NUMBERS

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There is no more impressive form of literature than the narrative epic poem. That combination of depth and breadth of conception which some have called sublimity has here found a natural and adequate expression. The theory of number is the epic poem of mathematics. The mutual reflection of the two arts will supply a sort of explanation in the intellectual dimension of the epic quality in both...

But the question, what is a number? is an invitation to analyze counting and to find out what sort of thing makes counting possible. In poetry I suppose the corresponding question would be, what makes recounting possible? The answer, if it were to be complete, would take us into the most abstract and subtle mathematical thought. But the key to the problem is the simplest sort of insight. The same peculiar combination of simplicity and subtlety is involved in the theory of narrative, but as everyone knows, insight here belongs to the most common of common-sense conceptions.

—Scott Buchanan, *Poetry and Mathematics* [2, p. 64]

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0. Numbers and sets

The set $\mathbb{N}$ of **natural numbers** is described by the so-called *Peano Axioms* [28]. These refer to the following three features of $\mathbb{N}$.

1. It has a distinguished **initial element,** denoted by 1 or 0, depending on the writer.
2. It has a distinguished operation of **succession,** which converts an element into its **successor.** The successor of x is often denoted by $x + 1$, though I shall also use $\mathsf{s}(x)$ and x^{s}.
3. In particular, the operation of succession takes a *single* argument (unlike addition or multiplication, which takes two arguments).

The class $\mathbf{V}$ of all *sets*[1] also has three features. I shall develop an analogy between the three features of $\mathbb{N}$ listed above and the following three features of $\mathbf{V}$.

1. It has a distinguished element, the empty set, denoted by $\varnothing$ or 0.
2. It has one kind or 'type' of non-empty set. This is by the **Extension Axiom,** according to which a set is determined solely by its elements (or lack of elements). If sets are considered as boxes of contents, then the contents may vary, but the boxes themselves are always cardboard, say,—never plastic or steel.
3. Each non-empty set has a single kind or 'grade' of element; the elements are not sorted into multiple compartments as in an angler's tackle-box.

Von Neumann's definition of the *ordinal numbers* [42] gives us, in particular, the natural numbers. The von Neumann definition of the natural numbers can be seen as a construction of $\mathbb{N}$ within $\mathbf{V}$ by means of the analogy between $\mathbb{N}$ and $\mathbf{V}$ just suggested. Indeed, $\mathbb{N}$ can be understood as a *free algebra* whose *signature* comprises the constant (or nullary function-symbol) 1 and the singulary[2] function-symbol s. Von Neumann's definition gives us the particular free algebra denoted by ω, in which 1 is interpreted as $\varnothing$, and s is interpreted as the function $x \mapsto x \cup \{x\}$. A similar set-theoretic construction will give us a free algebra in an arbitrary algebraic signature, as long as there is a *type* of set for each symbol in the signature, and sets having the type of an n-ary symbol have n *grades* of members. There is a corresponding generalization of the notion of an ordinal number as well. I work out the details of this construction at the end of this article, in Part 3. Meanwhile, in Part 1, I consider what I hold to be misconceptions about numbers. It was these considerations that led to the whole of this paper. In Part 2, I review the logical development of numbers from a historical perspective.

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[1] That is, *hereditary* sets, namely sets whose elements are hereditary sets [20, p. 9].

[2] Although the term 'unary' is commonly used now in this context, I follow Quine, Church [4, n. 29, p. 12], and Robinson [29, pp. 19, 23] in preferring the term *singulary.* The first five Latin distributive numbers are *singul-, bin-, tern-, quatern-,* and *quin-* [27], and these are apparently sources for the terms *binary, ternary, quaternary,* and *quinary.* The Latin cardinals are *un-, du-, tri-, quattuor, quinque.* (The hyphens stand for variable case-endings.)

Part 1

1. Numbers and geometry

It is worthwhile to pay close attention to the fundamental properties of $\mathbb{N}$. We may learn these wrongly in school. We may be taught—rightly—that $\mathbb{N}$ has the following properties:

1. It admits **proofs by induction:** every subset that contains 1 and that contains $k+1$ whenever it contains k is the whole set.
2. It is **well-ordered:** every non-empty subset has a least element.
3. It admits **proofs by 'complete' or 'strong' induction:** every subset that contains 1 and that contains $k+1$ whenever it contains $1, \dots, k$ is the whole set.
4. It admits **recursive definitions** of functions: on it, there is a unique function h such that $h(1) = a$ and $h(k+1) = f(h(k))$, where f is a given operation on a set that has a given element a.

We may also be taught that these properties are *equivalent.* But they are *not* equivalent. Indeed, it is 'not even wrong' to say that they are equivalent, since properties 1 and 4 are possible properties of algebras in the signature $\{1, \mathsf{s}\}$, while 2 is a possible property of ordered sets, and 3 is a possible property of ordered algebras.

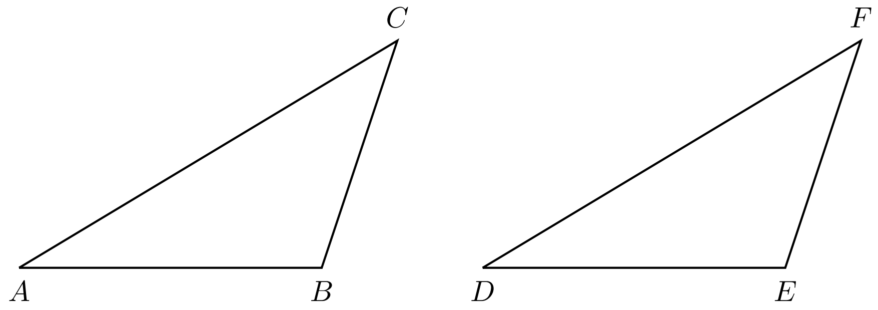
A well-ordered set without a greatest element can be made into an algebra in the signature $\{1, \mathsf{s}\}$ by letting 1 be the least element and letting $\mathsf{s}(x)$ be the least element that is greater than x; but this algebra need not admit induction in the sense of 1. Indeed, an example of such an algebra is the algebra of ordinals that are less than $\omega + \omega$. Conversely, an algebra in the signature $\{1, \mathsf{s}\}$ that admits induction need not admit an ordering $<$ such that $x < \mathsf{s}(x)$: finite cyclic groups provide examples.

The four properties listed above can *become* equivalent under additional assumptions; but these assumptions are not usually made explicit. Such assumptions are made explicit below in Theorems 2 and 5. I present these theorems, not as anything new, but as being insufficiently recognized.

Perhaps the real problem is a failure to distinguish between what I shall call *naturalistic* and *axiomatic* approaches to mathematics. In the former, mathematical objects exist out in nature; we note some of their properties, while perhaps overlooking other properties that are too obvious to mention. There is nothing wrong with this. Euclid's *Elements* [11, 12] is naturalistic, and beginning mathematics students would be better served by a course of reading Euclid than by a course in analytic geometry or in the abstract concepts of sets, relations, functions, and symbolic logic.

Not all of Euclid's propositions follow logically from his postulates and common notions. He never claims that they do.[3] His Proposition I.4 establishes the 'side-angle-side' condition for congruence of triangles, but the proof relies on 'applying' one triangle to another. Perhaps this 'application' is alluded to in his Common Notion 4: 'Things which coincide with one another are equal to one another.' Suppose indeed that in triangles ABC and DEF, sides AB and DE are equal, and AC and DF are equal, and angles BAC and EDF are equal. If the one triangle is 'applied' to the other, A to D and AB to DE, then B and E will coincide, and also C and F, by a sort of converse to Common

[3]Heath [11, I, pp. 221 f.] discusses the possibility that the the so-called Common Notions in the *Elements* are a later interpolation.



Notion 4 that Euclid does not make explicit: things that are equal can be made to coincide.[4] Then BC and EF coincide; indeed, as observed in a remark that is bracketed in [11], but omitted in [12], if the two straight lines do not coincide, then 'two straight lines will enclose a space: which is impossible.' Again, this is a principle so obvious as not to be worth treating as a postulate.

Without any explicit postulates, Euclid proves his number-theoretic propositions, from the Euclidean algorithm (VII.1–2) to the perfectness of the product of 2^n and $1 + 2 + 4 + \cdots + 2^n$ if the latter is prime (IX.36).

Hilbert's approach in *The Foundations of Geometry* [18] is axiomatic. Hilbert himself seems to see his project as the same as Euclid's:

> Geometry, like arithmetic, requires for its logical development only a small number of simple, fundamental principles. These fundamental principles are called the axioms of geometry. The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. This problem is tantamount to the logical analysis of our intuition of space.
>
> The following investigation is a new attempt to choose for geometry a *simple* and *complete* set of *independent* axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms.

However, it is not at all clear that Euclid is concerned with relations among axioms. Proving the *independence* of one axiom from others usually means showing that the latter have a model in which the former is false. Hilbert can do this, but Euclid makes no suggestion of different models of his postulates. Euclid makes observations and deductions about the world—the γῆ of γεωμετρία.[5] By constrast, while Hilbert's axioms

[4]Such coinciding, in the case of plane figures, may involve cutting and rearranging, as in the series of propositions that begins with I.35: parallelograms with the same base in the same parallels are equal.

[5]Euclid does not discuss his work, and in particular he does not use the word γεωμετρία. Herodotus does use the word, a century or two earlier:

> For this cause Egypt was intersected [by canals]. This king [Sesostris] moreover (so they said) divided the country among all the Egyptians by giving each an equal square

are obviously true in the world of Euclid, Hilbert nonetheless feels the need to establish their consistency by constructing a model. His model is an 'analytic' one (in the sense of analytic geometry): it is based on the field obtained from the number 1 by closing under $+$, $-$, $\times$, $\div$, and $x \mapsto \sqrt{1+x^2}$. For Hilbert, it seems, such a field is more real than Euclidean space, even though our intuition for fields, and especially for taking square roots in them, comes from geometry.

The approach to $\mathbb{N}$ in which properties 1–4 above are 'equivalent' is not quite naturalistic, not quite axiomatic. Considered as axioms in the sense of Hilbert, the properties are not meaningfully described as equivalent. But if the properties are to be understood just as properties of the numbers that we grew up counting, then it is also meaningless to say that the properties are equivalent: they are just properties of those numbers.

An axiomatic approach to mathematics may lead to a constructive approach, whereby we *create* our objects of study. Hilbert could have ignored his axioms and concentrated on his model of them, although this did not happen to be his purpose. Regarding $\mathbb{N}$, we may or may not wish ultimately to take a constructive approach, creating a model of the Peano Axioms by, for example, the method of von Neumann. If we do take this approach, then we presumably have recognized something more fundamental than the natural numbers: perhaps sets, which we may have approached naturalistically or axiomatically. But then the construction of the natural numbers may add to our understanding of, and confidence in, these sets. This is indeed how I see the von Neumann construction, along with the generalization in Part 3 below.

2. Numbers, quasi-axiomatically

Through Spivak's *Calculus* [35], I had my first encounters with serious mathematics. Because this book is worthy of scrutiny, I examine here its treatment of foundational matters. In the beginning, Spivak takes the real numbers as being out in nature. He just calls them 'numbers,' and in his first chapter he identifies twelve of their properties. These properties will, in his chapter 27, be recognized as the axioms for ordered fields. Meanwhile, in some proofs in his chapter 1, Spivak uses assumptions that he goes on to identify as having been unjustified. For example, to prove that, if a product is zero, then one of the factors must be zero, Spivak observes that, if $a \cdot b = 0$, and $a \neq 0$, then $a^{-1} \cdot (a \cdot b) = 0$. Thus he tacitly assumes that zero times anything is zero; on the next page, he comes clean and proves this. I find no fault here.

> parcel of land, and made this his source of revenue, appointing the payment of a yearly tax. And any man who was robbed by the river of a part of his land would come to Sesostris and declare what had befallen him; then the king would send men to look into it and measure the space by which the land was diminished, so that thereafter it should pay in proportion to the tax originally imposed. From this, to my way of thinking, the Greeks learnt the *art of measuring land* [γεωμετρίη]; the sunclock and the sundial, and the twelve divisions of the day, came to Hellas not from Egypt but from Babylonia. [17, 2.109]

For γεωμετρίη (the Ionic form of the word) here, other translators, as [16], just use 'geometry'. It is too simple to say that geometry developed from surveying, since it was not necessary for geometry as we know it to develop at all. On the other hand, if our experience of the world came entirely from gazing at the heavens, and if we nonetheless developed a geometry, then this would probably be what we in fact call spherical geometry.

