Edited from *The De Morgan Gazette* **8** no. 4 (2016) 27–78,
bit.ly/2hlyHzZ, ISSN 2053–1451

Thales and the Nine-point Conic

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December 23, 2016
Edited July 25, 2017

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Abstract

The nine-point circle is established by Euclidean means; the nine-point conic, Cartesian. Cartesian geometry is developed from Euclidean by means of Thales's Theorem. A theory of proportion is given, and Thales's Theorem proved, on the basis of Book I of Euclid's *Elements,* without the Archimedean assumption of Book V. Euclid's theory of areas is used, although this is obviated by Hilbert's theory of lengths. It is observed how Apollonius relies on Euclid's theory of areas. The historical foundations of the name of Thales's Theorem are considered. Thales is thought to have identified water as a universal substrate; his recognition of mathematical theorems as such represents a similar unification of things.

Preface

I review here the theorem, dating to the early nineteenth century, that a certain collection of nine points, associated with an arbitrary triangle, lie on a circle. Some of the nine points are defined by means of the orthocenter of the triangle. If the orthocenter is replaced by an arbitrary point, not collinear with any two vertices of the triangle, then the same nine points still lie on a conic section. This was recognized in the late nineteenth century, and I review this result too. All of this is an occasion to consider the historical and logical development of geometry, as seen in work of Thales, Euclid, Apollonius, Descartes, Hilbert, and Hartshorne. All of the proofs here are based ultimately, and usually explicitly, on Book I of Euclid's *Elements.* Even Descartes's "analytic" methods are so justified, although Descartes implicitly used the theory of proportion found in Euclid's Book V. In particular, Descartes relied on what is in some countries today called Thales's Theorem. Using only Book I of Euclid, and in particular its theory of areas, I give a proof of Thales's Theorem in the following form:

> If two triangles share a common angle, the bases of the triangles are parallel if and only if the rectangles bounded by alternate sides of the common angle are equal.

This allows an alternative development of a theory of proportion.

An even greater logical simplification is possible. As Hilbert shows, and as I review, using the simpler arguments of Hartshorne, one can actually *prove* Euclid's assumption that a *part* of a bounded planar region is less than the whole region. One still has to assume that a part of a bounded straight line is less than the whole; but one can now conceive of the theory of proportion as based entirely on the notion of length. This accords with Descartes's geometry, where length is the fundamental notion.

For Apollonius, by contrast, as I show with an example, areas are essential. His proofs about conic sections are based on areas. In Cartesian geometry, we assign letters to lengths, known and unknown, so as to be able to do computations without any visualization. This is how we obtain the nine-point conic.

We may however consider what is lost when we do not directly, *visually,* consider areas. Several theorems attributed to Thales seem to be due to a visual appreciation of symmetry. I end with this topic, one that I consider further in [59], an article not yet published when the present essay was published.

Aside from the formatting (I prefer A5 paper, so that two pages can easily be read side by side on a computer screen, or for that matter on an A4 printout)—aside from the formatting, the present version of my essay differs from the version published in the *De Morgan Gazette* as follows.

1. In the Introduction, To the last clause of the 23rd paragraph, I have supplied the missing "not," so that the clause now reads, "doing this does not represent the accomplishment of a preconceived goal."
2. Again, in the Introduction and throughout the essay, reference [59] is now to a published paper, not to the preliminary version on my own webpage.

3. In §3.2, in the paragraph before Theorem 6, to the sentence, "The definition also ensures that (3.5) *implies* (3.6)," I have added a second clause: "let $(x, y) = (c, d)$ in (3.7)."
4. In §3.3, the displayed proportion $(a : b) \mathbin{\&} (b : c) :: a : c$ is no longer numbered.
5. In the same section, I have deleted the second "this" in the garbled sentence, "Descartes shows this implicitly, this in order to solve ancient problems."
6. In §3.4, in the second paragraph, in the clause, "the point now designed by (x, y) is the one that was called $(x - hy/a, y)$ before," I have made the minus sign a plus sign.
7. In §4.2, I have enlarged and expanded Figure 4.8.
8. In the same section, I have changed "to" to "and" in the clause, "If d is the distance between the new origin to the intersection of the x'-axis with the y-axis."
9. Still in §4.2, the proportion (4.8) formerly had the same internal label as (4.6), which meant that the ensuing *proof* of (4.6) was wrongly justified by a reference to itself, rather than to (4.8).

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1 Introduction

According to Herodotus of Halicarnassus, a war was ended by a solar eclipse, and Thales of Miletus had predicted the year [41, I.74]. The war was between the Lydians and the Medes in Asia Minor; the year was the sixth of the war. The eclipse is thought to be the one that must have occurred on May 28 of the Julian calendar, in the year 585 before the Common Era [38, p. 15, n. 3].

The birth of Thales is sometimes assigned to the year 624 B.C.E. This is done in the "Thales" article on *Wikipedia* [74]; but it was also done in ancient times (according to the reckoning of years by Olympiads). The sole reason for this assignment seems to be the assumption that Thales must have been forty when he predicted the eclipse [45, p. 76, n. 1].

The nine-point circle was found early in the nineteenth century of the Common Era. The 9-point conic, a generalization, was found later in the century. The existence of the curves is proved with mathematics that can be traced to Thales and that is learned today in high school.

In the Euclidean plane, every triangle determines a few points individually: the incenter, circumcenter, orthocenter, and centroid. In addition to the vertices themselves, the triangle also determines various triples of points, such as those in Figure 1.1: the feet • of the altitudes, the midpoints ⧫ of the sides, and the midpoints ▲ of the straight lines running from the orthocenter to the three vertices. It turns out that

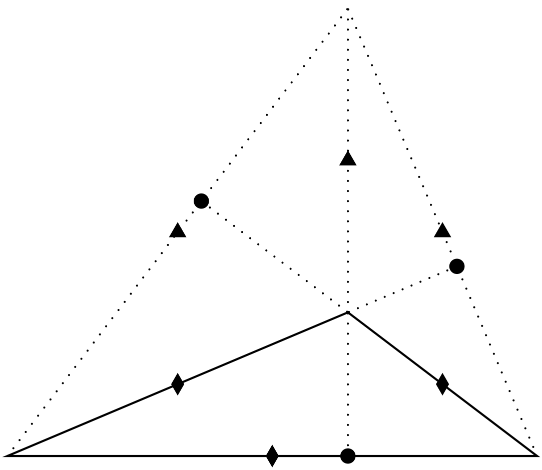


Figure 1.1: The nine points

these nine points lie on a circle, called the **nine-point circle** or **Euler circle** of the triangle. The discovery of this circle seems to have been published first by Brianchon and Poncelet in 1820–1 [13], then by Feuerbach in 1822 [30, 31].

Precise references for the discovery of the nine-point circle are given in Boyer's *History of Mathematics* [12, pp. 573–4] and Kline's *Mathematical Thought from Ancient to Modern Times* [46, p. 837]. However, as with the birth of Thales, precision is different from accuracy. Kline attributes the first publication on the nine-point circle to "Gergonne and Poncelet." In consulting his notes, Kline may have confused an author with the publisher, who was himself a mathematician. Boyer mentions that the joint paper of Brianchon and Poncelet was published in Gergonne's *Annales.* The confusion here is a reminder that even seemingly authoritative sources may be in error.

The mathematician is supposed to be a skeptic, accepting nothing before knowing its proof. In practice this does not always happen, even in mathematics. But the ideal should be maintained, even in subjects other than mathematics, like history.

There is however another side to this ideal. We shall refer often in this essay to Euclid's *Elements* [25, 26, 27, 28, 29], and especially to the first of its thirteen books. Euclid is accused of accepting things without proof. Two sections of Kline's history are called "The Merits and Defects of the *Elements*" [46, ch. 4, §10, p. 86] and "The Defects in Euclid" (ch. 42, §1, p. 1005). One of the supposed "defects" is,

> he uses dozens of assumptions that he never states and undoubtedly did not recognize.

