelian group E compose another abelian group, $\mathrm{Mult}(E)$. This group has an involution, $\mathsf{m} \mapsto \breve{\mathsf{m}}$, where $\breve{\mathsf{m}}(x, y) = \mathsf{m}(y, x)$. If we denote functional composition by c, so that $x \circ y = \mathsf{c}(x, y)$, then the bracket on $\mathrm{End}(E)$ is $\mathsf{c} - \breve{\mathsf{c}}$.

Other combinations of composition and its opposite are of interest. In particular, if xy is now $(\mathsf{c} + \breve{\mathsf{c}})(x, y)$, then we obtain the defining identities

$$xy = yx, \qquad\qquad (xy)x^2 = x(yx^2)$$

of a **Jordan ring.** However, if $(E, \cdot)$ is an arbitrary Jordan ring, then λ is *not* a homomorphism from $(E, \cdot)$ to $(\mathrm{End}(E), \circ + \breve{\circ})$, unless $x(yx^2) = x(yx^2) + yx^3$ identically on E.

Supposing λ is a homomorphism from an arbitrary ring $(E, \cdot)$ to $(\mathrm{End}(E), p\mathsf{c} - q\breve{\mathsf{c}})$ for some integers p and q, let us refer to $(E, \cdot)$ as a **(p, q)-ring.** So a ring is a (p, q)-ring just in case the equation

$$(xy)z = px(yz) - qy(xz) \tag{8}$$

is an identity on the ring. Then the associative rings are just the $(1, 0)$-rings. All Lie rings are $(1, 1)$-rings, and a $(1, 1)$-ring is a Lie ring if λ is injective on it. A Jordan ring is not generally a $(1, -1)$-ring. In particular, $(\mathrm{End}(E), p\mathsf{c} - q\breve{\mathsf{c}})$ itself is a (p, q)-ring for all abelian groups E when (p, q) is $(1, 0)$ or $(1, 1)$ or $(0, 0)$; the theorem below shows that this condition is necessary.

Given m in $\mathrm{Mult}(E)$ and a in E, let us denote by $\lambda^{\mathsf{m}}(a)$ the endomorphism $x \mapsto \mathsf{m}(a, x)$ of E. So λ^{m} is a homomorphism from E to $\mathrm{End}(E)$. We can now eliminate the variable z from (8), getting

$$\lambda^{\mathsf{m}}(\mathsf{m}(x, y)) = p \cdot \lambda^{\mathsf{m}}(x) \circ \lambda^{\mathsf{m}}(y) - q \cdot \lambda^{\mathsf{m}}(y) \circ \lambda^{\mathsf{m}}(x). \tag{9}$$

We can go further, eliminating (x, y) to get

$$\lambda^{\mathsf{m}} \circ \mathsf{m} = (p\mathsf{c} - q\breve{\mathsf{c}}) \circ (\lambda^{\mathsf{m}} \times \lambda^{\mathsf{m}}). \tag{10}$$

As a special case of this identity, we have

$$\lambda^{\mathsf{m}} \circ \mathsf{m} = \mathsf{m} \circ (\lambda^{\mathsf{m}} \times \lambda^{\mathsf{m}}) \tag{11}$$

on $\mathrm{End}(E)$, when m is c or b throughout; the theorem below is that c and b are the *only* combinations of c and $\breve{\mathsf{c}}$ for which (11) holds on every group $\mathrm{End}(E)$. The result can be obtained at length by replacing m in (11) with $p\mathsf{c} - q\breve{\mathsf{c}}$, applying each member to (x, y) and then z, and simplifying. Alternatively, one can proceed more systematically as follows.

Lemma. *For every abelian group E, on* $\mathrm{End}(E)$*, the following six identities hold:*

$$\begin{aligned}
\lambda^{\mathsf{c}} \circ \mathsf{c} &= \mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\mathsf{c}}),\\
\lambda^{\mathsf{c}} \circ \check{\mathsf{c}} &= \check{\mathsf{c}} \circ (\lambda^{\mathsf{c}} \times \lambda^{\mathsf{c}}),\\
\lambda^{\check{\mathsf{c}}} \circ \mathsf{c} &= \check{\mathsf{c}} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\check{\mathsf{c}}}),\\
\lambda^{\check{\mathsf{c}}} \circ \check{\mathsf{c}} &= \mathsf{c} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\check{\mathsf{c}}}),\\
\mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}}) &= \check{\mathsf{c}} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}}),\\
\mathsf{c} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}}) &= \check{\mathsf{c}} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}}).
\end{aligned}$$