The difficulties for me arise in Spivak's chapter 2. Here the various 'sorts' of numbers are introduced, starting with the natural numbers. The four properties of the natural numbers listed above in §1 are stated, and it is said that, from each of the first three properties, the others can be proved. For example, Spivak proves by induction that a set of natural numbers without a least element must be empty: 1 cannot be in the set, since otherwise it would be the least element; and for the same reason, if none of the numbers $1, \dots, k$ is in the set, then $k+1$ is not in the set.

The problem is that this argument uses more than induction: it uses that the set of natural numbers is ordered so that $n \leqslant k$ or $k+1 \leqslant n$ for all elements k and n. The existence of this ordering does not follow by induction alone. It *does* follow, if one has the natural numbers as real numbers. Again, for Spivak, the real numbers are just 'numbers', so the natural numbers must be among these; but Spivak does not emphasize this in the text. In exercise 25 at the end of chapter 2, an *inductive set* is defined as a set of numbers that contains 1 and is closed under addition of 1. The reader is invited to show that the set of numbers common to all inductive sets is inductive. It could be, but is not, made clear at this point that the properties called 1 and 4 in §1 above are equivalent properties of *inductive sets.*

Spivak's *Calculus* was a reference text and a source of exercises for a two-year high-school course taught by one Donald J. Brown.[6] The main text for the course was notes that Brown wrote and distributed, either as typed and mimeographed pages, or more usually as writing on the blackboard that we students copied down. Brown's notes are in some ways more formally rigorous than Spivak. From the beginning, they give the real numbers explicitly as composing the complete ordered field called $\mathbb{R}$. And yet the notes present the natural numbers as a set $\{1, 2, 3, \dots\}$ for which the 'fundamental axiom' is the 'Well Ordering Principle'. From this, the 'Principle of Mathematical Induction' is obtained as a theorem; it is left as an exercise to obtain the Well Ordering Principle as a theorem from the Principle of Induction. Here again, proofs must rely on hidden assumptions. At the relevant point in my copy of Brown's text, I find the following remark, apparently added by me during the course: 'If the given "definition" of $\mathbb{N}$ means anything, it must be the equivalent of the Principle of Mathematical Induction. Thus the P. of M.I. need not be a theorem, and the W.O.P. need not be an axiom.'

Part 2

3. Numbers, recursively

Let us forget about $\mathbb{R}$ and try to start from scratch. We should distinguish clearly between recursion from induction. **Recursion** is a method of *defining* sets and functions; **induction** is a method of *proving* that one set is equal to another.[7] We can try to understand $\mathbb{N}$ as having the following **recursive definition:**

(1) $\mathbb{N}$ contains 1;
(2) if $\mathbb{N}$ contains x, then $\mathbb{N}$ contains x^{s}.

Since this is explicitly a definition, there is really no need for a third condition, though it is sometimes given:

[6] This was at St. Albans School in Washington, D.C., 1981–3.
[7] One text that makes the distinction clear is Enderton [10, §1.2, pp. 22-30].

(3) $\mathbb{N}$ contains nothing but what it is required to contain by (1) and (2).

Again, I hold this third condition to be redundant for the same reason that the words *only if* are redundant when, for example, we define a natural number as *prime* if and only if it has just two distinct divisors.

The recursive definition of $\mathbb{N}$ means no more nor less than that, if A is a subset of $\mathbb{N}$ that contains 1 and that contains the successors of all of its elements, then $A = \mathbb{N}$. In short, the definition means that $\mathbb{N}$ **admits proofs by induction.** This can also be expressed by writing

$$\mathbb{N} = \{1, 1^{\mathsf{s}}, 1^{\mathsf{ss}}, \dots\}.$$

The recursive definition of $\mathbb{N}$ does not bring $\mathbb{N}$ into existence. Rather, it picks out $\mathbb{N}$ from among some things that are assumed to exist already. The definition assumes that there is *some* set that contains 1 and is closed under s. Then $\mathbb{N}$ is the smallest such set, namely the intersection of the collection of such sets.

Definitions such as the one just given have been found objectionable for being 'impredicative.' Zermelo [48, p. 191, n. 8] reports that Poincaré had such an objection to Zermelo's proof of the Schroeder–Bernstein Theorem. Let us have a look at this proof.

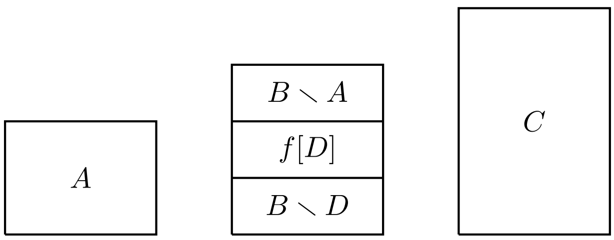
Suppose A, B, and C are sets such that $A \subseteq B \subseteq C$, and f is a bijection from C onto A. We aim to find a bijection g from B onto A. If there is any hope of finding such g, then probably we should have

$$g(x) = \begin{cases} f(x), & \text{if } x \in D, \\ x, & \text{if } x \in B \smallsetminus D, \end{cases} \tag{i}$$

for some subset D of B. In this case, since the range of g is to be (a subset of) A, we must have in particular $B \smallsetminus D \subseteq A$ and so $B \smallsetminus A \subseteq D$. Injectivity of g will follow from having $f[D] \subseteq D$. Let D now be the *smallest* X such that

$$(B \smallsetminus A) \cup f[X] \subseteq X.$$

That is, let D be the intersection of the set of all such sets X. We obtain g as desired. Indeed, when D is such, then $(B \smallsetminus A) \cup f[D] = D$. Since also $F[D] \subseteq A$, we have that $B \smallsetminus A$ and $f[D]$ are disjoint. Therefore B is the disjoint union of $B \smallsetminus A$, $f[D]$, and $B \smallsetminus D$, as in the figure below. We conclude $(B \smallsetminus D) \cup f[D] = A$, so g is indeed surjective onto A.



Such is the proof in the article [47, p. 208] of Zermelo giving his axioms of set-theory, although Zermelo does not give the argument in the heuristic[8] style that I have used.

[8]The historically correct word is *analytic,* in view of remarks like this of Pappus:

> Now *analysis* [ἀνάλυσις] is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which

The argument might be said to assume the existence of what it purports to establish the existence *of,* since D belongs to the set of which it is the intersection.

For an alternative heuristic argument, again suppose g is as in (**i**), so that

$$g[B] = f[D] \cup (B \smallsetminus D). \tag{ii}$$

Say $g[B] = A$. Then in particular $g[B] \subseteq A$, so by (**ii**) we have $B \smallsetminus D \subseteq A$, hence

$$B \smallsetminus A \subseteq D. \tag{iii}$$

Also, $A \subseteq g[B]$, but by (**ii**) we have $g[B] \subseteq f[B] \cup (B \smallsetminus D)$, hence

$$A \smallsetminus f[B] \subseteq B \smallsetminus D. \tag{iv}$$

Since $(B \smallsetminus A) \cup (A \smallsetminus f[B]) = B \smallsetminus f[B]$, we now have

$$\begin{aligned} g[B \smallsetminus f[B]] &= g[(B \smallsetminus A) \cup (A \smallsetminus f[B])] \\ &= g[B \smallsetminus A] \cup g[A \smallsetminus f[B]] \\ &= f[B \smallsetminus A] \cup (A \smallsetminus f[B]) && \text{[by (iii) and (iv)]} \\ &= (f[B] \smallsetminus f[A]) \cup (A \smallsetminus f[B]) = A \smallsetminus f[A]. \end{aligned}$$

Assuming g is injective, we conclude $g[f[B]] = f[A]$. Now we can proceed as before, but with $f[B]$ and $f[A]$ in place of B and A. In particular, we replace (**ii**) with

$$g[f[B]] = f[D \cap f[B]] \cup (f[B] \smallsetminus D)$$

and obtain $f[B] \smallsetminus f[A] \subseteq D$, and so forth. We find ultimately

$$\bigcup\{B \smallsetminus A, f[B] \smallsetminus f[A], f[f[B]] \smallsetminus f[f[A]], \dots\} \subseteq D.$$

Conversely, to ensure that $g[B] = A$, it suffices to let D be this union. Indeed, this is the same D found in Zermelo's argument above, only now it is found through a **recursive** procedure, as $\bigcup\{B_n \smallsetminus A_n : n \in \mathbb{N}\}$, where

$$B_1 = B, \qquad A_1 = A, \qquad B_{\mathsf{s}(n)} = f[B_n], \qquad A_{\mathsf{s}(n)} = f[A_n].$$

Is this better than finding D as the intersection of a set that contains D? A problem with the recursive approach is that the existence of the function $n \mapsto (A_n, B_n)$ is not justified merely by the possibility of defining $\mathbb{N}$ itself recursively as we did above.

I propose to refer to any set[9] that has a distinguished element and a distinguished singulary operation as an **iterative structure.**[10] The distinguished element can be called 1, and the operation, s; or they can be called $1^{\mathfrak{A}}$ and $\mathsf{s}^{\mathfrak{A}}$, to avoid ambiguity, if the structure itself is $\mathfrak{A}$, but there are other iterative structures around.[11] An *iterative* structure meets minimal requirements for repeated activity: it gives us something—s—to do, and something—1—to do it to.

> is admitted as a result of synthesis [συνθέσις]; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about... and such a method we call analysis, as being a reverse solution [ἀνάπαλιν λύσις].[39, p. 601]

The style of geometry pioneered by Descartes is not so much analytic as *algebraic.*

[9] Later the possibility of using a proper class here will be considered.

[10] Stoll [36, §2.1, p. 58] uses the term 'unary system'.

[11] This is like calling me David, unless there is another David present, in which case I may become David Pierce.

Every iterative structure has a unique smallest substructure, which is the intersection of the collection of all substructures;[12] this intersection therefore admits induction and can be recursively defined. In particular, $\mathbb{N}$ is such an intersection; but $\mathbb{N}$ has properties beyond admitting induction. In particular, if $\mathfrak{A}$ is another iterative structure, then there is a unique homomorphism h from $\mathbb{N}$ to $\mathfrak{A}$, given by the **recursive definition**

(1) $h(1) = 1$, that is, $h(1^{\mathbb{N}}) = 1^{\mathfrak{A}}$;
(2) $h(x^{\mathsf{s}}) = h(x)^{\mathsf{s}}$, that is, $h(\mathsf{s}^{\mathbb{N}}(x)) = \mathsf{s}^{\mathfrak{A}}(h(x))$.

In short,

$$h = \{(1,1), (1^{\mathsf{s}}, 1^{\mathsf{s}}), (1^{\mathsf{ss}}, 1^{\mathsf{ss}}), \dots\}, \qquad \textbf{(v)}$$

where the left-hand entries in the ordered pairs are from $\mathbb{N}$; the right-hand, from A.

We have now merely *asserted* the possibility of defining functions recursively on $\mathbb{N}$. Arithmetic follows from this assertion: the binary operations of addition, multiplication, and exponentiation can be defined recursively in their right-hand arguments. First we have

$$m + 1 = m^{\mathsf{s}}, \qquad m + x^{\mathsf{s}} = (m + x)^{\mathsf{s}}, \qquad \textbf{(vi)}$$

that is, the function $x \mapsto m + x$ is the unique homomorphism from $(\mathbb{N}, 1, \mathsf{s})$ to $(\mathbb{N}, m^{\mathsf{s}}, \mathsf{s})$. Then s is $x \mapsto x + 1$, and we continue with the definitions

$$m \cdot 1 = m, \qquad m \cdot (x + 1) = m \cdot x + m; \qquad \textbf{(vii)}$$

$$m^1 = m, \qquad m^{x+1} = m^x \cdot m; \qquad \textbf{(viii)}$$

that is, the functions $x \mapsto m \cdot x$ and $x \mapsto m^x$ are the unique homomorphisms from $(\mathbb{N}, 1, \mathsf{s})$ to $(\mathbb{N}, m, x \mapsto x + m)$ and $(\mathbb{N}, m, x \mapsto x \cdot m)$ respectively.