We all do this, all the time; and it is not a defect. We cannot state everything that we assume; even the possibility of stating things is based on assumptions about language itself. We try to state some of our assumptions, in order to question them, as when we encounter a problem with the ordinary course of life. By the account of R. G. Collingwood [18], the attempt to work out our fundamental assumptions is metaphysics.

Herodotus says the Greeks learned geometry from the Egyptians, who needed it in order to measure how much land they lost to the flooding of the Nile. Herodotus's word *γεωμετρίη* means also surveying, or "the art of measuring land" [41, II.109]. According to David Fowler in *The Mathematics of Plato's Academy* [32, §7.1(d), pp. 231–4; §8.1, pp. 279–81], the Egyptians defined the area of a quadrilateral field as the product of the averages of the pairs of opposite sides.

The Egyptian rule is not strictly accurate. Book I of the *Elements* corrects the error. The climax of the book is the demonstration, in Proposition 45, that every straight-sided field is *exactly* equal to a certain parallelogram with a given side.

Euclid's demonstrations take place in a world where, as Archimedes postulates [6, p. 36],

> among unequal [magnitudes], the greater exceeds the smaller by such [a difference] that is capable, added itself to itself, of exceeding everything set forth . . .

This is the world in which the theory of proportion in Book V of the *Elements* is valid. A theory of proportion is needed for the Cartesian geometry whereby the nine-point conic is established.

One can develop a theory of proportion that does not require the Archimedean assumption. Is it a defect that Euclid does not do this? We shall do it, trying to put into the theory only enough to make **Thales's Theorem** true. This is the theorem that, if a straight line cuts two sides of a triangle, it cuts them proportionally if and only if it is parallel to the third side. I shall call this Thales's Theorem for convenience, and because it is so called in some countries today [53]. There is also some historical basis for the name: we shall investigate how much.

The present essay is based in part on notes prepared originally for one of several twenty-minute talks at the Thales Buluşması (Thales Meeting), held in the Roman theater in the ruins of Thales's home town, September 24, 2016. The event was arranged by the Tourism Research Society (Turizm Araştırmaları Derneği, TURAD) and the office of the mayor of Didim. Part of the Aydın province of Turkey, the district of Didim encompasses the ancient Ionian cities of Priene and Miletus, along with the temple of Didyma. This temple was linked to Miletus, and Herodotus refers to the temple under the name of the family of priests, the Branchidae.

My essay is based also on notes from a course on Pappus's

Theorem and projective geometry given at my home university, Mimar Sinan, in Istanbul, and at the Nesin Mathematics Village, near the Ionian city of Ephesus.

To seal the Peace of the Eclipse, the Lydian King Alyattes gave his daughter Aryenis to Astyages, son of the Median King Cyaxares [41, I.74]. It is not clear whether Aryenis was the mother of Astyages's daughter Mandane, whom Astyages married to the Persian Cambyses, and whose son Astyages tried to murder, because of the Magi's unfavorable interpretation of certain dreams [41, I.107–8]. That son was Cyrus, who survived and grew up to conquer his grandfather. Again Herodotus is not clear that this was the reason for the quarrel with Cyrus by Croesus [41, I.75], who was son and successor of Alyattes and thus brother of Astyages's consort Aryenis. But Croesus was advised by the oracles at Delphi and Amphiaraus that, if he attacked Persia, a great empire would be destroyed, and that he should make friends with the mightiest of the Greeks [41, I.52–3]. Perhaps it was in obedience to this oracle that Croesus sought the alliance with Miletus mentioned by Diogenes Laertius, who reports that Thales frustrated the plan, and "this proved the salvation of the city when Cyrus obtained the victory" [24, I.25]. Nonetheless, Herodotus reports a general Greek belief—which he does not accept—that Thales helped Croesus's army march to Persia by diverting the River Halys (today's Kızılırmak) around them [41, I.75]. But Croesus was defeated, and thus his own great empire was destroyed.

When the victorious Cyrus returned east from the Lydian capital of Sardis, he left behind a Persian called Tabulus to rule, but a Lydian called Pactyes to be treasurer [41, I.153]. Pactyes mounted a rebellion, but it failed, and he sought asylum in the Aeolian city of Cyme. The Cymaeans were told by the oracle at Didyma to give him up [41, I.157–9]. In disbelief, a Cymaean called Aristodicus began driving away the birds that nested around the temple.

> But while he so did, a voice (they say) came out of the inner shrine calling to Aristodicus, and saying, "Thou wickedest of men, wherefore darest thou do this? wilt thou rob my temple of those that take refuge with me?" Then Aristodicus had his answer ready: "O King," said he, "wilt thou thus save thine own suppliants, yet bid the men of Cyme deliver up theirs?" But the god made answer, "Yea, I do bid them, that ye may the sooner perish for your impiety, and never again come to inquire of my oracle concerning the giving up of them that seek refuge with you."

As the temple survives today, so does the sense of the injunction of the oracle, in a Turkish saying [50, p. 108]:

İsteyenin bir yüzü kara, vermeyenin iki yüzü.
Who asks has a black face, but who does not give has two.

From his studies of art, history, and philosophy, Collingwood concluded that "all history is the history of thought" [17, p. 110]. As a form of thought, mathematics has a history. Unfortunately this is forgotten in some *Wikipedia* articles, where definitions and results may be laid out as if they have been understood since the beginning of time. We can all rectify this situation, if we will, by contributing to the encyclopedia. On May 13, 2013, to the article "Pappus's Hexagon Theorem" [73], I added a section called "Origins," giving Pappus's own proof. The theorem can be seen as lying behind Cartesian geometry.

In his *Geometry* of 1637, Descartes takes inspiration from Pappus, whom he quotes in Latin, presumably from Commandinus's edition of 1588 [21, p. 6, n. 9]; the 1886 French edition of the *Geometry* has a footnote [23, p. 7], seemingly in Descartes's voice, although other footnotes are obviously from an editor: "I cite rather the Latin version than the Greek text, so that everybody will understand me more easily."

The admirable *Princeton Companion to Mathematics* [36, pp. 47–76] says a lot about where mathematics is now in its history. In one chapter, editor Timothy Gowers discusses "The General Goals of Mathematical Research." He divides these goals among nine sections: (1) Solving Equations, (2) Classifying, (3) Generalizing, (4) Discovering Patterns, (5) Explaining Apparent Coincidences, (6) Counting and Measuring, (7) Determining Whether Different Mathematical Properties Are Compatible, (8) Working with Arguments That Are Not Fully Rigorous, (9) Finding Explicit Proofs and Algorithms. These are some goals of research *today.* There is a tenth section of the chapter, but its title is general: "What Do You Find in a Mathematical Paper?" As Gowers says, the kind of paper that he means is one written on a pattern established in the twentieth century.

The *Princeton Companion* is expressly not an encyclopedia. One must not expect every species of mathematics to meet one or more of Gowers's enumerated goals. Geometrical theorems like that of the nine-point circle do not really seem to meet the goals. They are old-fashioned. The nine-point circle itself is not the *explanation* of a coincidence; it *is* the coincidence that a certain set of nine points all happen to lie at the same distance from a tenth point. A proof of this coincidence may be all the explanation there is. The proof might be described as *explicit,* in the sense of showing how that tenth point can be found; but in this case, there can be no other kind of proof. From any *three* points of a circle, the center can be found.