Proof. The first identity expresses the associativity of $(\mathrm{End}(E), \mathsf{c})$. Continuing down the line, we have

$$\begin{aligned}
\lambda^{\mathsf{c}} \circ \check{\mathsf{c}}(x, y)(z) &= \lambda^{\mathsf{c}}(y \circ x)(z) = y \circ x \circ z,\\
\check{\mathsf{c}} \circ (\lambda^{\mathsf{c}} \times \lambda^{\mathsf{c}})(x, y)(z) &= \check{\mathsf{c}}(\lambda^{\mathsf{c}}(x), \lambda^{\mathsf{c}}(y))(z) = (\lambda^{\mathsf{c}}(y) \circ \lambda^{\mathsf{c}}(x))(z) = y \circ x \circ z,
\end{aligned}$$

and so on:

$$\begin{aligned}
\lambda^{\mathsf{c}} \circ \check{\mathsf{c}}\,(x, y)(z) &= y \circ x \circ z = \check{\mathsf{c}} \circ (\lambda^{\mathsf{c}} \times \lambda^{\mathsf{c}})\,(x, y)(z),\\
\lambda^{\check{\mathsf{c}}} \circ \mathsf{c}\,(x, y)(z) &= z \circ x \circ y = \check{\mathsf{c}} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\check{\mathsf{c}}})\,(x, y)(z),\\
\lambda^{\check{\mathsf{c}}} \circ \check{\mathsf{c}}\,(x, y)(z) &= z \circ y \circ x = \mathsf{c} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\check{\mathsf{c}}})\,(x, y)(z),\\
\mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}})\,(x, y)(z) &= x \circ z \circ y = \check{\mathsf{c}} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}})\,(x, y)(z),\\
\mathsf{c} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}})\,(x, y)(z) &= y \circ z \circ x = \check{\mathsf{c}} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}})\,(x, y)(z).
\end{aligned}$$ □

It will also be useful to note that the various combinations of binary operations into ternary (or higher) operations can be related by means of permutations. Indeed, for any abelian group E, for any n in ω, there is a left action of $\mathrm{Sym}(n)$ on E^n by permutation of coordinates: Elements $\vec{x}$ of E^n are functions $i \mapsto x_i$ on n, that is, on $\{0, \dots, n-1\}$; elements of $\mathrm{Sym}(n)$ permute this set; then

$$\sigma(\vec{x}) = \vec{x} \circ \sigma^{-1}.$$

Each function $\vec{x} \mapsto \sigma(\vec{x})$ is an endomorphism of E^n. Thus $\mathrm{Sym}(n)$ can be considered as a subset of $\mathrm{End}(E^n)$, though not a sub*group*: $\mathrm{Sym}(n)$ is a subgroup of the multiplicative semigroup of $(\mathrm{End}(E), \circ)$. For example, if $\mathsf{m} \in \mathrm{Mult}(E)$, then $\check{\mathsf{m}}$ can be understood as $\mathsf{m} \circ (0\ 1)$. Treating the equations in the lemma as identities of ternary operations on $\mathrm{End}(E)$, we have for example

$$\lambda^{\check{\mathsf{c}}} \circ \mathsf{c} = \lambda^{\mathsf{c}} \circ \mathsf{c} \circ (0\ 1\ 2).$$

Theorem 2. *Suppose $\mathscr{R}$ is a class of rings that, for some integers p and q, for all abelian groups E, contains* $(\mathrm{End}(E), p\mathsf{c} + q\check{\mathsf{c}})$*. The following are equivalent.*

1. *For every $(E, \cdot)$ in $\mathscr{R}$, the homomorphism λ from E to* $\mathrm{End}(E)$ *is a homomorphism from $(E, \cdot)$ to* $(\mathrm{End}(E), p\mathsf{c} + q\check{\mathsf{c}})$*.*
2. $(p, q) \in \{(1, 0), (1, 1), (0, 0)\}$.