Skolem [32] develops arithmetic by recursion and induction alone, though without discussing the logical relations between these two methods and without mentioning homomorphisms as such. Skolem's purpose is to avoid logical quantifiers. Instead of defining $m < n$ to mean $m + x = n$ for *some* x, he defines it by saying that $m < 1$ never, and $m < k + 1$ if $m < k$ or $m = k$. To see this definition in terms of homomorphisms, let us denote the set $\{0, 1\}$ by $\mathbb{B}$ for Boole.[13] Every binary relation R on $\mathbb{N}$ corresponds to a binary *characteristic function* χ_R from $\mathbb{N}$ into $\mathbb{B}$ given by

$$\chi_R(x, y) = 1 \iff x \mathrel{R} y.$$

We are defining the function $x \mapsto \chi_<(m, x)$ from $\mathbb{N}$ to $\mathbb{B}$, which is described by the table below. Unless $m = 1$, the function is *not* a homomorphism from $(\mathbb{N}, 1, \mathsf{s})$ into

x	1	2	…	$m-1$	m	$m+1$	$m+2$	…
$\chi_<(m,x)$	0	0	…	0	0	1	1	…

$(\mathbb{B}, 0, t)$ for any choice of t, simply because there is no operation t on $\mathbb{B}$ that takes each

[12] This assumes that the iterative structure is based on a set, rather than a proper class.

[13] From Boole himself [1, p. 41]: 'We have seen that the symbols of Logic are subject to the special law,

$$x^2 = x.$$

Now of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law.'

entry in the bottom row of the table to the next entry. However, there is an operation on $\mathbb{N} \times \mathbb{B}$ that takes each *column* of the table to the next column, namely $(x, y) \mapsto \big(x + 1, \max\big(y, \chi_{=}(m, x)\big)\big)$. Call this operation t; then (assuming $m \neq 1$) the function $x \mapsto \big(x, \chi_{<}(m, x)\big)$ is the unique homomorphism from $(\mathbb{N}, 1, \mathsf{s})$ to $\big(\mathbb{N} \times \mathbb{B}, (1, 0), t\big)$.

If f is an arbitrary function from $\mathbb{N}$ to an arbitrary set A, then the function $x \mapsto \big(x, f(x)\big)$ is a homomorphism from $(\mathbb{N}, 1, \mathsf{s})$ to $\Big(\mathbb{N}\times A, \big(1, f(1)\big), (x, y) \mapsto \big(x+1, f(x+1)\big)\Big)$. But this would be a truly circular way to define f. The operations on $\mathbb{N}$ that Gödel [14] calls **recursive** are not defined as homomorphisms. Rather, the collection of recursive operations can itself be defined **recursively** as follows (here I follow Gödel in letting the initial element of $\mathbb{N}$ be 0):

(1) constant operations and s are recursive;
(2) compositions of recursive operations are recursive;
(3) if ψ is an $(n-1)$-ary, and μ an $(n+1)$-ary, recursive operation for some positive n, then the n-ary operation φ is recursive, where, for each n-tuple $\boldsymbol{a}$ of natural numbers, the function $x \mapsto \big(x, \varphi(x, \boldsymbol{a})\big)$ is the unique homomorphism from $(\mathbb{N}, 0, \mathsf{s})$ to $\Big(\mathbb{N} \times \mathbb{N}, \big(0, \psi(\boldsymbol{a})\big), (y, z) \mapsto \big(y + 1, \mu(y, z, \boldsymbol{a})\big)\Big)$.

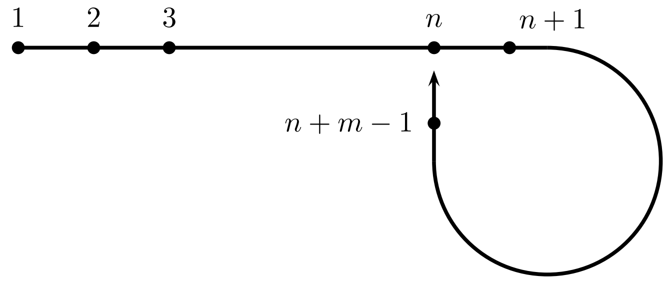
Thus the recursive operations compose a sort of algebra which is closed both under composition and under the operation that produces φ from ψ and μ.

4. Numbers, axiomatically

Why does $\mathbb{N}$ admit recursive definitions of functions? The function h in (**v**) can be understood as the smallest subset of $\mathbb{N} \times A$ that contains $(1, 1)$ and is closed under the operation $(x, y) \mapsto (x^{\mathsf{s}}, y^{\mathsf{s}})$. This set is a *relation* R from $\mathbb{N}$ to A, and it follows by induction that, for each x in $\mathbb{N}$, there is at least one y in A such that $x\ R\ y$. If this y is always unique, then R is the desired homomorphism h. However, uniqueness of y does not follow by induction alone: it is necessary, as well as sufficient, that 1 be not a successor in $\mathbb{N}$ and that immediate predecessors, when they do exist, be unique. This is Theorem 2 below.

Dedekind [7, II (130), pp. 88–90] observes the sufficiency of the additional conditions, along with the necessity of *some* additional condition. Indeed, let us say that an arbitrary iterative structure $\mathfrak{A}$ **admits recursion** if, for every iterative structure $\mathfrak{B}$, there is a unique homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. In particular, $\mathbb{N}$ admits recursion. If also $\mathfrak{A}$ admits recursion, then the homomorphism from $\mathfrak{A}$ to $\mathbb{N}$ must be the inverse of the homomorphism from $\mathbb{N}$ to $\mathfrak{A}$. Indeed, the composition (in either sense) of the two homomorphisms is a homomorphism, but so is the identity, hence these must be equal. Therefore $\mathfrak{A}$ and $\mathbb{N}$ are isomorphic, and $\mathfrak{A}$ admits induction, simply because $\mathbb{N}$ does. However, as Dedekind notes, the converse does not follow; $\mathfrak{A}$ may admit induction, but not recursion. Indeed, for any natural number n, the quotient $\mathbb{Z}/(n)$ can be considered as an iterative structure whose initial element is 0 and in which the successor of x is $x + 1$. Then $\mathbb{Z}/(n)$ admits induction; but there is no homomorphism from $\mathbb{Z}/(n)$ to $\mathbb{Z}/(m)$ unless $m \mid n$.

As Landau [22, Thms 4 & 28; see also pp. ix. f.] shows implicitly, the recursive definitions of addition and multiplication in (**vi**) and (**vii**) are in fact valid on arbitrary iterative structures that admit induction. Such structures may be finite, as in the figure below (where n may be 1, and m may be 0). Henkin [15] makes the point explicit, while



observing that the recursive definition of exponentiation fails in some such structures. Indeed, since for Henkin the initial element of $\mathbb{N}$ is 0, his definition of exponentiation is

$$x^0 = \mathsf{s}(0), \qquad x^{\mathsf{s}(u)} = x^u \cdot x,$$

which always fails in $\mathbb{Z}/(n)$ in case $n > 1$, simply because the definition requires $0^0 = 1$, but $0^n = 0$. However, if we keep 1 as the initial element of $\mathbb{N}$, then the following curiosity arises.

Theorem 1. *On $\mathbb{Z}/(n)$, the recursive definition* (**viii**) *of exponentiation is valid if and only if $n \in \{1, 2, 6, 42, 1806\}$.*

Proof. The desired n are just such that

$$x^{n+1} \equiv x \pmod{n}.$$

Such n are found by Zagier [46], and earlier by Dyer-Bennet [9], as I learned by entering the sequence 1, 2, 6, 42, 1806 at [34]. Alternatively, such n must be squarefree, since if p is prime and $p^2 \mid n$, then $(n/p)^2 \equiv 0 \pmod{n}$, although $n/p \not\equiv 0 \pmod{n}$. So we want to find the squarefree n whose prime factors p are such that $x^{n+1} \equiv x \pmod{p}$, that is, either $p \mid x$ or $x^n \equiv 1 \pmod{p}$. Since x here might be a primitive root of p, we just want n such that $p-1 \mid n$ whenever p is a prime factor of n. Let us refer to a prime p as **good** if $p-1$ is squarefree and all prime factors of $p-1$ are good. Then all prime factors of n must be good. This definition of good primes can be understood as being **recursive.** Indeed, the empty set consists of good primes. If A is a finite set of good primes, then every prime of the form

$$1 + \prod_{p \in B} p,$$

where $B \subseteq A$, is good. Then let A' comprise these primes, along with the primes in A. We have a recursively defined function $x \mapsto A_x$ on $\mathbb{N}$, where $A_1 = \varnothing$ and $A_{x+1} = {A_x}'$. The set of good primes is the union of the sets A_x. We compute

$$A_2 = \{2\}, \qquad A_3 = \{2, 3\}, \qquad A_4 = \{2, 3, 7\}, \qquad A_5 = \{2, 3, 7, 43\},$$

but then the sequence stops growing, since

$$2 \cdot 43 + 1 = 87 = 3 \cdot 29; \qquad 2 \cdot 7 \cdot 43 + 1 = 603 = 3^2 \cdot 67;$$
$$2 \cdot 3 \cdot 43 + 1 = 259 = 7 \cdot 37; \qquad 2 \cdot 3 \cdot 7 \cdot 43 + 1 = 1807 = 13 \cdot 139.$$

So the set of good primes is $\{2, 3, 7, 43\}$. In this set, we have

$$p < q \iff p \mid q - 1.$$

Hence the set of desired n is $\{1,\ 2,\ 2\cdot 3,\ 2\cdot 3\cdot 7,\ 2\cdot 3\cdot 7\cdot 43\}$, which is as claimed. □

The sequence $(2, 3, 7, 43)$ of good primes is the beginning of the sequence $(a_1, a_2, \dots)$, where $a_{n+1} = a_1 \cdot a_2 \cdots a_n + 1$. This sequence arises from what Mazur [26] calls the *self-proving* formulation of Proposition IX.20 of Euclid's *Elements.* Euclid's formulation, in Heath's translation, is, 'Prime numbers are more than any assigned multitude of prime numbers.'[14] In these terms, the self-proving formulation is that, if a multitude of primes be assigned, then the product of its members, plus one, is a number whose prime factors—of which there is at least one—are not in the assigned multitude.

Again, as an iterative structure, $\mathbb{N}$ admits recursive definition of functions because of the theorem below. The theorem should be standard. Dedekind [7, II (126), p. 85] proves the reverse implication, as does, more recently, Stoll [36, ch. 2, Thm 1.2, p. 61]. Mac Lane and Birkhoff [25] mention the theorem, apparently attributing the forward direction to Lawvere: they refer to admission of recursion by an iterative structure as the Peano–Lawvere Axiom. (Lawvere and Rosebrugh [23] call it the Dedekind–Peano Axiom.)

Theorem 2 (Recursion)**.** *For an iterative structure to admit recursion, it is sufficient and necessary that*

(1) it admit induction,
(2) 1 *be not a successor, and*
(3) every successor have a unique immediate predecessor.

An iterative structure meeting the three conditions listed in the theorem is what Dedekind [7, II (71), p. 67] calls a **simply infinite system;** it is what is axiomatized by the so-called Peano Axioms [28]. Peano seems to assume that the Recursion Theorem is obvious, or at least that recursive definitions are obviously justified by induction alone. They are not, as Landau points out [22, pp. ix–x]; but the confusion continues to be made.[15]

To establish the *sufficiency* of the conditions given in the theorem, we can use them as suggested at the head of this section. Namely, we can show that the unique homomorphism from $\mathbb{N}$ to an arbitrary iterative structure $\mathfrak{A}$ is the intersection h of the set of all *relations* from $\mathbb{N}$ to A that contain $(1, 1)$ and are closed under $(x, y) \mapsto (x^{\mathsf{s}}, y^{\mathsf{s}})$. This requires assuming that $\mathbb{N}$ itself is indeed a set, as opposed to a proper class. As discussed in the next section, one might wish to avoid doing this. In that case, one can obtain h as the union of the set comprising every relation R from $\mathbb{N}$ to A such that, if $x \mathrel{R} y$, then either $(x, y) = (1, 1)$ or else $(x, y) = (u^{\mathsf{s}}, v^{\mathsf{s}})$ for some (u, v) such that $u \mathrel{R} v$.

5. Numbers, constructively

The next two sections have aspects of an historical review of set-theory. However, I refer only to articles and books that I have direct access to (mainly, though not exclusively,

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[14] *Οἵ πρῶτοι ἀριθμοι πλείους εἰσὶ παντὸς τοῦ προτεθέντος πλήθους πρώτων ἀριθμῶν* [6]. Euclid says not that there are *infinitely many* prime numbers, but that there are more than can be told.