One can *generalize* the nine-point circle, obtaining the **nine-point conic,** which is determined by any four points in the Euclidean plane, provided no three are collinear. As in Fig-

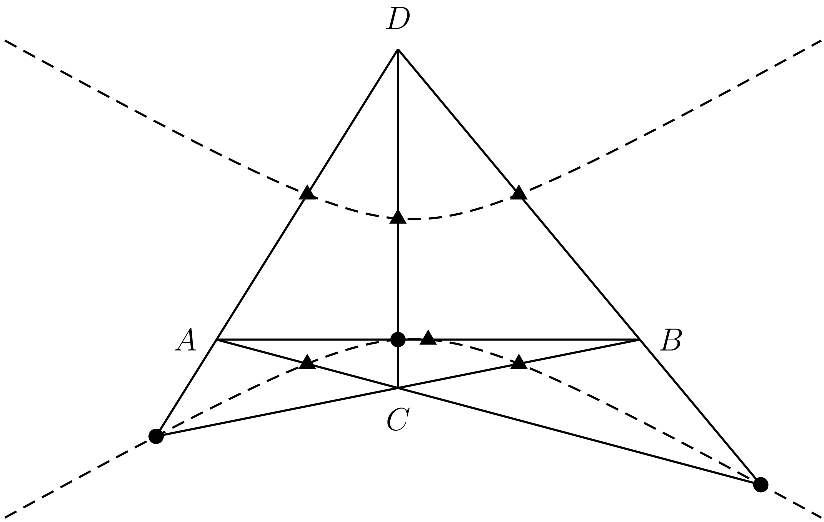


Figure 1.2: A nine-point hyperbola

ures 1.2 and 1.3, where the four points are A, B, C, and D, the conic passes through the midpoints ▲ of the straight lines bounded by the six pairs formed out of the four points, and it also passes through the intersection points • of the pairs of straight lines that together pass through all four points.

The discovery of the nine-point circle itself would seem not to be the accomplishment of any particular goal, beyond simple enjoyment. Indeed, one might make an alternative list of goals of mathematical research: (1) personal satisfaction, (2) satisfying collaboration with friends and colleagues, (3) *impressing* those friends and colleagues, (4) serving science and industry, (5) winning a grant, (6) earning a promotion, (7) finding a job in the first place. These may be goals in any academic pursuit. But none of them can come to be recognized as goals unless the first one or two have actually been

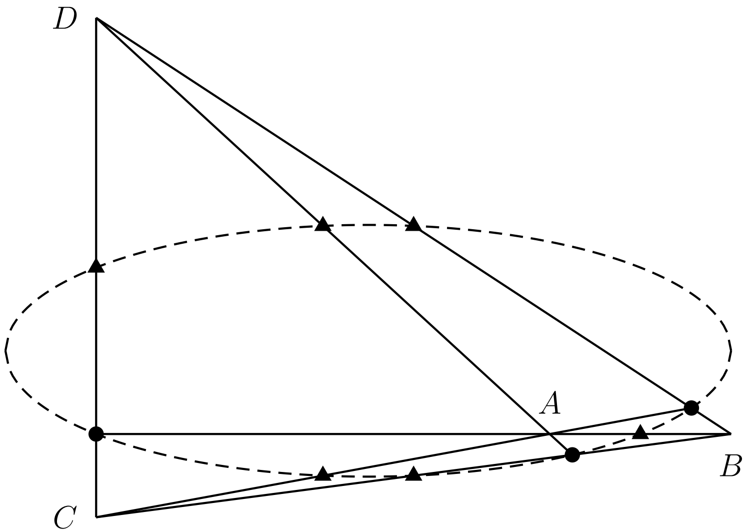


Figure 1.3: A nine-point ellipse

achieved. First you have to find out by chance that something is worth doing for its own sake, before you can put it to some other use.

The present essay is an illustration of the general point. A fellow alumnus of St John's College [56] expressed to me, along with other alumni and alumnae, the pleasure of learning the nine-point circle. Not having done so before, I learned it too, and also the nine-point conic. I wrote out proofs for my own satisfaction. The proof of the circle uses the theorem that a straight line bisecting two sides of a triangle is parallel to the third. This is a special case of Thales's Theorem. The attribution to Thales actually obscures some interesting mathematics; so I started writing about this, using the notes I mentioned. Thales's Theorem allows a theoretical justification of the multiplication that Descartes defines in order to introduce algebra to geometry. The nine-point conic is an excellent illustration of the power of Descartes's geometry; Thales's Theorem can make the geometry rigorous. However, David Hilbert takes another approach.

The first-year students in my department in Istanbul read Euclid for themselves in their first semester. They learn implicitly about Descartes in their second semester, in lectures on analytic geometry. I have read Pappus with older students too, as I mentioned. All of the courses have been an opportunity and an impetus to clarify the transition from Euclid to Descartes. The nine-point circle and conic provide an occasion to bring the ideas together; but doing this does not represent the accomplishment of a preconceived goal.

The high-school geometry course that I took in 1980–1 in Washington could have included the nine-point circle. The course was based on the text by Weeks and Adkins, who taught proofs in the two-column, "statement–reason" format [72, pp. 47–8]. A 1982 edition of the text is apparently still available, albeit from a small publisher. The persistence of the text is satisfying, though I was not satisfied by the book as a student. The tedious style had me wondering why we did not just read Euclid's *Elements.* I did this on my own, and I did it three years later as a student at St John's College.

After experiencing Euclid, both as student and teacher, I have gone back to detect foundational weaknesses in the Weeks–Adkins text. One of them is the confusion of equality with sameness. I discuss this in detail elsewhere [59]. The distinction between equality and sameness is important, in geometry if not in algebra. In geometry, equal line segments have the same *length*; but the line segments are still not in any sense the same segment. An isosceles triangle has *two* equal sides. But in Euclid, two ratios are never equal, although they may be the *same.* This helps clarify what can be meant by a ratio in the first place. In modern terms, a ratio is an equivalence class; so any definition of ratio must respect this.

Another weakness of Weeks and Adkins has been shared by most modern books, as far as I know, since Descartes's *Geometry.* The weakness lies in the treatment of Thales's Theorem. The *meaning* of the theorem, the *truth* of the theorem, and the *use* of the theorem to justify algebra—none of these are obvious. In *The Foundations of Geometry,* Hilbert recognizes the need for work here, and he does the work [43, pp. 24–33]. Weeks and Adkins recognize the need too, but only to the extent of proving Thales's Theorem for commensurable divisions, then mentioning that there is an incommensurable case. They do this in a section labelled [B] for difficulty and omissibility [72, pp. v, 212–4]. They say,

> Ideas involved in proofs of theorems for incommensurable segments are too difficult for this stage of our mathematics.

Such condescension is annoying; but in any case, we shall establish Thales's Theorem as Hilbert does, in the sense of not using the Archimedean assumption that underlies Euclid's notion of commensurability.

First we shall use the theory of areas as developed in Book I of the *Elements.* This relies on Common Notion 5: the whole is greater than the part, not only when the whole is a bounded straight line, but also when it is a bounded region of the plane. When point B lies between A and C on a straight line, then AC is greater than AB; and when two rectangles share a base, the rectangle with the greater height is the greater. Hilbert shows how to *prove* the latter assertion from the former. He does this by developing an "algebra of segments."

We shall review this "algebra of segments"; but first we shall focus on the "algebra of areas." It is not really algebra, in the sense of relying on strings of juxtaposed symbols; it relies on an understanding of pictures. In *The Shaping of Deduction in Greek Mathematics* [49, p. 23], Reviel Netz examines how the diagram of an ancient Greek mathematical proposition is not always recoverable from the text alone. The diagram is an integral part of the proposition, even its "metonym": it stands for the entire proposition the way the enunciation of the proposition stands today [49, p. 38]. In the summary of Euclid called the *Bones* [28], both the enunciations and the diagrams of the propositions of the *Elements* are given. Unlike, say, Homer's *Iliad,* Euclid's *Elements* is not a work that one understands through hearing it recited by a blind poet.

The same will be true for the present essay, if only because I have not wanted to take the trouble to write out everything in words. The needs of blind readers should be respected; but this might be done best with tactile diagrams, which could benefit sighted readers as well. What Collingwood writes in *The Principles of Art* [15, pp. 146–7] about painting applies also to mathematics:

> The forgotten truth about painting which was rediscovered by what may be called the Cézanne–Berenson approach to it was that the spectator's experience on looking at a picture is not a specifically visual experience at all . . . It does not belong to sight alone, it belongs also (and on some occasions even more essentially) to touch . . . [Berenson] is thinking, or thinking in the main, of distance and space and mass: not of touch sensations, but of motor sensations such as we experience by using our muscles and moving our limbs. But these are not actual motor sensations, they are imaginary motor sensations. In order to enjoy them when looking at a Masaccio we need not walk straight through the picture, or even stride about the gallery; what we are doing is to imagine ourselves as moving in these ways.