Proof. Suppose $(p, q) \in \mathbb{Z}^2$. It suffices to show that $(\mathrm{End}(E), p\mathsf{c} + q\check{\mathsf{c}})$ is a (p, q)-ring for all E if and only if $(p, q) \in \{(1, 0), (1, 1), (0, 0)\}$. Considering Identity (9), reformulated as (10), we have to show that the identity

$$(p\lambda^{\mathsf{c}} - q\lambda^{\check{\mathsf{c}}}) \circ (p\mathsf{c} - q\check{\mathsf{c}}) = (p\mathsf{c} - q\check{\mathsf{c}}) \circ [(p\lambda^{\mathsf{c}} - q\lambda^{\check{\mathsf{c}}}) \times (p\lambda^{\mathsf{c}} - q\lambda^{\check{\mathsf{c}}})]$$

holds on all endomorphism-groups $\mathrm{End}(E)$ if and only if (p, q) is $(1, 0)$, $(1, 1)$, or $(0, 0)$. We compute:

$$(p\lambda^{\mathsf{c}} - q\lambda^{\check{\mathsf{c}}}) \circ (p\mathsf{c} - q\check{\mathsf{c}}) - (p\mathsf{c} - q\check{\mathsf{c}}) \circ [(p\lambda^{\mathsf{c}} - q\lambda^{\check{\mathsf{c}}}) \times (p\lambda^{\mathsf{c}} - q\lambda^{\check{\mathsf{c}}})]$$

$$= p^2\lambda^{\mathsf{c}} \circ \mathsf{c} - pq\lambda^{\mathsf{c}} \circ \check{\mathsf{c}} - pq\lambda^{\check{\mathsf{c}}} \circ \mathsf{c} + q^2\lambda^{\check{\mathsf{c}}} \circ \check{\mathsf{c}} - \\ - (p\mathsf{c} - q\check{\mathsf{c}}) \circ (p^2\mathsf{c} \times \mathsf{c} - pq\mathsf{c} \times \check{\mathsf{c}} - pq\check{\mathsf{c}} \times \mathsf{c} + q^2\check{\mathsf{c}} \times \check{\mathsf{c}})$$

$$= p^2\lambda^{\mathsf{c}} \circ \mathsf{c} - pq\lambda^{\mathsf{c}} \circ \check{\mathsf{c}} - pq\lambda^{\check{\mathsf{c}}} \circ \mathsf{c} + q^2\lambda^{\check{\mathsf{c}}} \circ \check{\mathsf{c}} - \\ - p^3\mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\mathsf{c}}) + p^2q\mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}}) + p^2q\mathsf{c} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}}) - pq^2\mathsf{c} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\check{\mathsf{c}}}) + \\ + p^2q\check{\mathsf{c}} \circ (\lambda^{\mathsf{c}} \times \lambda^{\mathsf{c}}) - pq^2\check{\mathsf{c}} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}}) - pq^2\check{\mathsf{c}} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}}) + q^3\check{\mathsf{c}} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\check{\mathsf{c}}}))$$

$$= p^2\lambda^{\mathsf{c}} \circ \mathsf{c} - pq\lambda^{\mathsf{c}} \circ \check{\mathsf{c}} - pq\lambda^{\check{\mathsf{c}}} \circ \mathsf{c} + q^2\lambda^{\check{\mathsf{c}}} \circ \check{\mathsf{c}} - \\ - p^3\lambda^{\mathsf{c}} \circ \mathsf{c} + p^2q\mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}}) + p^2q\mathsf{c} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}}) - pq^2\lambda^{\check{\mathsf{c}}} \circ \check{\mathsf{c}} + \\ + p^2q\lambda^{\mathsf{c}} \circ \check{\mathsf{c}} - pq^2\mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}}) - pq^2\mathsf{c} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}}) + q^3\lambda^{\check{\mathsf{c}}} \circ \mathsf{c})$$

$$= (p^2 - p^3)\lambda^{\mathsf{c}} \circ \mathsf{c} + (p^2q - pq)\lambda^{\mathsf{c}} \circ \check{\mathsf{c}} + (q^3 - pq)\lambda^{\check{\mathsf{c}}} \circ \mathsf{c} + (q^2 - pq^2)\lambda^{\check{\mathsf{c}}} \circ \check{\mathsf{c}} + \\ + (p^2q - pq^2)\mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}}) + (p^2q - pq^2)\check{\mathsf{c}} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}})$$

by the lemma. The five distinct coefficients here are:

$$p^2(1 - p), \quad pq(p - 1), \quad q(q^2 - p), \quad q^2(1 - p), \quad pq(p - q).$$