[15] Indeed, Mac Lane and Birkhoff [25] seem to invite the student to make the confusion. They give the Peano Axioms and then immediately use *recursion* to define the iterates of a permutation of a set. Only later is the equivalence of recursion to the axioms mentioned. Another writer who invites confusion is Burris [3, Appendix B, p. 391]: he states the Peano Axioms and then immediately defines addition recursively without justification, although he claims to be following the presentation of Dedekind.

in van Heijenoort's anthology [40]); therefore my review is incomplete, as the references in Fraenkel *et al.* [13] and in Levy [24] make me aware.

In a letter to a skeptic called Keferstein, Dedekind [8] explains his work as an answer to the question:

> What are the mutually independent fundamental properties of the sequence $\mathbb{N}$, that is, those properties that are not derivable from one another, but from which all others follow?[16]

Having discovered these properties and made them the defining properties of a simply infinite system, Dedekind feels the need to ask,

> does such a system *exist* at all in the realm of our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions.

Dedekind finds his proof in 'the totality S of all things, which can be objects of my thought' [7, II (66), p. 64]: this is closed under the injective but non-surjective operation that converts a thought into the thought that it *is* a thought. But I will say with Bob Dylan, 'You don't need a weatherman to know which way the wind blows.' The consistency of Dedekind's definition of a simply infinite system is self-evident. If we do construct an example, this will tend to confirm, if anything, the validity of our method of construction, rather than the consistency of the definition itself.

The existence of a simply infinite system with an underlying *set* is the Axiom of Infinity of set-theory. I take set-theory to begin with the Extension Axiom mentioned in §0, along with the **'Comprehension Axiom':** every property determines a set, namely the set of objects with the property. Russell shows this 'axiom' to be false in his letter to Frege:

> Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain conditions a definable collection does not form a totality. [31]

It is common now to refer to Russell's totalities[17] as **sets,** and to his definable collections as **classes.** Then all sets are classes. The 'Comprehension Axiom' is the converse; but it is false. If alternative axioms for set-theory can be proposed, and these allow construction of a satisfactory example of a simply infinite system, then the construction will not really justify the Peano Axioms; rather, it will tend to justify the set-theory axioms.

Zermelo [47] offers seven axioms[18] for sets; they can be expressed as follows:

1. **Extension** (as in §0).
2. **Elementary Sets:** classes $\varnothing$, $\{x\}$, and $\{x, y\}$, with no, one, and two elements, are sets.
3. **Separation:** subclasses of sets are sets.
4. **Power Set:** the class $\mathscr{P}(x)$ of subsets of a set x is a set.
5. **Union:** the union $\bigcup x$ of a set x is a set.

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[16] I take the liberty of writing $\mathbb{N}$ where van Heijenoort's text has simply N.

[17] Actually the term 'totality' in the quotation is a translation from Russell's German.

[18] Like Zermelo, I shall not bother to distinguish axioms from axiom *schemes.*

6. **Choice:** the union of a set of nonempty disjoint sets has a subset that has exactly one element in common with each of those disjoint sets.
7. **Infinity:** there is a set that contains $\varnothing$ and closed under the operation $x \mapsto \{x\}$ of singleton formation.

By the Axioms of Union and Elementary Sets, two sets x and y have a *union,* $x \cup y$, namely $\bigcup\{x, y\}$. By the Power Set Axiom, for every set x, there is a set, namely $\mathscr{P}(x)$, which has greater cardinality by Cantor's Theorem.[19] In particular, the sequence

$$x, \qquad \mathscr{P}(x), \qquad \mathscr{P}(\mathscr{P}(x)), \qquad \mathscr{P}(\mathscr{P}(\mathscr{P}(x))), \qquad \dots \tag{ix}$$

is of strictly increasing cardinality, although we have not formulated a precise notion of sequence.

By the Separation Axiom, the *intersection* $\bigcap \boldsymbol{C}$ of a nonempty class $\boldsymbol{C}$ is a set. Hence, by the Axiom of Infinity, the intersection of the class of sets that contain $\varnothing$ and are closed under $x \mapsto \{x\}$ is a set Ω. Therefore $\big(\Omega, \varnothing, x \mapsto \{x\}\big)$ is a simply infinite system. In particular, it admits induction, and we may express this by writing

$$\Omega = \Big\{\varnothing, \{\varnothing\}, \big\{\{\varnothing\}\big\}, \dots\Big\}. \tag{x}$$

It would however be more satisfactory to obtain Ω, as a class, *without* assuming the Axiom of Infinity. To this end, let us denote by

$$\mathbf{Zm}$$

the class comprising every set whose every element is either $\varnothing$ or $\{x\}$ for some x in the set. Then $\mathbf{Zm}$ contains the sets $\varnothing$, $\{\varnothing\}$, $\big\{\varnothing, \{\varnothing\}\big\}$, $\Big\{\varnothing, \{\varnothing\}, \big\{\{\varnothing\}\big\}\Big\}$, and so forth. The hope is that $\bigcup \mathbf{Zm}$ is the desired class Ω.

To tell whether the hope is realized, we can note first that induction makes sense for proper classes. In particular, suppose $\bigcup \mathbf{Zm}$ has a subclass $\boldsymbol{D}$ that contains $\varnothing$ and is closed under $x \mapsto \{x\}$. Then on the class $\bigcup \mathbf{Zm} \smallsetminus \boldsymbol{D}$, the function $\{x\} \mapsto x$ of unique-element extraction is well-defined, and the class is closed under this function. If we could conclude that $\bigcup \mathbf{Zm} \smallsetminus \boldsymbol{D}$ must be empty—which we cannot yet do—then $\big(\bigcup \mathbf{Zm}, \varnothing, x \mapsto \{x\}\big)$ would admit induction.

Skolem [33, §4, pp. 296–7] observes that Zermelo's axioms do not guarantee the existence of a set comprising the terms of the sequence in (**ix**). He remedies this with the following axiom (also discovered by Fraenkel):

8. **Replacement:** the image of a set under a function is a set.

Skolem makes the notion of a class precise: it is defined by a formula in (what we call) the first-order logic of the signature $\{\in\}$. These formulas can be defined **recursively:**

1. Equations $x = y$ and 'memberships' $x \in y$ are formulas.
2. if x is a variable, and φ and ψ are formulas, then $\neg\varphi$, $(\varphi \Rightarrow \psi)$, and $\exists x\ \varphi$ are formulas.

More precisely, a formula in *one* free variable defines a class, while a formula in *two* free variables defines a binary relation, which may be a function. A binary relation becomes

[19] The theorem is so called by Zermelo [47, p. 211].

identified with a class when, with Kuratowski [21, p.171],[20] we define ordered pairs by the identity

$$(x, y) = \{\{x\}, \{x, y\}\}. \qquad \textbf{(xi)}$$

If we are willing to accept a recursive definition like the definition of formulas in the signature $\{\in\}$, then we should accept a definition of natural numbers as strings of the form $\mathtt{s}\cdots\mathtt{s}1$ (or perhaps instead of the form $1\cdots 1$). However, such definitions do not give us natural numbers or formulas as composing *sets*. Nor is this required for the sake of formulating set-theory in the first place.

A class on which the function $\{x\} \mapsto x$ is a well-defined operation is one example of a class whose every element has nonempty intersection with the class. Skolem [33, §6, p. 298] perceives the possibility, along with the non-necessity, of such classes: this suggests non-categoricity of the axioms so far. Von Neumann [41, §5, pp.413–4] more explicitly proposes excluding such classes with a new axiom. His axiom can be understood as that there is no sequence $(a, a', a'', \dots)$ such that $a' \in a$, and $a'' \in a'$, and so forth. Since we are now engaged in determining how such a sequence can be formulated in the first place, let us introduce the new axiom in the following form.

9. **Foundation:** membership is well-founded on every nonempty class, that is, some element is disjoint from the class.[21]

With the new axiom then, the definition of Ω in (**x**) as the union $\bigcup \mathbf{Zm}$ works, in that $\bigcup \mathbf{Zm}$ admits induction by the argument above. However, it is not very satisfying to have to rely on the new axiom. We can avoid using the axiom as such by incorporating it into the definition of $\mathbf{Zm}$.

Theorem 3. *Let* $\mathbf{Zm}'$ *comprise those elements of* $\mathbf{Zm}$ *whose every subset* x *is disjoint from some element of* x. *Then* $(\bigcup \mathbf{Zm}', \varnothing, x \mapsto \{x\})$ *is a simply infinite system, even without recourse to the Axioms of Infinity and Foundation.*

Proof. If $A \in \mathbf{Zm}'$, and $\boldsymbol{D}$ is a subclass of $\bigcup \mathbf{Zm}'$ that contains $\varnothing$ and is closed under $x \mapsto \{x\}$, then on $A \cap (\bigcup \mathbf{Zm}' \smallsetminus \boldsymbol{D})$, the operation $\{x\} \mapsto x$ is well-defined, so $A \cap (\bigcup \mathbf{Zm}' \smallsetminus \boldsymbol{D})$ must be empty. Since A was arbitrary, $\bigcup \mathbf{Zm}' \smallsetminus \boldsymbol{D}$ is empty. □

It may appear that the definition of Ω as $\bigcup \mathbf{Zm}'$ still does not yield what is desired. Foundation excludes sets on which the operation $\{x\} \mapsto x$ is well-defined; but suppose A

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[20] An earlier definition of ordered pairs is Wiener's [45]: $(x, y) = \Big\{\{\{x\}, \varnothing\}, \{\{y\}\}\Big\}$, though Wiener calls it $\iota`(\iota`\iota`x \cup \iota`\Lambda) \cup \iota`\iota`\iota`y$, using the notation of Whitehead and Russell [43, pp. 28–37].

[21] The Foundation Axiom has a *set* form, namely that that membership is well-founded on every nonempty *set*. See Levy [24, II.7, p. 72] on the equivalence of the two forms in the presence of the other axioms. However, in the absence of the Axiom of Infinity, the set form of Foundation does not imply the class form given above. A proof that this is so can be adapted from a proof given by Cohen [5, II.5, p. 72] to show the independence of the set form of Foundation from the other axioms. Suppose we do have ω in the usual sense, as at (**xiv**) below. Let C be a set $\{a_x \colon x \in \omega\}$ indexed by ω such that no member is an element of another member. For example, each a_n could be $\{n+1\}$. If a is an arbitrary set, let $\mathscr{P}_{\mathrm{f}}(x)$ denote the set of *finite* subsets of a. Now let $C^* = \bigcup\{C, \mathscr{P}_{\mathrm{f}}(C), \mathscr{P}_{\mathrm{f}}(\mathscr{P}_{\mathrm{f}}(C)), \dots\}$. Define $<$ on C^* so that $a < b$ if and only if either $a \in b$, or else $a = a_{k+1}$ and $b = a_k$ for some k in ω. Then $(C^*, <)$ is a model of the first eight numbered axioms above, except Infinity; and it is a model of the set form of Foundation. But it is not a model of the class form of Foundation. Indeed, in $(C^*, <)$, we can define the class $\boldsymbol{D}$ of all sets A such that, if $b \in A$, then $b \subseteq A$. Then $\boldsymbol{D} = \bigcup\{\varnothing, \mathscr{P}(\varnothing), \mathscr{P}(\mathscr{P}(\varnothing)), \dots\}$, and $C^* \smallsetminus \boldsymbol{D}$ is a counterexample to Foundation.

is a nonempty 'collection' on which $\{x\} \mapsto x$ is well-defined. Then the union $A \cup \Omega$ may belong to $\mathbf{Zm}'$, unless A can be obtained as a subclass of this union. If we can obtain Ω by other means, then we can obtain A as $(A \cup \Omega) \smallsetminus \Omega$; but this just begs the question of whether we have Ω as a class in the first place.

If we think we need to fix this problem, we might try letting $\mathbf{Zm}''$ comprise the *finite* elements of $\mathbf{Zm}'$. Here we can use for example the definition mentioned by Whitehead and Russell [44, ∗120.23],[22] whereby B is finite if every subset of $\mathscr{P}(B)$ contains B, provided the subset contains $\varnothing$ and is closed under the operation $x \mapsto \{c\} \cup x$ for each c in B. If $\mathbf{Zm}''$ had a finite element $A \cup \Omega_0$, where A violated Foundation, then we could remove the elements of Ω_0 one by one, thus obtaining A, in violation of Foundation. However, we still have no guarantee that we can do this, unless we already have Ω_0 as a class.