To imagine these movements, I would add, one needs some experience of making them. Doing mathematics requires some kind of imaginative understanding of what the mathematics is about. This understanding may be engendered by drawings of triangles and circles; but then it might just as well, if not better, be engendered by triangles and circles that can be held and manipulated.

Descartes develops a kind of mathematics that might seem to require a minimum of imagination. If you have no idea of the points that you are looking for, you can just call them (x, y) and proceed. Pappus describes the general method [52, 634, p. 82]:

> Now, analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method 'analysis', as if to say *anapalin lysis* (reduction backward).

The derivation of the nine-point conic will be by Cartesian analysis.

In Rule Four of the posthumously published *Rules for the Direction of the Mind* [22, 373, p. 17], Descartes writes of a method that is

> so useful . . . that without it the pursuit of learning would, I think, be more harmful than profitable. Hence I can readily believe that the great minds of the past were to some extent aware of it, guided to it even by nature alone . . . This is our experience in the simplest of sciences, arithmetic and geometry: we are well aware that the geometers of antiquity

> employed a sort of analysis which they went on to apply to the solution of every problem, though they begrudged revealing it to posterity. At the present time a sort of arithmetic called "algebra" is flourishing, and this is achieving for numbers what the ancients did for figures . . . But if one attends closely to my meaning, one will readily see that ordinary mathematics is far from my mind here, that it is quite another discipline I am expounding, and that these illustrations are more its outer garments than its inner parts . . . Indeed, one can even see some traces of this true mathematics, I think, in Pappus and Diophantus who, though not of that earliest antiquity, lived many centuries before our time. But I have come to think that these writers themselves, with a kind of pernicious cunning, later suppressed this mathematics as, notoriously, many inventors are known to have done where their own discoveries are concerned . . . In the present age some very gifted men have tried to revive this method, for the method seems to me to be none other than the art which goes by the outlandish name of "algebra"—or at least it would be if algebra were divested of the multiplicity of numbers and imprehensible figures which overwhelm it and instead possessed that abundance of clarity and simplicity which I believe true mathematics ought to have.

Possibly Apollonius is, for Descartes, of "earliest antiquity"; but in any case he precedes Pappus and Diophantus by centuries. He may have a secret weapon in coming up with his propositions about conic sections; but *pace* Descartes, I do not think it is Cartesian analysis. One cannot have a method for finding things, unless one already has—or *somebody* has—a good idea of what one wants to find in the first place. As if opening boxes to see what is inside, Apollonius slices cones. This is why we can now write down equations and call them conic sections.

Today we think of conic sections as having *axes*: one for the parabola, and two each for the ellipse and hyperbola. The notion comes from Apollonius; but for him, an axis is just a special case of a *diameter.* A diameter of a conic section bisects certain chords of the section that are all parallel to one another. In Book I of the *Conics* [2, 3], Apollonius shows that *every* straight line through the center of an ellipse or hyperbola is a diameter in this sense; and every straight line parallel to the axis is a diameter of a parabola. One can give a proof by formal change of coordinates; but the proof of Apollonius involves areas, and it does not seem likely that this is his translation of the former proof. In any case, for comparison, we shall set down both proofs, for the parabola at least.

In introducing the nine-point circle near the beginning of his *Introduction to Geometry* [19, p. 18], Coxeter quotes Pedoe on the same subject from *Circles* [54, p. 1]:

> This [nine-point] circle is the first really exciting one to appear in any course on elementary geometry.

I am not sure whether to read this as encouragement to learn the nine-point circle, or as disparagement of the education that the student might have had to endure, in order to be able to learn the circle. In any case, all Euclidean circles are the same in isolation. In Book III of the *Elements* are the theorems that every angle in a semicircle is right (III.31) and that the parts of intersecting chords of a circle bound equal rectangles (III.35). The former theorem is elsewhere attributed to Thales. Do not both theorems count as exciting? The nine-point circle is exciting for combining the triangles of Book I with the circles of Book III.

2 The Nine-point Circle

2.1 Centers of a triangle

The angle bisectors of a triangle, and the perpendicular bisectors of the sides of the triangle, meet respectively at single points, called today the **incenter** and **circumcenter** of the triangle [72, pp. 187–8]. This is an implicit consequence of *Elements* IV.4 and 5, where circles are respectively inscribed in, and circumscribed about, a triangle; the centers of these circles are the points just mentioned.

The concurrence of the altitudes of a triangle is used in the *Book of Lemmas.* The book is attributed ultimately to Archimedes, and Heiberg includes a Latin rendition in his own edition of Archimedes [4, p. 427]. However, the book comes down to us originally in Arabic. Its Proposition 4 [5, p. 304–5] concerns a semicircle with two semicircles removed, as in Figure 2.1; the text quotes Archimedes as having called the shape an **arbelos,** or shoemaker's knife. In the only such instance that I know of, the big Liddell–Scott lexicon [47, p. 235] illustrates the ἄρβηλος entry with a picture of the shape. The term and the shape came to my attention, before the high-school course that I mentioned, in a "Mathematical Games" column of Martin Gardner [34, ch. 10, p. 149]. The second theorem that Gardner mentions is Proposition 4 of the *Book of Lemmas*: the arbelos $ABCD$ is equal to the circle whose

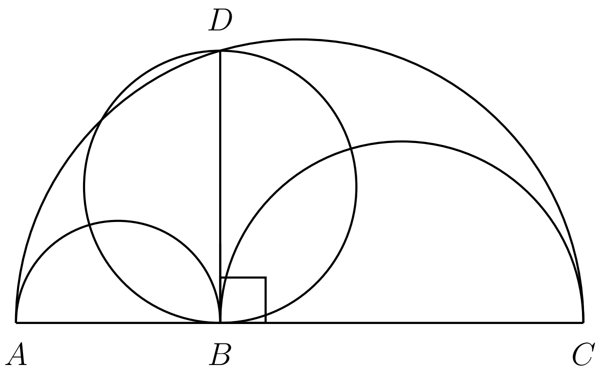


Figure 2.1: The arbelos

diameter is BD. Proposition 5 is that the circles inscribed in the two parts into which the arbelos is cut by BD are equal; the proof appeals to the theorem that the altitudes of a triangle concur at a point.

Today that point is the **orthocenter** of the triangle, and its existence follows from that of the circumcenter. In Figure 2.2, where the sides of triangle GHK are parallel to the respective sides of ABC, the altitudes AD, BE, and CF of ABC *are* the perpendicular bisectors of the sides of GHK. Since these perpendicular bisectors concur at L, so do the altitudes of ABC.

In Propositions 13 and 14 of *On the Equilibrium of Planes* [5, p. 198–201], Archimedes shows that the center of gravity of a triangle must lie on a median, and therefore must lie at the intersection of two medians. Implicitly then, the three medians must concur at a point, which we call the **centroid,** though Archimedes's language suggests that this is known independently. The existence of the centroid follows from the special case of Thales's Theorem that we shall want anyway,

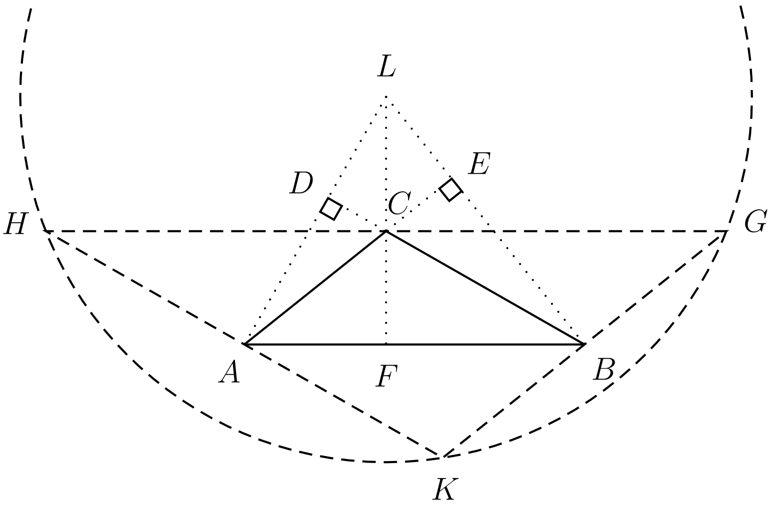


Figure 2.2: Concurrence of altitudes

to prove the nine-point circle.