They are all 0 if (p, q) is $(1, 0)$, $(1, 1)$, or $(0, 0)$. Conversely if they are all 0, then either $p = 0 = q$, or $p = 1$ and $q \in \{0, 1\}$. This completes the proof, provided there is at least one abelian group E such that no non-trivial combination of the six ternary operations $\lambda^{\mathsf{c}} \circ \mathsf{c}$, $\lambda^{\mathsf{c}} \circ \check{\mathsf{c}}$, $\lambda^{\check{\mathsf{c}}} \circ \mathsf{c}$, $\lambda^{\check{\mathsf{c}}} \circ \check{\mathsf{c}}$, $\check{\mathsf{c}} \circ (\lambda^{\check{\mathsf{c}}} \times \lambda^{\mathsf{c}})$, and $\mathsf{c} \circ (\lambda^{\mathsf{c}} \times \lambda^{\check{\mathsf{c}}})$ on $\mathrm{End}(E)$ is 0.

Such a group will be $\mathbb{Z}^4$. If $k < 3$, let f_k be the element $(k\ 3)$ of $\mathrm{Sym}(4)$, considered as an element of $\mathrm{End}(\mathbb{Z}^4)$. If $\sigma \in \mathrm{Sym}(3)$, considered as an element of $\mathrm{End}((\mathrm{End}(\mathbb{Z}^4))^3)$, then

$$\sigma(f_0, f_1, f_2) = (f_i, f_j, f_k)$$

for some distinct i, j, and k in 3; and

$$\lambda^{\mathsf{c}} \circ \mathsf{c} \circ \sigma(f_0, f_1, f_2) = \lambda^{\mathsf{c}} \circ \mathsf{c}\,(f_i, f_j, f_k) = (i\ 3) \circ (j\ 3) \circ (k\ 3) = (3\ k\ j\ i).$$

No non-trivial $\mathbb{Z}$-linear combination of these elements of $\mathrm{End}(\mathbb{Z}^4)$ is 0. Indeed, these elements can be listed as

$$(3\ 2\ 1\ 0), \qquad (3\ 1\ 2\ 0), \qquad (3\ 2\ 0\ 1), \qquad (3\ 0\ 2\ 1), \qquad (3\ 1\ 0\ 2), \qquad (3\ 0\ 1\ 2).$$

A combination of them takes (x_0, x_1, x_2, x_3) to

$$a(x_1, x_2, x_3, x_0) + b(x_2, x_3, x_1, x_0) + c(x_2, x_0, x_3, x_1) + \\ + d(x_3, x_2, x_0, x_1) + e(x_1, x_3, x_0, x_2) + f(x_3, x_0, x_1, x_2),$$

which is

$$((a+e)x_1+(b+c)x_2+(d+f)x_3,\ (c+f)x_0+(a+d)x_2+(b+e)x_3,\\ (d+e)x_0+(b+f)x_1+(a+c)x_3,\ (a+b)x_0+(c+d)x_1+(e+f)x_2).$$

If this is always 0, then in particular the coefficients $a+e$, $a+d$, and $d+e$ are 0, so a, d, and e are 0; likewise $b+c$, $c+f$, and $b+f$ are 0, so b, c, and f are 0. But each of the six functions $\lambda^{\mathsf{c}}\circ\mathsf{c}$, $\lambda^{\mathsf{c}}\circ\check{\mathsf{c}}$, $\lambda^{\check{\mathsf{c}}}\circ\mathsf{c}$, $\lambda^{\check{\mathsf{c}}}\circ\check{\mathsf{c}}$, $\mathsf{c}\circ(\lambda^{\mathsf{c}}\times\lambda^{\check{\mathsf{c}}})$, and $\check{\mathsf{c}}\circ(\lambda^{\check{\mathsf{c}}}\times\lambda^{\mathsf{c}})$ on $\mathrm{End}(\mathbb{Z}^4)$ is $\mathsf{c}\circ\mathsf{c}\circ\sigma$ for some σ in $\mathrm{Sym}(3)$. Thus $\mathbb{Z}^4$ is as desired. □

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