The problem here is the one that Skolem hit upon: set-theory is not categorical. Indeed, suppose the formula $\varphi(x)$ defines the class Ω as in ($\mathbf{x}$), and introduce constants c, c', c'', ... Then the sentences

$$\varphi(c), \qquad c' \in c, \qquad \varphi(c'), \qquad c'' \in c', \qquad \varphi(c''), \qquad \dots$$

are consistent with set-theory, in that no contradiction is derivable from them that is not already derivable from the axioms. We must settle for observing that, in any model of set theory, φ defines a class that acts *as if* it were as in ($\mathbf{x}$). From outside the model, we may see an infinite descending chain of elements of the model; but the chain is not a class of the model.[23]

6. Ordinals

As sets may be gathered into classes, so classes may be gathered into—let us say **families.** For example, given an equivalence-relation on a class, we have the family of equivalence-classes of members of that class.

Given well-ordered sets $\mathfrak{A}$ and $\mathfrak{B}$, let us say $\mathfrak{A} < \mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to a proper initial segment of $\mathfrak{B}$. Then exactly one of $x < y$, $x \cong y$, and $y < x$ holds for all well-ordered sets x and y. Hence the relation $<$ induces a well-ordering of the family of isomorphism-classes of well-ordered sets.

It may be desirable to find a well-ordered class that is isomorphic to the family of isomorphism-classes of well-ordered sets. This will be

$$\mathbf{ON},$$

the class of **ordinal numbers.** Then every well-ordered set will be isomorphic both to a unique ordinal and to the well-ordered set of its predecessors in $\mathbf{ON}$. With this motivation and observation, von Neumann [42] obtains $\mathbf{ON}$ so that each element *is* the set of its predecessors.

[22]I learn the reference from Suppes [37, p. 102]; various definitions of finite sets are discussed by Tarski [38].

[23]Edward Nelson *recommends* conceiving some natural numbers as *nonstandard* and as composing an infinite descending chain. See Chapter 1, 'Internal set theory,' of a once-proposed book (`http://www.math.princeton.edu/~nelson/books.html`, accessed March 30, 2009).

Suppose $\mathfrak{A}$ is an **order,** or **ordered set,** $(A, <)$: by this I mean simply that $<$ is irreflexive and transitive on the set A.[24] A **section** of $\mathfrak{A}$ is the set of predecessors in A of a particular element of A. Let the set of predecessors of x be denoted by $\operatorname{pred}(x)$, so that

$$\operatorname{pred}(x) = \{y \in A \colon y < x\}.$$

since it is the set of predecessors of x in A. Then $\mathfrak{A}$ **admits induction** if the only subset B of A for which

$$\operatorname{pred}(x) \subseteq B \implies x \in B$$

for all x in A is A itself. There are various ways to define recursion in ordered sets; it is simplest for present purposes to say that $\mathfrak{A}$ **admits recursion** if, for every class $\boldsymbol{C}$, for every function $\boldsymbol{F}$ from $\mathscr{P}(\boldsymbol{C})$ to $\boldsymbol{C}$, there is a unique function g from A to $\boldsymbol{C}$ given by

$$g(x) = \boldsymbol{F}(\{g(y) \colon y < x\}) = \boldsymbol{F}(g[\operatorname{pred}(x)]). \tag{\textbf{xii}}$$

As with Theorem 2, the following should be standard. Indeed, possibly awareness of the following has led to the misconception that recursion and induction are equivalent properties of iterative structures.

Theorem 4 (Recursion)**.** *For* total *orders, admission of recursion and induction are equivalent to each other and to being well-ordered.* □

If $\mathfrak{A}$ is well-ordered, and in (**xii**) we let $\boldsymbol{F}$ be the identity on the universe $\mathbf{V}$, then

$$g(x) = \{g(y) \colon y < x\} = g[\operatorname{pred}(x)]. \tag{\textbf{xiii}}$$

In this case, $(g[A], \in)$ is the ordinal number isomorphic to $\mathfrak{A}$; indeed, this is so by von Neumann's original *definition* of the ordinal numbers [42, p. 348]. Alternative, equivalent definitions include R. Robinson's [30], namely that an ordinal is a set that both is *totally* ordered by membership and *includes* all of its elements. This definition of ordinals assumes the Foundation Axiom, which ensures that each ordinal is *well*-ordered by membership. A set that includes all of its elements is a **transitive** set; then another definition of ordinals, again assuming Foundation, is that they are transitive sets of transitive sets.

The class **ON** has the injective operation $x \mapsto x \cup \{x\}$ of succession as well as the element $\varnothing$, which is not a successor. Thus **ON** meets two of the three conditions for being a simply infinite system. It fails to meet the third condition, of admitting induction, if there are ordinals other than $\varnothing$ that are not successors. Such ordinals are **limits.** The class of ordinals that neither *are* limits nor *contain* limits is called

$$\omega. \tag{\textbf{xiv}}$$

This is an initial segment of **ON**, so it is either **ON** itself or a member of it. In any case, $(\omega, \varnothing, x \mapsto x \cup \{x\})$ is a simply infinite system. We could let the Axiom of Infinity be that ω is a set.

Fraenkel *et al.* [13, ch. II, §3, 6, p. 49] are vague on whether this Axiom can be expressed in the form 'Such-and-such a class is a set.' They list three forms of the Axiom, called VIa, b, and c, namely: there are sets containing $\varnothing$ and closed under $x \mapsto \{x\}$, $x \mapsto x \cup \{x\}$, and $(x, y) \mapsto x \cup \{y\}$ respectively. They write:

[24] So $<$ is what is sometimes called a *strict partial ordering,* although it might be total.

> In our heuristic classification of the axioms, as to whether they are instances of the axiom schema of comprehension or not, the axiom of infinity can, to some extent, be viewed as such an instance. Each of VIa and VIb can be stated as 'there exists a set which consists of all natural numbers', for the respective notion of natural number, and VIc can be stated similarly.

The authors seem to be treating 'being a natural number' as a condition defining a class in the original imprecise sense. However, the authors *do* have the precise notion of a condition as being given by a formula in the first-order logic of $\in$. I do not know why they do not directly address the question of whether, without the Axiom of Infinity, there is still a formula defining the natural numbers.

The Recursion Theorem for iterative structures (Theorem 2 above) can be extended to include the following, which gives conditions under which the equivalence of properties 1 and 2 in §1 can be established.

Theorem 5. *The following are equivalent conditions on an iterative structure.*

1. *It is a simply infinite system.*
2. *It can be well-ordered so that* $x < x^{\mathsf{s}}$*, and every element besides* 1 *is* x^{s} *for some element* x*.*
3. *It admits induction and can be ordered so that* $x < x^{\mathsf{s}}$*.*

Proof. Every iterative structure satisfying 1 is isomorphic to ω, which satisfies 2.

Suppose $\mathfrak{B}$ satisfies 2, and $\mathfrak{A}$ is a substructure. If $B \smallsetminus A$ were non-empty, then its least element would be c^{s} for some c; but $c < c^{\mathsf{s}}$, so $c \in A$, hence $c^{\mathsf{s}} \in A$. Therefore $B \smallsetminus A$ is empty. Thus $\mathfrak{B}$ satisfies 3.

Suppose finally $\mathfrak{A}$ satisfies 3. As noted in §3, addition can be defined in $\mathfrak{A}$ to satisfy (**vi**). Again by induction, the ordering of $\mathfrak{A}$ satisfies $x < x + y$. Let h be the homomorphism from $(\mathbb{N}, 1, \mathsf{s})$ to $\mathfrak{A}$. Then h is surjective by induction in $\mathfrak{A}$, and $h(x + y) = h(x) + h(y)$ by induction on y in $\mathbb{N}$. Moreover, in $\mathbb{N}$, if $x < y$, then $x + z = y$ for some z, so $h(x)+h(z) = h(y)$ and therefore $h(x) < h(y)$. Thus h is injective, so h is an isomorphism, and therefore $\mathfrak{A}$, like $\mathbb{N}$, satisfies 1. □

Ordinals suggest the following alternative form of Theorem 3, that is, another way to obtain the class Ω in (**x**).

Theorem 6. *Let* $\boldsymbol{C}$ *comprise every set*

(1) *whose every element is* $\varnothing$ *or* $\{x\}$ *for some element* x and
(2) *which has a well-ordering* $<$ *such that* $x < \{x\}$ *for each element* x *such that* $\{x\}$ *is also an element.*

Then $(\bigcup \boldsymbol{C}, \varnothing, x \mapsto \{x\})$ *is a simply infinite system, even without recourse to the Axioms of Infinity and Foundation.*

Proof. Suppose $A \in \boldsymbol{C}$. Then $\{x\}$ is the successor of x in the well-ordering of A; that is, there is no y such that $x < y < \{x\}$. Indeed, if there is, let x be least such that there is; and then let y be least. But $y \neq \varnothing$, since $\varnothing$ must be the least element of A. Hence $y = \{z\}$ for some z, and then $z < x$ by minimality of y, so $z < x < \{z\}$, contrary to the minimality of x.

Now suppose also $B \in \boldsymbol{C}$. Then one of A and B is an initial segment of the other, when both are considered as well-ordered sets. Indeed, one of them, say A, is *isomorphic* to an initial segment of the other, B. If the isomorphism is h, then there can be no least element x of A such that $h(x) \neq x$, since if there is, then $x = \{y\}$ for some y, so $h(x) = h(\{y\}) = \{h(y)\} = \{y\} = x$ (since $\{y\}$ succeeds y, and $\{h(y)\}$ succeeds $h(y)$, in the respective orderings).

It now follows that $\bigcup \boldsymbol{C}$ satisfies condition 2 in Theorem 5. □

Having **ON**, we can define the function **R** on **ON** recursively by[25]

$$\mathbf{R}(\alpha) = \bigcup\{\mathscr{P}(\mathbf{R}(\beta)) \colon \beta < \alpha\}.$$

Then $\bigcup \mathbf{R}[\mathbf{ON}]$ is called **WF**, the class of **well-founded** sets; it is a model of the set-theoretic axioms, including Foundation, regardless of whether the latter 'really' holds; so we might as well assume that it does hold. Then **WF** is the class of *all* sets.

Similarly, there is a function **L** on **ON** such that $\bigcup \mathbf{L}[\mathbf{ON}]$—the class of **constructible** sets—is a model of all of the set-theoretic axioms, including Choice, regardless of whether the latter 'really' holds; so we may assume that it does hold.[26]

Now all of the set-theoretic axioms are consistent with those that can be expressed in the form 'Such-and-such a class is a set', along with Extension.

Part 3

7. Algebras and orderings

Having ω, we can define arbitrary structures. This is done at the beginning of any model-theory book (such as Hodges [19]), except that these books generally assume that structures and their signatures are based on *sets*.[27] That assumption is not made here.

If $\boldsymbol{C}$ is a class, and $n \in \omega$, then $\boldsymbol{C}^n$ can be understood as the class of functions from n into $\boldsymbol{C}$. (Such functions are indeed sets, by the Replacement Axiom, so there can be a class of them.) An **n-ary relation** on $\boldsymbol{C}$ is a subclass of $\boldsymbol{C}^n$. In particular, a **nullary** (0-ary) relation is either $\varnothing$ or $\{\varnothing\}$, that is, 0 or 1 (the sets composing $\mathbb{B}$ as used in §3). An **n-ary operation** on $\boldsymbol{C}$ is a function from $\boldsymbol{C}^n$ into $\boldsymbol{C}$. In the most precise sense, a **signature** can be defined as two disjoint classes—of **predicates** and **function-symbols** respectively—together with a function from their union into ω. A signature $(\mathscr{S}_{\mathrm{p}}, \mathscr{S}_{\mathrm{f}}, \mathrm{ar})$ may be written more simply as $\mathscr{S}$, and $\mathscr{S}$ may be treated as $\mathscr{S}_{\mathrm{p}} \cup \mathscr{S}_{\mathrm{f}}$; then a **structure** with this signature is a class $\boldsymbol{C}$ together with a function $s \mapsto s^{\mathfrak{C}}$ on $\mathscr{S}$ such that if $\mathrm{ar}(s) = n$, then $s^{\mathfrak{C}}$ is an n-ary relation (when $s \in \mathscr{S}_{\mathrm{p}}$) or operation (when $s \in \mathscr{S}_{\mathrm{f}}$) on $\boldsymbol{C}$. The class $\boldsymbol{C}$ is then the **universe** of the structure, and $s^{\mathfrak{C}}$ is the **interpretation** of s in the structure; but as noted in §3, we may refer to the interpretation as s also. I follow the tradition of denoting a structure by the Fraktur form of the letter used for its universe.

An **algebra** is a structure whose signature has only function-symbols. If $\mathfrak{A}$ and $\mathfrak{B}$ are algebras with the same signature, then a homomorphism from the former to the latter

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[25] I use the notation of Kunen [20, p. 95]. It seems the original definition is by von Neumann.