Theorem 1. *A straight line bisecting a side of a triangle bisects a second side if and only if the cutting line is parallel to the third side.*

Proof. In triangle ABC in Figure 2.3, let D be the midpoint of side AB (*Elements* I.10), and let DE and DF be drawn parallel to the other sides (*Elements* I.31). Then triangles ADE and DBF have equal sides between equal angles (*Elements* I.29), so the triangles are congruent (*Elements* I.26). In particular, $AE = DF$. But in the parallelogram $CEDF$, $DF = EC$ (*Elements* I.34). Thus $AE = EC$. Therefore DE is the bisector of two sides of ABC. Conversely, since there is only one such bisector, it must be the parallel to the third side. □

For completeness, we establish the centroid. In Figure 2.4,

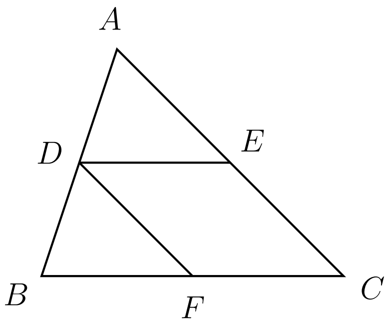


Figure 2.3: Bisector of two sides of a triangle

if AD and CF are medians of triangle ABC, and BH is drawn parallel to AD, then, by passing though G, BE bisects CH by the theorem just proved, and the angles FBH and FAG are equal by *Elements* I.27. Since the vertical angles BFH and AFG are equal (*Elements* I.15), and $BF = AF$, the triangles BFH and AFG must be congruent (*Elements* I.26), and in particular $BH = AG$. Since these straight lines are also parallel, so are AH and BE (*Elements* I.33). Again by Theorem 1, BE bisects AC.

2.2 The angle in the semicircle

We shall want to know that the angle in a semicircle is right. This is Proposition III.31 of Euclid; but an attribution to Thales is passed along by Diogenes Laertius, the biographer of philosophers [24, I.24–5]:

> Pamphila says that, having learnt geometry from the Egyptians, he [Thales] was the first to inscribe in a circle a right-angled triangle, whereupon he sacrificed an ox. Others say it was Pythagoras, among them being Apollodorus the cal-

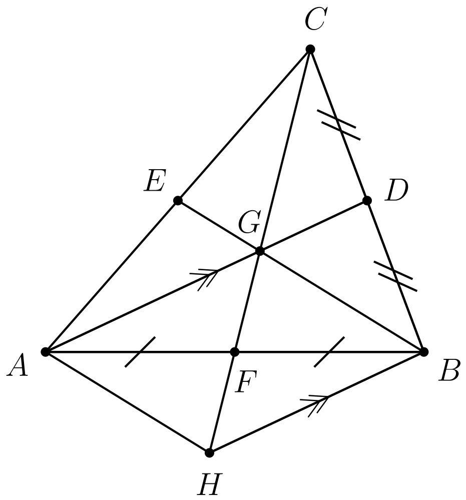


Figure 2.4: Concurrence of medians

culator.

The theorem is easily proved by means of *Elements* I.32: the angles of a triangle are together equal to two right angles. Let the side BC of triangle ABC in Figure 2.5 be bisected at D, and let AD be drawn. If A lies on the circle with diameter BC, then the triangles ABD and ACD are isosceles, so their base angles are equal, by *Elements* I.5: this is also attributed to Thales, as we shall discuss later. Meanwhile, the four base angles being together equal to two right angles, the two of them that make up angle BAC must together be right. The converse follows from *Elements* I.21, which has no other obvious use: an angle like BEC inscribed in BAC is greater than BAC; circumscribed, like BFC, less.

Thales may have established *Elements* III.31; but it is hard to attribute to him the proof based on I.32 when Proclus, in

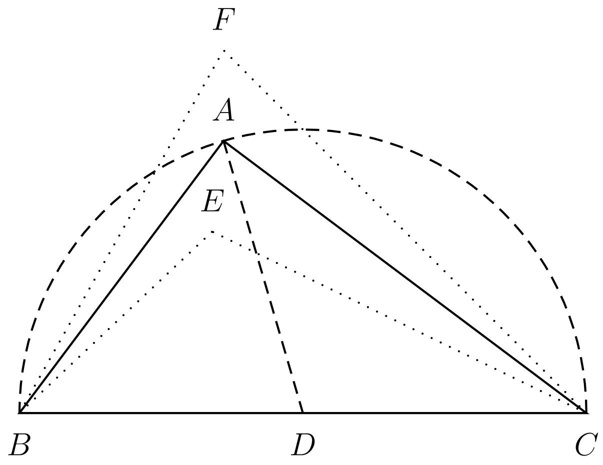
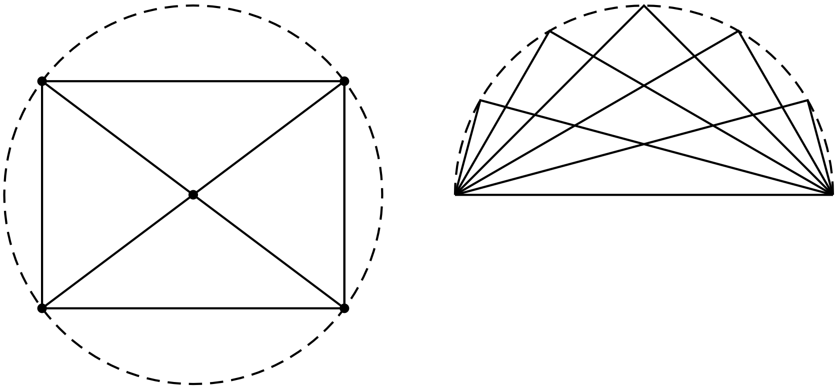


Figure 2.5: The angle in the semicircle

his *Commentary on the First Book of Euclid's* Elements, attributes this result to the Pythagoreans [62, 379.2], who came after Thales. Proclus cites the now-lost history of geometry by Eudemus, who was apparently a student of Aristotle.

In his *History of Greek Mathematics,* Heath [39, pp. 136–7] proposes an elaborate argument for III.31 not using the general theorem about the sum of the angles of a triangle. If a rectangle exists, one can prove that the diagonals intersect at a point equidistant from the four vertices, so that they lie on a circle whose center is that intersection point, as in Figure 2.6a. In particular then, a right angle is inscribed in a semicircle.

It seems to me one might just as well draw two diameters of a circle and observe that their endpoints, by symmetry, are the vertices of an equiangular quadrilateral. This quadrilateral must then be a rectangle: that is, the four equal angles of the quadrilateral must together make a circle. This can be inferred from the observation that equiangular quadrilaterals



(a) A rectangle in a circle (b) A locus of right angles

Figure 2.6: Right angles and circles

can be used to tile floors.

Should the existence of rectangles be counted thus, not as a theorem, but as an observation, if not a postulate? In *A Short History of Greek Mathematics,* which is earlier than Heath's history, Gow [35, p. 144] passes along a couple of ideas of one Dr Allman about inductive reasoning. From floor tiles, one may induce, as above, that the angle in a semicircle is right. By observation, one may find that the locus of apices of right triangles whose bases (the hypotenuses) are all the same given segment is a semicircle, as in Figure 2.6b.

2.3 The nine-point circle

Theorem 2 (Nine-point Circle)**.** *In any triangle, the midpoints of the sides, the feet of the altitudes, and the midpoints of the straight lines drawn from the orthocenter to the vertices*

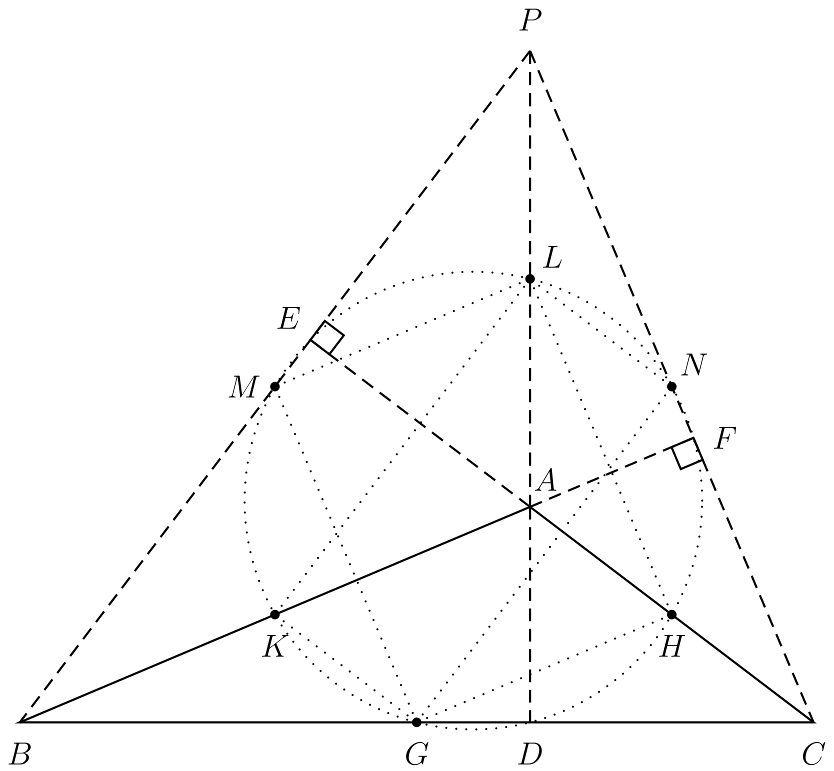


Figure 2.7: The nine-point circle

lie on a single circle.

Proof. Suppose the triangle is ABC in Figure 2.7. Let the altitudes BE and CF be dropped (*Elements* I.12); their intersection point is P. Let AP be drawn and extended as needed, so as to meet BC at D. We already know that P is the orthocenter of ABC, so AD must be at right angles to BC; in fact we shall prove this independently.

Bisect BC, CA, and AB at G, H, and K respectively, and bisect PA, PB, and PC at L, M, and N respectively. By

Theorem 1, GK and LN are parallel to AC; so they are parallel to one another, by *Elements* I.30. Similarly KL and NG are parallel to one another, being parallel to BE. Then the quadrilateral $GKLN$ is a parallelogram; it is a rectangle, by *Elements* I.29, since AC and BE are at right angles to one another. Likewise $GHLM$ is a rectangle. The two rectangles have common diagonal GL, and so the circle with diameter GL also passes through the remaining vertices of the rectangles, by the converse of *Elements* III.31, discussed above. Similarly, the respective diagonals KN and HM of the two rectangles must be diameters of the circle, and so $KHNM$ is a rectangle; this yields that AD is at right angles to BC. The circle must pass through E, since angle MEH is right, and MH is a diameter; likewise the circle passes through F and G. □

The Nine-point Circle Theorem is symmetric in the vertices and orthocenter of the triangle. These four points have the property that the straight line through any two of them passes through neither of the other two; moreorever, the line is at right angles to the straight line through the remaining two vertices. In other words, the points are the vertices of a **complete quadrangle,** and each of its three pairs of opposite sides are at right angles to one another. The intersection of a pair of opposite sides being called a **diagonal point,** a single circle passes through the three of these and the midpoints of the six sides.

We proceed to the case of a complete quadrangle whose opposite sides need not be at right angles. There will be a single *conic section* passing through the three diagonal points and the midpoints of the six sides. The proof will use Cartesian geometry, as founded on Thales's Theorem.

3 The Nine-point Conic

3.1 Thales's Theorem

A rudimentary form of Thales's Theorem is mentioned in the fanciful dialogue by Plutarch called *Dinner of the Seven Wise Men* [61, §2, pp. 351–3]. Here the character of Neiloxenus says *of* and *to* Thales,

> he does not try to avoid, as the rest of you do, being a friend of kings and being called such. In your case, for instance, the king [of Egypt] finds much to admire in you, and in particular he was immensely pleased with your method of measuring the pyramid, because, without making any ado or asking for any instrument, you simply set your walking-stick upright at the edge of the shadow which the pyramid cast, and, two triangles being formed by the intercepting of the sun's rays, **you demonstrated that the height of the pyramid bore the same relation to the length of the stick as the one shadow to the other.**

The word translated as "relation" here, in the sentence that I have emboldened, is *λόγος*. This is usually translated as "ratio" in mathematics. The whole sentence is

ἔδειξας
ὃν ἡ σκιὰ πρὸς τὴν σκιὰν λόγον εἶχε
τὴν πυραμίδα πρὸς τὴν βακτηρίαν ἔχουσαν [60];

stretching the bounds of English style, one might render this literally as

You showed,
what ratio the shadow had to the shadow,
the pyramid [as] having to the staff.

It would be clearer to reverse the order of the last two lines. If the pyramid's height and shadow have lengths P and L, the shadow and height being measured from the center of the base, while the lengths of Thales's height and shadow are p and ℓ, then we may write the claim as

$$P : p : : L : \ell. \tag{3.1}$$

If not theoretically, this must mean practically

$$P \cdot \ell = L \cdot p, \tag{3.2}$$

that is, the rectangle of dimensions P and ℓ is equal to the rectangle of dimensions L and p. Then what is being attributed to Thales is something like the rectangular case of *Elements* I.43:

> In any parallelogram the complements of the parallelograms about the diameter are equal to one another.

Thus in the parallelogram $ABCD$ in Figure 3.1, where AGC is a straight line, the parallelograms BG and GD are equal. I pause to note that Euclid's two-letter notation for parallelograms here is not at all ambiguous in Euclid's Greek, where a diagonal of a parallelogram may be ἡ AB, while the parallelogram itself is τὸ AB; the articles ἡ and τό are feminine and neuter respectively. Euclid's word παραπλήρωμα for complement is neuter, like παραλληλόγραμμον "parallelogram" itself, while γραμμή "line" is feminine. This observation, made in [57], is based on similar observations by Reviel Netz [49].

In Figure 3.1, the parallelograms BG and GD are equal because they are the result of subtracting equal triangles from

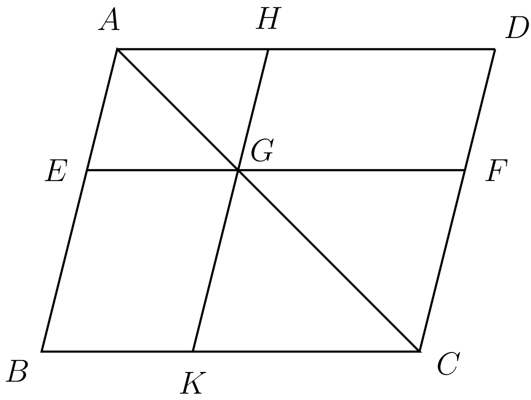


Figure 3.1: *Elements* I.43

equal triangles; the equalities of the triangles ($AEG = GHA$ and so forth) are by *Elements* I.34. The theorem that Plutarch attributes to Thales may now simply be the following.