[26] For the same reason, we may assume that the Generalized Continuum Hypothesis holds; but we don't.

[27] Ziegler [49] treats the 'monster model' of a complete theory as based on a proper class; but the signature is still a set.

is a function h from A to B, inducing a function h from A^n to B^n for each n in ω, such that

$$h(F^{\mathfrak{A}}(\boldsymbol{x})) = F^{\mathfrak{B}}(h(\boldsymbol{x})) \tag{xv}$$

for all $\boldsymbol{x}$ in A^n, for all n-ary F from the signature, for all n in ω. In case $n = 0$, (**xv**) becomes $h(F^{\mathfrak{A}}) = F^{\mathfrak{B}}$. An algebra **admits recursion** if from it to every algebra in the same signature there is a unique homomorphism; an algebra **admits induction** if it has no proper sub-algebra. I do not know who first formulated the following generalization of Theorem 2, though Enderton [10, p. 27] alludes to such a theorem:

Theorem 7 (Recursion)**.** *An algebra admits recursion if and only if it admits induction and its distinguished operations are injective and have disjoint ranges.* □

In a signature with no nullary function-symbols, the only algebras that admit induction are empty. An arbitrary algebra that admits recursion can be called **free.** All free algebras in the same signature are isomorphic.

If the universe of an algebra $\mathfrak{B}$ is a set, then $\mathfrak{B}$ has a subalgebra $\mathfrak{A}$ that admits induction. Indeed, the universe of $\mathfrak{A}$ is the intersection of the set of universes of subalgebras of $\mathfrak{B}$. In short, $\mathfrak{A}$ is the smallest subalgebra of $\mathfrak{B}$. If the universe of $\mathfrak{B}$ is a proper class, then we cannot obtain $\mathfrak{A}$ in this way. We might use a trick to obtain $\mathfrak{A}$, as in Theorem 3 or 6. It would be more efficient to obtain a *free* algebra $\mathfrak{F}$ in the same signature, since then $\mathfrak{A}$ will be the image of $\mathfrak{F}$ in $\mathfrak{B}$.

In an arbitrary algebraic signature $\mathscr{S}$, we shall obtain a free algebra as a subalgebra of the algebra $\mathfrak{S}(\mathscr{S})$ of **strings** of $\mathscr{S}$. The universe of $\mathfrak{S}$ is $\mathscr{S}^{<\omega}$, namely $\bigcup\{\mathscr{S}^n : n \in \omega\}$; but in this context an element $x \mapsto s_k$ of $\mathscr{S}^n$ is written as $s_0 \cdots s_{n-1}$. The **length** of every element of $\mathscr{S}^n$ is n. If F is an n-ary symbol of $\mathscr{S}$, then $F^{\mathfrak{S}}$ is the operation

$$(t_0, \dots, t_{n-1}) \longmapsto Ft_0 \cdots t_{n-1} \tag{xvi}$$

of concatenation on $\mathfrak{S}$. Here the length of $Ft_0 \cdots t_{n-1}$ is one plus the sum of the lengths of the t_k. Because the class ω of possible lengths is well-ordered, we have the following. The theorem is a requirement for doing symbolic logic, but perhaps not every logician bothers to work out the proof.

Theorem 8. $\mathfrak{S}(\mathscr{S})$ *has a subalgebra that is a free algebra.*

Proof. Let $\boldsymbol{C}$ be the class of subsets b of $\mathscr{S}^{<\omega}$ such that every element of b is $Ft_0 \cdots t_{n-1}$ for some elements t_k of b, for some n-ary F in $\mathscr{S}$, for some n in ω. Let $A = \bigcup \boldsymbol{C}$. Then A is the universe of a subalgebra $\mathfrak{A}$ of $\mathfrak{S}(\mathscr{S})$, since if F is an n-ary element of $\mathscr{S}$, and $(t_k : k < n) \in A^n$, then each t_k is in some b_k in $\boldsymbol{C}$, and then

$$\bigcup\{b_k : k < n\} \cup \{Ft_0 \cdots t_{n-1}\} \in \boldsymbol{C},$$

so $Ft_0 \cdots t_{n-1}$ is in A.

The algebra $\mathfrak{A}$ admits induction. Indeed, if not, then it has a proper subalgebra with universe B. An element of $A \smallsetminus B$ of minimal length has the form $Ft_0 \cdots t_{n-1}$, where each t_k must therefore belongs to B; but then $Ft_0 \cdots t_{n-1}$ must also be in B.

Immediately the operations $F^{\mathfrak{S}}$ have disjoint ranges.

Proper initial substrings of elements of A are not elements of A. Indeed, suppose this is so for every element of A that is shorter than the element $Ft_0 \cdots t_{n-1}$. If $Ft_0 \cdots t_{n-1}$ has an initial segment that is in A, then this segment must have the form $Fu_0 \cdots u_{n-1}$, where one of t_0 and u_0 is an initial segment of the other, so by inductive hypothesis they are both equal. Likewise, if, for some k in n, we have that t_i and u_i are equal when $i < k$, then $t_k = u_k$. By induction in n, we have $Ft_0 \cdots t_{n-1} = Fu_0 \cdots u_{n-1}$. By induction on lengths, proper initial segments of elements of A are not elements of A.

Consequently each operation $F^{\mathfrak{S}}$ on A is injective. By Theorem 7 then, $\mathfrak{A}$ does indeed admit recursion. □

The free algebra guaranteed by the theorem can be denoted by

$$\mathrm{Tm}^0(\mathscr{S});$$

it comprises the **closed terms** of $\mathscr{S}$ in **Łukasiewicz** or **'Polish' notation** [4, n. 91, p. 38]. Again, it was because we already had ω as a class that we could obtain $\mathrm{Tm}^0(\mathscr{S})$ without working out a full analogue of Theorem 3 or 6. However, I wish now, in the remaining sections, to work out an analogue of ω itself in the signature $\mathscr{S}$.

8. Numbers, generalized

If $\mathscr{S}$ is an arbitrary algebraic signature, we want to obtain a class $\omega_{\mathscr{S}}$ that is to ω as $\mathscr{S}$ is to $\{1, \mathsf{s}\}$. We can consider ω to arise as follows.

One starts with the assumption that there is *some* simply infinite system $(\mathbb{N}, 1, \mathsf{s})$. With some difficulty, one shows that there is a total ordering of $\mathbb{N}$ such that $x < x^{\mathsf{s}}$ for all x in $\mathbb{N}$; then one shows that $\mathbb{N}$ is well-ordered by $<$. Now one has a function g on $\mathbb{N}$ as in (**xiii**), so that $g(x) = g[\mathrm{pred}(x)]$. One then computes

$$\begin{aligned} g(1) &= g[\varnothing] \\ &= \varnothing, \end{aligned} \qquad \begin{aligned} g(x+1) &= g[\mathrm{pred}(x+1)] \\ &= g[\mathrm{pred}(x) \cup \{x\}] \\ &= g[\mathrm{pred}(x)] \cup g[\{x\}] \\ &= g(x) \cup \{g(x)\}. \end{aligned} \tag{xvii}$$

Let $x \cup \{x\}$ be denoted by

$$x'.$$

Thus g is a homomorphism from $(\mathbb{N}, 1, \mathsf{s})$ to $(\mathbf{V}, \varnothing, ')$, and the latter has a substructure with universe $g[\mathbb{N}]$ that admits induction. By the Foundation Axiom, the operation $x \mapsto x'$ on $\mathbf{V}$ is injective. Therefore g is an embedding, and $(g[\mathbb{N}], \varnothing, ')$ is a simply infinite system. The ordering of $g[\mathbb{N}]$ induced from $\mathbb{N}$ by g is membership. By induction, each element of $g[\mathbb{N}]$ is transitive. One then defines $\mathbf{ON}$ to consist of the transitive sets that are well-ordered by membership. Then $(\mathbf{ON}, \varnothing, ')$ is an iterative structure; moreover, $\mathbf{ON}$ is transitive and well-ordered by membership. An element of $\mathbf{ON}$ that neither is $\varnothing$ nor belongs to the image of $x \mapsto x'$ is a **limit.** One defines ω as the class of all elements of $\mathbf{ON}$ that neither are limits nor contain limits. Then ω contains $\varnothing$ and is closed under $x \mapsto x'$, but no proper subclass $\boldsymbol{C}$ of ω is the universe of a substructure of $(\omega, \varnothing, ')$, since the least element of $\omega \smallsetminus \boldsymbol{C}$ is either $\varnothing$ or else α' for some α in $\boldsymbol{C}$. Therefore one has recovered $g[\mathbb{N}]$ as ω. In particular, there is no need to use the Peano Axioms to define an ordering that well-orders $\mathbb{N}$; such an ordering is induced from ω.

The ordering of $\mathbb{N}$ can also be defined recursively, without reference to ω, by

$$\mathrm{pred}(1) = \varnothing, \qquad \mathrm{pred}(n+1) = \mathrm{pred}(n) \cup \{n\}. \tag{xviii}$$

We want similarly to define an ordering on the free algebra $\mathrm{Tm}^0(\mathscr{S})$. Meanwhile, as a first generalization of (**xvii**), on $\mathrm{Tm}^0(\mathscr{S})$ we have the **height** function, hgt, given recursively by

$$\mathrm{hgt}(Ft_0\cdots t_{n-1}) = \bigcup\bigl\{\mathrm{hgt}(t_k)\cup\{\mathrm{hgt}(t_k)\}\colon k<n\bigr\}. \tag{xix}$$

By induction, $\mathrm{hgt}(x)$ is in ω, and $\mathrm{hgt}(Ft_0\cdots t_{n-1})$ is the greatest of the numbers $\mathrm{hgt}(t_k)+1$ (so it is 0 if $n=0$). In a generalization of (**xviii**), we can make the recursive definitions

$$\mathrm{pred}(Ft_0\cdots t_{n-1}) = \bigcup\{\mathrm{pred}(t_k)\cup\{t_k\} : k<n\},$$

$$\mathrm{pred}_k(Ft_0\cdots t_{n-1}) = \begin{cases}\mathrm{pred}(t_k)\cup\{t_k\}, & \text{if } k<n,\\ \varnothing, & \text{if } k\geqslant n.\end{cases}$$

Immediately,

$$\mathrm{pred}(Ft_0\cdots t_{n-1}) = \bigcup\{\mathrm{pred}_k(Ft_0\cdots t_{n-1})\colon k\in\omega\}. \tag{xx}$$

Let us also write

$$t<u \iff t\in\mathrm{pred}(u), \qquad t<_k u \iff t\in\mathrm{pred}_k(u). \tag{xxi}$$

An ordering **directs** a class if every finite subset has an upper bound in the class with respect to the ordering. An ordering is **well-founded** on a class if every section of the class is a set and every subset of the class has a minimal element.

Lemma 1. *On* $\mathrm{Tm}^0(\mathscr{S})$*:*

(1) *$x<y$ if and only if $x<_k y$ for some k in ω;*
(2) *if $x<y$ and $y<_k z$, then $x<_k z$;*
(3) *$<$ and $<_k$ are well-founded orderings;*
(4) *$<$ directs $\mathrm{pred}_k(x)$;*
(5) *t_k is maximal with respect to $<$ in $\mathrm{pred}_k(Ft_0\cdots t_{n-1})$, assuming $k<n$.*

Proof. Condition (1) is a restatement of (**xx**), and (2) is

$$y<_k z \implies \mathrm{pred}(y)\subseteq\mathrm{pred}_k(z),$$

which is established by induction on z. Indeed, suppose the claim holds when $z\in\{t_0,\dots,t_{n-1}\}$, but now $y<Ft_0\cdots t_{n-1}$. Then either $y<_k t_k$ or $y=t_k$ for some k in n. In the former case, by inductive hypothesis and (1), $\mathrm{pred}(y)\subseteq\mathrm{pred}_k(t_k)\subseteq\mathrm{pred}(t_k)$. Hence, in either case,

$$\mathrm{pred}(y)\subseteq\mathrm{pred}(t_k)\subseteq\mathrm{pred}_k(Ft_0\cdots t_{n-1}),$$

so the claim holds when $z=Ft_0\cdots t_{n-1}$.

By (1) and (2), the relations $<$ and $<_k$ are transitive. To complete (3), we observe by induction on u that

$$v<u \implies \mathrm{hgt}(v)\in\mathrm{hgt}(u).$$

As $\in$ is irreflexive, so is $<$, and hence so is $<_k$, by (1). As $\in$ is well-founded, and the classes $\mathrm{pred}(y)$ are all sets, $<$ and $<_k$ are well-founded.