Theorem 3. *In equiangular right triangles, the rectangles bounded by alternate legs are equal.*

Proof. Let the equiangular right triangles be ABC and AEF in Figure 3.2. We shall show

$$AF \cdot AB = AE \cdot AC. \tag{3.3}$$

It will be enough to show $AF \cdot EB = AE \cdot FC$, since we can then add the common rectangle $AE \cdot AF$ to either side. Let rectangles AG and AH be completed, as by the method whereby Euclid constructs a square in *Elements* I.46; this gives us also the rectangle $AELF$. Let BH and CG be extended to meet at D; by *Elements* I.43, it will be enough to show that the diagonal AL, extended, passes through D, or in other words, L lies on the diagonal AD of the rectangle $ABDC$. But

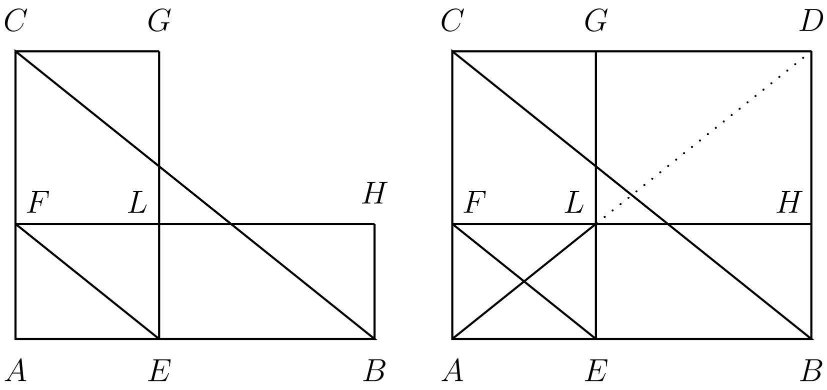


Figure 3.2: The rectangular case of Thales's Theorem

triangles AFE and FAL are congruent by *Elements* I.4, and likewise triangles ACB and CAD are congruent. Thus

$$\angle FAL = \angle AFE = \angle ACB = \angle CAD,$$

which is what we wanted to show. □

The foregoing is a special case of the following theorem, which does not require the special case in its proof.

Theorem 4. *In triangles that share an angle, the parallelograms in this angle that are bounded by alternate sides of the angle are equal if and only if the triangles are equiangular.*

Proof. Let the triangles be ABC and AEF in Figure 3.3, and let the parallelograms AG and AH be completed. Let the diagonals CE and BF be drawn. If the triangles are equiangular, then we have

$$FE \parallel CB \tag{3.4}$$

by *Elements* I.27. In this case, as in Euclid's proof of VI.2, the triangles FEC and EFB are equal by I.37. Adding triangle

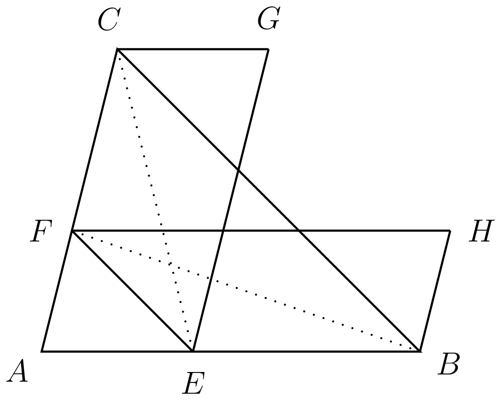


Figure 3.3: Intermediate form of Thales's Theorem

AEF in common, we obtain the equality of the triangles AEC and AFB. This gives us the equality of their doubles; and these, by I.34, are just the parallelograms AG and AH.

Conversely, if these parallelograms are equal, then so are their halves, the triangles AEC and AFB; hence the triangles FEC and EFB are equal, so (3.4) holds by *Elements* I.39, and then the original triangles ABC and AEF are equiangular by *Elements* I.29. $\square$

Proclus [62, 352.15] inadvertently gives evidence that Thales could *use* Theorem 3, if not Theorem 4. Discussing *Elements* I.26, which is the triangle-congruence theorem whose two parts are now abbreviated as A.S.A. and A.A.S. [72, p. 62], and which we used earlier to prove the concurrence of the medians of a triangle, Proclus says,

> Eudemus in his history of geometry attributes the theorem itself to Thales, saying that the method by which he is reported to have determined the distance of ships at sea shows that he must have used it.

If Thales really used Euclid's I.26 for measuring distances of ships, it may indeed have been by the method that Heath suggests [39, pp. 132–3]: climb a tower, note the angle of depression of the ship, then find an object on land at the same angle. The object's distance is that of the ship. This obviates any need to know the height of the tower, or to know proportions. Supposedly one of Napoleon's engineers measured the width of a river this way.

Nonetheless, Gow [35, p. 141] observes plausibly that the method that Heath will propose is not generally practical. Thales must have had the more refined method of similar triangles:

> It is hardly credible that, in order to ascertain the distance of the ship, the observer should have thought it necessary to reproduce and measure on land, in the horizontal plane, the enormous triangle which he constructed in imagination in a perpendicular plane over the sea. Such an undertaking would have been so inconvenient and wearisome as to deprive Thales' discovery of its practical value. It is therefore probable that Thales knew another geometrical proposition: viz. 'that the sides of equiangular triangles are proportional.' (Euc. VI. 4.)

But Proposition VI.4 is overkill for measuring distances. All one needs is the case of right triangles, in the form of Theorem 3 above. This must be the real theorem that Gow goes on to discuss:

> And here no doubt we have the real import of those Egyptian calculations of *seqt,* which Ahmes introduces as exercises in arithmetic. The *seqt* or ratio, between the distance of the ship and the height of the watch-tower is the same as that between the corresponding sides of any small but similar triangle. The discovery, therefore, attributed to Thales is

> probably of Egyptian origin, for it is difficult to see what other use the Egyptians could have made of their *seqt,* when found. It may nevertheless be true that the proposition, Euc. VI. 4, was not known, as now stated, either to the Egyptians or to Thales. It would have been sufficient for their purposes to know, inductively, that the *seqts* of equiangular triangles were the same.

Gow is right that Euclid's Proposition VI.4 need not have been known. But what he seems to mean is that the Egyptians and Thales need only have had a knack for applying the theorem, without having stepped back to recognize the theorem as such. This may be so; but there is no reason to think they had a knack for applying the theorem in full generality.

To establish that theorem, Thales's Theorem, in full generality, we shall prove that, in the proof of Theorem 3, equation (3.3) still holds, even when applied to Figure 3.3 in the proof of Theorem 4. To do this, we shall rely on the converse of *Elements* I.43: in Figure 3.1, if the parallelograms BG and GD are equal, then the point G must lie on the diagonal AD. We can prove this by contradiction, or by contraposition. If G did not lie on the diagonal, then we should be in the situation of Figure 3.4, where now parallelograms BN and ND are equal, but BG is part of BN, and ND is part of GD, so BG is less than GD, by Euclid's Common Notion 5.

In Euclid's proof of the Pythagorean Theorem, *Elements* I.47, three auxiliary straight lines concur. Heath [26, Vol. 1, p. 367] passes along Hero's proof of this, including, as a lemma, the converse of I.43. Hero's proof is direct, but relies on *Elements* I.39: equal triangles lying on the same side of the same base are in the same parallels. We used this in proving Theorem 4, and it is the converse of I.37; Euclid proves it by contradiction, using Common Notion 5.