Finally, both (4) and (5) follow from the observation that, by definition, t_k is the *maximum* element of $\mathrm{pred}_k(Ft_0\cdots t_{n-1})$ with respect to $<$, if $k<n$. □

Proving the lemma would take more work if the corresponding ordering $\in$ of ω were not available. One could use the ordering on $\mathbb{N}$ as defined by (**xviii**); but proving directly that it is a well-ordering takes more work than proving the same for $\in$ on ω. In any case, we can now prove a generalization of Theorem 5.

Theorem 9. *An algebra in a signature $\mathscr{S}$ is free if and only if*

(1) *it has orderings $<$ and $<_k$ for each k in ω, with corresponding sets of predecessors as in* (**xxi**)*, such that*
  (a) *$x < y$ if and only if $x <_k y$ for some k in ω,*
  (b) *if $x < y$ and $y <_k z$, then $x <_k z$,*
  (c) *$<$ directs* $\operatorname{pred}_k(x)$*,*
  (d) *x_k is a maximal element with respect to $<$ of* $\operatorname{pred}_k(F(x_0, \dots, x_{n-1}))$*, assuming $k < n$,*

(2) *the distinguished operations have disjoint ranges, and*

(3) *one of the following:*
  (a) *the union of these ranges is the whole underlying class of the algebra, and $<$ is well-founded, or*
  (b) *the algebra admits induction.*

Proof. If an algebra in $\mathscr{S}$ is free, then it is isomorphic to $\mathrm{Tm}^0(\mathscr{S})$, so (1) follows from the lemma. Also (2) and (3a) follow from this and Theorem 7.

Suppose $\mathfrak{B}$ satisfies (1), (2), and (3a), and $\mathfrak{A}$ is a subalgebra. If $B \smallsetminus A$ is nonempty, then it has a minimal element of the form $F(x_0, \dots, x_{n-1})$; but then the x_k must be in A, and then so must $F(x_0, \dots, x_{n-1})$ be. Thus $\mathfrak{B} = \mathfrak{A}$, so it satisfies (3b).

Finally, suppose $\mathfrak{A}$ satisfies (1), (2), and (3b). By Theorem 7, to conclude that $\mathfrak{A}$ is free, it is enough to show that each $F^{\mathfrak{A}}$ is injective. By induction in $\mathfrak{A}$, the homomorphism h from $\mathrm{Tm}^0(\mathscr{S})$ to $\mathfrak{A}$ is surjective. If some $F^{\mathfrak{A}}$ is not injective, then there is some *minimal* $Ft_0 \cdots t_{n-1}$ in $\mathrm{Tm}^0(\mathscr{S})$ for which

$$F^{\mathfrak{A}}(h(t_0), \dots, h(t_{n-1})) = h(Ft_0 \cdots t_{n-1}) = h(Fu_0 \cdots u_{n-1}) = F^{\mathfrak{A}}(h(u_0), \dots, h(u_{n-1}))$$

for some $Fu_0 \cdots u_{n-1}$, although $h(t_k) \neq h(u_k)$ for some k. But $h(t_k)$ and $h(u_k)$ are maximal, and the set of them has an upper bound, in $\operatorname{pred}_k(h(Ft_0 \cdots t_{n-1}))$; hence they are equal. This contradiction shows $\mathfrak{A}$ is free. □

Suppose x is an $(n+1)$-tuple $(x_0, \dots, x_n)$ for some n in ω. Let us say that the **type** of x is x_n; if $k < n$, let us say that the elements of x of **grade** k are the elements of x_k. We may use the following notation:

$$y \in_k (x_0, \dots, x_n) \iff k < n \mathbin{\&} y \in x_k,$$

$$\mathscr{G}(x) = \begin{cases} \bigcup\{x_k \colon k < n\}, & \text{if } x = (x_0, \dots, x_n) \text{ for some } n \text{ in } \omega \text{ and some } x_k, \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$y \in' x \iff y \in \mathscr{G}(x).$$

Here $\mathscr{G}(x)$ is the set of 'graded' elements of x. Any elements of x_n, as such, are left out.

The notion of well-foundedness makes sense for arbitrary binary relations,[28] such as $\in'$. Indeed, a binary relation $\boldsymbol{R}$ is **well-founded** on a class $\boldsymbol{C}$ if

[28]This observation, with the ensuing definition, is apparently due to Zermelo [24, II.5.1, p. 63].

(1) the class $\{x\colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} a\}$ is a set whenever $a \in \boldsymbol{C}$, and
(2) if b is a nonempty subset of $\boldsymbol{C}$, then b has a **minimal** element with respect to $\boldsymbol{R}$, that is, an element d such that
$$b \cap \{x\colon x \mathrel{\boldsymbol{R}} d\} = \varnothing.$$

By the axioms of Infinity and Choice, condition (2) is equivalent to

(2*) there is no sequence $(a_n\colon n \in \omega)$ of elements of $\boldsymbol{C}$ such that $a_{n+1} \mathrel{\boldsymbol{R}} a_n$ for each n in ω.

In particular, well-founded relations are irreflexive. The Foundation Axiom is just that membership is well-founded on $\mathbf{V}$. If $\boldsymbol{R}$ is well-founded on $\boldsymbol{C}$, then $(\boldsymbol{C}, \boldsymbol{R})$ **admits induction** in the sense that the only subclass $\boldsymbol{D}$ of $\boldsymbol{C}$ for which
$$\boldsymbol{C} \cap \{x\colon x \mathrel{\boldsymbol{R}} a\} \subseteq \boldsymbol{D} \implies a \in \boldsymbol{D}$$
for all a in $\boldsymbol{C}$ is $\boldsymbol{C}$ itself.

Theorem 10. *The relation $\in'$ is well-founded on $\mathbf{V}$.*

Proof. Using the definition (**xi**) for ordered pairs, we have
$$(x_0, \dots, x_n) = \Big\{\big\{\{0\}, \{0, x_0\}\big\}, \dots, \big\{\{n\}, \{n, x_n\}\big\}\Big\},$$
$$\bigcup\bigcup(x_0, \dots, x_n) = \{0, \dots, n, x_0, \dots, x_n\},$$
so $\{y\colon \in' x\} \subseteq \bigcup\bigcup\bigcup x$, a set. Suppose there were a sequence $(x_n\colon n \in \omega)$ such that always $x_{n+1} \in' x_n$. Then always $x_{n+1} \in \bigcup\bigcup\bigcup x_n$. But then, assuming $x_n \in \mathbf{R}(\alpha_n)$, we should have also $\bigcup\bigcup\bigcup x_n \in \mathbf{R}(\alpha_n)$, and then $x_{n+1} \in \mathbf{R}(\alpha_{n+1})$ for some α_{n+1} that was strictly less than α_n. There is no such sequence $(\alpha_n\colon n \in \omega)$ of ordinals. □

For a better generalization of (**xvii**) than (**xix**), we make $\mathbf{V}$ into an $\mathscr{S}$-algebra by defining
$$F^{\mathbf{V}}(x_0, \dots, x_{n-1}) = (\mathscr{G}(x_0) \cup \{x_0\}, \dots, \mathscr{G}(x_{n-1}) \cup \{x_{n-1}\}, F) \qquad \textbf{(xxii)}$$
for all F in $\mathscr{S}$.

Theorem 11. *The operations $F^{\mathbf{V}}$ are injective and have disjoint ranges.*

Proof. That the ranges are disjoint is immediate from the definitions. If
$$F^{\mathbf{V}}(x_0, \dots, x_{n-1}) = F^{\mathbf{V}}(y_0, \dots, y_{n-1}),$$
but $x_k \neq y_k$ for some k, then, since
$$\mathscr{G}(x_k) \cup \{x_k\} = \mathscr{G}(y_k) \cup \{y_k\},$$
we have $x_k \in \mathscr{G}(y_k)$ and $y_k \in \mathscr{G}(x_k)$, that is, $x_k \in' y_k$ and $y_k \in' x_k$. But this contradicts Theorem 10. □

Considering $\mathbf{V}$ as an $\mathscr{S}$-algebra, we can now define
$$\omega_{\mathscr{S}}$$
as the homomorphic image of $\mathrm{Tm}^0(\mathscr{S})$ in $\mathbf{V}$. Then $\omega_{\mathscr{S}}$ is free by Theorems 7 and 11. But I propose to obtain $\omega_{\mathscr{S}}$ alternatively as a certain subclass of a class $\mathbf{ON}_{\mathscr{S}}$, just as ω is obtained from $\mathbf{ON}$.

9. Ordinals, generalized

Let us define

$$\operatorname{pred}_k(x) = \{y \colon y \in_k x\},$$

so that

$$\operatorname{pred}_k((x_0, \dots, x_n)) = \begin{cases} x_k, & \text{if } k < n, \\ \varnothing, & \text{otherwise,} \end{cases}$$

and

$$\mathscr{G}(x) = \bigcup\{\operatorname{pred}_k(x) \colon k \in \omega\}.$$

Let us say that x is **k-transitive** if

$$y \in \operatorname{pred}_k(x) \implies \mathscr{G}(y) \subseteq \operatorname{pred}_k(x).$$

In order to define $\mathbf{ON}_{\mathscr{S}}$, we first define a class $\mathbf{D}_{\mathscr{S}}$, which comprises all x such that

(1) x is an $(n+1)$-tuple having the type of an n-ary element of $\mathscr{S}$ for some n in ω, and then $\operatorname{pred}_k(x) \neq \varnothing$ whenever $k < n$;
(2) each element y of $\mathscr{G}(x)$ is an $(m+1)$-tuple having the type of an m-ary element of $\mathscr{S}$ for some m in ω, and then $\operatorname{pred}_\ell(y) \neq \varnothing$ whenever $\ell < m$;
(3) x is k-transitive for each k in ω;
(4) each element of $\mathscr{G}(x)$ is k-transitive for each k in ω.

We shall presently define $\mathbf{ON}_{\mathscr{S}}$ as a subclass of $\mathbf{D}_{\mathscr{S}}$. Meanwhile, we already have some analogues to properties of $\mathbf{ON}$.

Lemma 2. *The relation $\in'$ is transitive on $\mathbf{D}_{\mathscr{S}}$.*

Proof. If $\mathbf{D}_{\mathscr{S}}$ contains x, y, and z, where $y \in' x$ and $z \in' y$, then $y \in \mathscr{G}(x)$, so $y \in \operatorname{pred}_k(x)$ for some k, and therefore $\mathscr{G}(y) \subseteq \operatorname{pred}_k(x) \subseteq \mathscr{G}(x)$ by (3); but also $z \in \mathscr{G}(y)$, so $z \in \mathscr{G}(x)$, that is, $z \in' x$. □

Also, like $\mathbf{ON}$ itself, $\mathbf{D}_{\mathscr{S}}$ has a kind of transitivity:

Lemma 3. *If $x \in \mathbf{D}_{\mathscr{S}}$, then $\mathscr{G}(x) \subseteq \mathbf{D}_{\mathscr{S}}$.*

Proof. Suppose $x \in \mathbf{D}_{\mathscr{S}}$ and $y \in \mathscr{G}(x)$. Then y satisfies (1), by (2) for x. Also y satisfies (3), by (4) for x. Finally, $y \in \operatorname{pred}_k(x)$ for some k in ω, so by (3) for x we have $\mathscr{G}(y) \subseteq \operatorname{pred}_k(x)$, hence $\mathscr{G}(y) \subseteq \mathscr{G}(x)$. So y satisfies (2) and (4). Therefore $y \in \mathbf{D}_{\mathscr{S}}$. □

When $\mathscr{S} = \{1, \mathsf{s}\}$, then $\mathbf{D}_{\mathscr{S}}$ is not really anything new. Indeed, informally,

$$\mathbf{D}_{\{1,\mathsf{s}\}} = \Big\{(1), (\{(1)\}, \mathsf{s}), \Big(\Big\{(1), (\{(1)\}, \mathsf{s})\Big\}, \mathsf{s}\Big), \dots\Big\}.$$

Using the definition in (**xxii**), we have:

Theorem 12. $(\mathbf{D}_{\{1,\mathsf{s}\}}, 1^{\mathbf{V}}, \mathsf{s}^{\mathbf{V}}) \cong (\mathbf{ON}, \varnothing, \alpha \mapsto \alpha + 1)$.

Proof. Note that $1^{\mathbf{V}}$ is the 1-tuple (1). From $\mathbf{ON}$ to $\mathbf{V}$, there is a function $\boldsymbol{H}$ defined recursively by

$$\boldsymbol{H}(\alpha) = \begin{cases} (1), & \text{if } \alpha = \varnothing; \\ (\boldsymbol{H}[\alpha], \mathsf{s}), & \text{if } \alpha \neq \varnothing. \end{cases}$$

Then

$$\mathscr{G}(\boldsymbol{H}(\alpha)) = \operatorname{pred}_0(\boldsymbol{H}(\alpha)) = \boldsymbol{H}[\alpha]. \qquad (\mathbf{xxiii})$$

Therefore

$$\boldsymbol{H}(\alpha+1) = (\boldsymbol{H}[\alpha] \cup \{\boldsymbol{H}(\alpha)\}, \mathsf{s}) = (\mathscr{G}(\boldsymbol{H}(\alpha)) \cup \{\boldsymbol{H}(\alpha)\}, \mathsf{s}) = \mathsf{s}^{\mathbf{V}}(\boldsymbol{H}(\alpha)),$$

which means $\boldsymbol{H}$ is a homomorphism of $\{1, \mathsf{s}\}$-algebras. If $\alpha < \beta$, then $\boldsymbol{H}(\alpha) \in \boldsymbol{H}[\beta]$, so $\boldsymbol{H}(\alpha) \in' \boldsymbol{H}(\beta)$ by (**xxiii**). Therefore $\boldsymbol{H}$ is order-preserving and, in particular, injective. It remains to show that the range of $\boldsymbol{H}$ is $\mathbf{D}_{\{1,\mathsf{s}\}}$.

In the definition of $\mathbf{D}_{\{1,\mathsf{s}\}}$, conditions (1) and (2) hold for elements of $\boldsymbol{H}[\mathbf{ON}]$ by definition of $\boldsymbol{H}$. Then (3) follows; that is, $\boldsymbol{H}(\beta)$ is always k-transitive. Indeed, it is trivially so, if $\beta = 0$ or $k > 0$; and by (**xxiii**), we have that, if $x \in \operatorname{pred}_0(\boldsymbol{H}(\beta))$, then $x = \boldsymbol{H}(\alpha)$ for some α, so

$$\mathscr{G}(x) = \boldsymbol{H}[\alpha] \subseteq \boldsymbol{H}[\beta] = \operatorname{pred}_0(\boldsymbol{H}(\beta)).$$

Since each element of $\mathscr{G}(\boldsymbol{H}(\beta))$ is some $\boldsymbol{H}(\alpha)$, it is k-transitive too: thus (4). So $\boldsymbol{H}$ maps $\mathbf{ON}$ into $\mathbf{D}_{\{1,\mathsf{s}\}}$.

By induction in $\mathbf{D}_{\{1,\mathsf{s}\}}$ with respect to the well-founded relation $\in'$, we establish $\mathbf{D}_{\{1,\mathsf{s}\}} \subseteq \boldsymbol{H}[\mathbf{ON}]$. Suppose $x \in \mathbf{D}_{\{1,\mathsf{s}\}}$. Then $\mathscr{G}(x) \subseteq \mathbf{D}_{\{1,\mathsf{s}\}}$ by the lemma. As an inductive hypothesis, suppose $\mathscr{G}(x) \subseteq \boldsymbol{H}[\mathbf{ON}]$. If $\boldsymbol{H}(\beta) \in \mathscr{G}(x)$, and $\alpha < \beta$, then, since $\boldsymbol{H}(\alpha) \in \mathscr{G}(\boldsymbol{H}(\beta))$ by (**xxiii**), we have $\boldsymbol{H}(\alpha) \in \mathscr{G}(x)$ by the 0-transitivity of x. Thus $\{\alpha \colon \boldsymbol{H}(\alpha) \in \mathscr{G}(x)\}$ is transitive, so it is an ordinal β, and $\boldsymbol{H}[\beta] = \mathscr{G}(x)$. If $\beta = \varnothing$, then $x = (1) = \boldsymbol{H}(\beta)$ by the second part of condition (1) in the definition of $\mathbf{D}_{\{1,\mathsf{s}\}}$; if $\beta \neq \varnothing$, then $\mathscr{G}(x) \neq \varnothing$, so x can only be $(\mathscr{G}(x), \mathsf{s})$, which is also $\boldsymbol{H}(\beta)$. So $\boldsymbol{H}$ is an isomorphism between $\mathbf{ON}$ and $\mathbf{D}_{\{1,\mathsf{s}\}}$. □

An element x of $\mathbf{D}_{\mathscr{S}}$ can be called a **limit** if some set $\operatorname{pred}_k(x)$ has no maximal element with respect to $\in'$. In general, there may be x in $\mathbf{D}_{\mathscr{S}}$ such that neither x nor any element of $\mathscr{G}(x)$ is a limit, but still x is not in $\omega_{\mathscr{S}}$. Indeed, if $\mathscr{S} = \{a, b, \mathsf{s}\}$, where a and b are nullary, and s is singulary as usual, then $\mathbf{D}_{\{1,\mathsf{s}\}}$ contains $\big(\{(a), (b)\}, \mathsf{s}\big)$, which is not in $\omega_{\mathscr{S}}$. So we let

$$\mathbf{ON}_{\mathscr{S}}$$

denote the class of x in $\mathbf{D}_{\mathscr{S}}$ that meet the additional conditions

(5) $\in'$ directs each set $\operatorname{pred}_k(x)$;
(6) $\in'$ directs each set $\operatorname{pred}_\ell(y)$ for each y in $\mathscr{G}(x)$.

We already know from Theorem 10 that $\in'$ is well-founded on $\mathbf{ON}_{\mathscr{S}}$ and its elements; but this relies on the axioms of Foundation, Infinity, and Choice. To avoid this reliance, we can impose one additional condition on the elements x of $\mathbf{ON}_{\mathscr{S}}$:

(7) $\in'$ is well-founded on $\mathscr{G}(x)$.

Theorem 13. *The relation $\in'$ is a well-founded ordering of $\mathbf{ON}_{\mathscr{S}}$.*

Proof. By Lemma 2, $\in'$ is transitive on $\mathbf{ON}_{\mathscr{S}}$. If $x \in \mathbf{ON}_{\mathscr{S}}$, then, because $\in'$ is well-founded on $\mathscr{G}(x)$ by (7), in particular $x \notin' x$. Therefore $\in'$ orders $\mathbf{ON}_{\mathscr{S}}$. Also $\{y \colon y \in \mathbf{ON}_{\mathscr{S}} \ \&\ y \in' x\}$ is a set, being a subclass of $\mathscr{G}(x)$. Suppose $b \subseteq \mathbf{ON}_{\mathscr{S}}$ and $x \in b$. If x is not a minimal element of b with respect to $\in'$, then $b \cap \{y \colon y \in' x\}$, being a subset of $\mathscr{G}(x)$, has a minimal element; but this minimal element is minimal in b as well, by transitivity of $\in'$. Therefore $\in'$ is well-founded on $\mathbf{ON}_{\mathscr{S}}$. □

Theorem 14. *If $x \in \mathbf{ON}_{\mathscr{S}}$, then $\mathscr{G}(x) \subseteq \mathbf{ON}_{\mathscr{S}}$.*

Proof. Suppose $x \in \mathbf{ON}_{\mathscr{S}}$ and $y \in \mathscr{G}(x)$. By Lemma 3, we know $y \in \mathbf{D}_{\mathscr{S}}$. By (6) for x, we know y satisfies (5); by (3) also, y satisfies (6). Finally, since $\mathscr{G}(y) \subseteq \mathscr{G}(x)$ by (3), we have (7) for y. □

Theorem 15. *$\mathbf{ON}_{\mathscr{S}}$ is an $\mathscr{S}$-subalgebra of $\mathbf{V}$.*

Proof. Suppose $x_k \in \mathbf{ON}_{\mathscr{S}}$ when $k < n$, and let $y = F^{\mathbf{V}}(x_0, \dots, x_{n-1})$. We show that y satisfies the defining conditions of $\mathbf{ON}_{\mathscr{S}}$.

1. First, y satisfies (1) by definition, that is, (**xxii**).
2. Suppose $z \in \mathscr{G}(y)$; equivalently, $z \in \operatorname{pred}_k(y)$ for some k in n. But

$$\operatorname{pred}_k(y) = \mathscr{G}(x_k) \cup \{x_k\}, \tag{\textbf{xxiv}}$$

so in particular either $z = x_k$ or $z \in \mathscr{G}(x_k)$. By (1) and (2) respectively for x_k, we have that y satisfies (2).

3. Also, if $z = x_k$, then

$$\mathscr{G}(z) = \mathscr{G}(x_k) \subseteq \operatorname{pred}_k(y),$$

while if $z \in \mathscr{G}(x_k)$, then $z \in \operatorname{pred}_\ell(x_k)$ for some ℓ, so that

$$\mathscr{G}(z) \subseteq \operatorname{pred}_\ell(x_k) \subseteq \mathscr{G}(x_k) \subseteq \operatorname{pred}_k(y)$$

by (3) for x_k and (**xxiv**). Thus y satisfies (3).

4. If $z = x_k$, then z is ℓ-transitive for each ℓ in ω by (3) for x_k, while if $z \in \mathscr{G}(x_k)$, then z is ℓ-transitive for each ℓ in ω by (4) for x_k. Therefore y satisfies (4), and so $y \in \mathbf{D}_{\mathscr{S}}$.
5. By (**xxiv**), we have that x_k is the greatest element of $\operatorname{pred}_k(y)$ with respect to $\in'$, so this ordering directs $\operatorname{pred}_k(y)$. Thus y satisfies (5).
6. If $z = x_k$, then $\in'$ directs $\operatorname{pred}_\ell(z)$ by (5) for x_k, while if $z \in \mathscr{G}(x_k)$, then $\in'$ directs $\operatorname{pred}_\ell(z)$ by (6) for x_k. Therefore y satisfies (6).
7. We have $\mathscr{G}(y) = \bigcup\{\mathscr{G}(x_k) \cup \{x_k\} \colon k < n\}$. By Theorems 13 and 14, since $\in'$ is well-founded on each $\mathscr{G}(x_k)$, it is also well-founded on each set $\mathscr{G}(x_k) \cup \{x_k\}$ and hence on their union. So y is in $\mathbf{ON}_{\mathscr{S}}$. □

Now $\omega_{\mathscr{S}}$ can be understood as the image of $\mathrm{Tm}^0(\mathscr{S})$ in $\mathbf{ON}_{\mathscr{S}}$. But again, we want to obtain $\omega_{\mathscr{S}}$ independently from $\mathrm{Tm}^0(\mathscr{S})$.

Theorem 16. *$\omega_{\mathscr{S}}$ consists of those x in $\mathbf{ON}_{\mathscr{S}}$ such that neither x nor any element of $\mathscr{G}(x)$ is a limit.*

Proof. By Theorem 13, we can argue by induction. Suppose x_k meets the conditions when $k < n$. Let $z = F^{\mathbf{V}}(x_0, \dots, x_{n-1})$. Each x_k is a maximal element of $\operatorname{pred}_k(z)$, so z is not a limit. Each element of $\mathscr{G}(z)$ is either x_k or an element of $\mathscr{G}(x_k)$ for some k in n, so it is not a limit either. Thus z meets the conditions.

To prove the converse, suppose if possible that $(x_0, \dots, x_{n-1}, F)$ is a counterexample that is minimal with respect to $\in'$. Then each of the x_k has a maximal element, u_k, with respect to $\in'$. But $\in'$ directs x_k, so u_k is the greatest element, hence $x_k \subseteq \mathscr{G}(u_k) \cup \{u_k\}$. Also $\mathscr{G}(u_k) \subseteq x_k$ by the k-transitivity of $(x_0, \dots, x_{n-1}, F)$. Thus

$$(x_0, \dots, x_{n-1}, F) = (\mathscr{G}(u_0) \cup \{u_0\}, \dots, \mathscr{G}(u_{n-1}) \cup \{u_{n-1}\}, F) = F^{\mathbf{V}}(u_0, \dots, u_{n-1}).$$

But also, neither u_k nor any element of $\mathscr{G}(u_k)$ is a limit, so $u_k \in \omega_{\mathscr{S}}$ by minimality of $(x_0, \dots, x_{n-1}, F)$, and therefore $F^{\mathbf{V}}(u_0, \dots, u_{n-1}) \in \omega_{\mathscr{S}}$. □

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