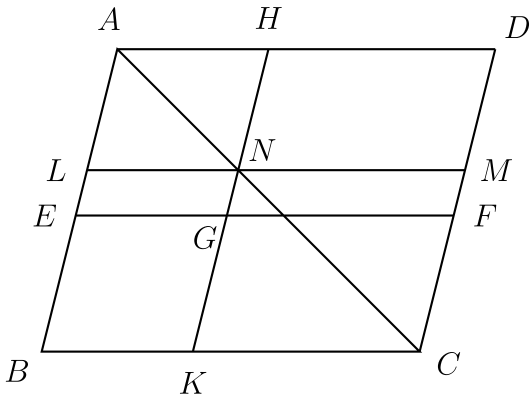


Figure 3.4: Converse of *Elements* I.43

Theorem 5. *If two parallelograms share an angle, the parallelograms are equal if and only if the rectangles bounded by the same sides are equal.*

Proof. Let the parallelograms be AG and AH in Figure 3.5. Supposing them equal, we prove (3.3), namely $AF \cdot AB = AE \cdot AC$. Let the parallelogram $ABDC$ be completed. By the converse of *Elements* I.43, the point L lies on the diagonal AD. Now erect the perpendiculars AR and AM (using *Elements* I.10), and make them equal to AE and AF respectively, as by drawing circles. Each of the parallelograms NE and SF is equal to a rectangle of sides equal to AE and AF, by *Elements* I.35. Therefore A lies on the diagonal LX, again by the converse of *Elements* I.43, so A and L both lie on the diagonal DX of the large parallelogram. Consequently, the parallelograms SC and NB are equal; but they are also equal to $AE \cdot AC$ and $AF \cdot AB$ respectively. The converse is similar. □

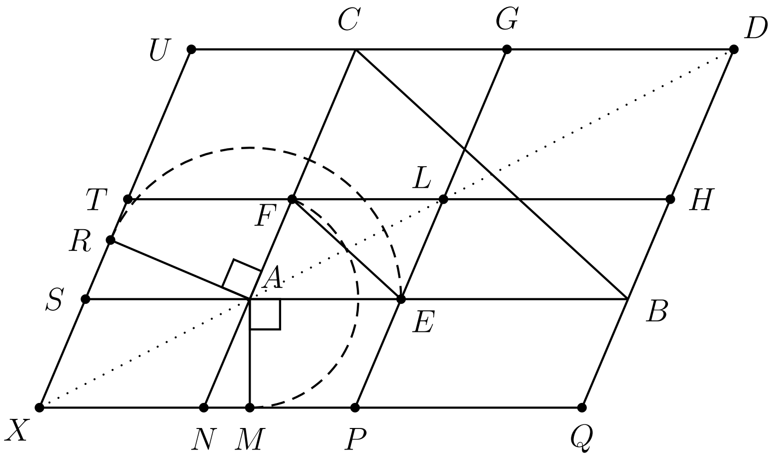


Figure 3.5: Thales's Theorem

Theorems 4 and 5 together are what we are calling Thales's Theorem, provided we can establish *Elements* VI.16:

> If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and [conversely].

To do this, we need a proper theoretical definition of proportion.

3.2 Thales and Desargues

I suggested that the equivalence of the proportion (3.1) with the equation (3.2) was "practical." To read the colons in the expression

$$a : b : : c : d, \tag{3.5}$$

we can say any of the following:

- a, b, c, and d are proportional;

- a is to b as c is to d;
- the ratio of a to b is the same as the ratio of c to d.

From the last clause, we can delete the phrase "the same as": this is effectively what Plutarch does in the passage quoted above, although the translator puts the phrase back in. The equation

$$a \cdot d = b \cdot c. \tag{3.6}$$

does not in itself express a property of the ordered pair (a, b) that is the same as the corresponding property of (c, d); it expresses only a relation between the pairs. Immediately we have $a \cdot b = b \cdot a$, and if (3.6) holds, so does $c \cdot b = d \cdot a$; so the relation expressed by (3.6) between (a, b) and (c, d) is reflexive and symmetric. The relation is not obviously transitive: if (3.6) holds, and $c \cdot f = d \cdot e$, it is not obvious that $a \cdot f = b \cdot e$. It would be *true,* for example, if we allowed passage to a fourth dimension, obtaining from the hypotheses

$$a \cdot d \cdot c \cdot f = b \cdot c \cdot d \cdot e,$$

whence, presumably, the desired conclusion would follow; but this would be a theorem. Therefore (3.6) alone cannot constitute the *definition* of (3.5). I have argued this elsewhere in the context of Euclid's number theory [58].

Suppose now that each of the letters in (3.5) stands for a **length:** not a number, but the class of Euclidean bounded straight lines that are equal to a particular straight line. I propose to define (3.5) to mean that for all lengths x and y,

$$b \cdot x = a \cdot y \iff d \cdot x = c \cdot y. \tag{3.7}$$

In other words, (3.5) means that the sets

$$\{(x, y) \colon b \cdot x = a \cdot y\}, \qquad \{(x, y) \colon d \cdot x = c \cdot y\}$$

are the same. This definition ensures *logically* that the relation of having the same ratio is transitive, as any relation described as a sameness should be. The definition also ensures that (3.5) *implies* (3.6): let $(x, y) = (c, d)$ in (3.7). The definition avoids the "Archimedean" assumption required by the definition attributed to Eudoxus, found in Book V of the *Elements*. However, we need to prove that (3.6) implies (3.5). This implication is *Elements* VI.16 for the new definition of proportion.

Theorem 6. *If, of four lengths, the rectangle bounded by the extremes is equal to the rectangle bounded by the means, then the lengths are in proportion.*

Proof. Supposing we are given four lengths a, b, c, d such that (3.6) holds, we want to show (3.5), as defined by (3.7). It is enough to show that, if two lengths e and f are such that

$$a \cdot f = b \cdot e, \tag{3.8}$$

then

$$c \cdot f = d \cdot e. \tag{3.9}$$

In Figure 3.6, let AH and AL have lengths c and e respectively. Draw AG parallel to HL (*Elements* I.31), and let AG have length a. Complete the parallelogram $ABDC$ so that GB and HC have lengths b and d respectively. By (3.6) and Theorem 5, the parallelograms GP and HN are equal. Now complete the parallelogram $ABFE$, and denote by g the length of LE. The parallelograms GR and GP are equal, by *Elements* I.35. Both LM and HK have length a (*Elements* I.34), so parallelograms LQ and HN are equal, by *Elements* I.36. Thus GR and LQ are equal. Hence, by Theorem 5, we have

$$a \cdot g = b \cdot e,$$

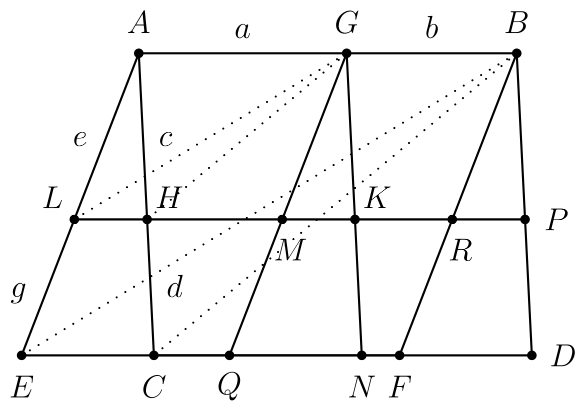


Figure 3.6: Transitivity

so $a \cdot g = a \cdot f$ by (3.8), and therefore $g = f$ by Common Notion 5. Since $LH \parallel EC$, we have (3.9) by Theorem 5. $\square$

There are easy consequences, corresponding to Propositions 16 and 22 of *Elements* Book V; but these propositions are not themselves so easy to prove with the Eudoxan definition of proportion.

Corollary 1 (Alternation)**.**

$$a : b : : c : d \implies a : c : : b : d.$$

Proof. Each proportion is equivalent to $a \cdot d = b \cdot c$. $\square$

In a note on V.16, Heath [26, Vol. 2, pp. 165–6] observes that the proposition is easier to prove when—as for us—the magnitudes being considered are lengths. He quotes the text-book of Smith and Bryant [67, pp. 298–9], which derives the special case from VI.1: parallelograms and triangles under the same height are to one another as their bases. We might write this as

$$a : b : : a \cdot c : b \cdot c.$$

One could take this as a *definition* of proportion; but then one has the problem of transitivity, as before. If

$$a : b : : c \cdot e : d \cdot e$$

for some e, meaning $c \cdot e = a \cdot f$ and $d \cdot e = b \cdot f$ for some f, one has to show the same for arbitrary e.

Corollary 2 (Cancellation).

$$\left.\begin{aligned} a : b &: : d : e \\ b : c &: : e : f \end{aligned}\right\} \implies a : c : : d : f.$$

Proof. Under the hypothesis, by alternation, $a : d : : b : e$ and $b : e : : c : f$. Then $a : d : : c : f$, since sameness of ratio is transitive by definition. $\square$

Theorems 4, 5, and 6 together constitute Thales's Theorem. In proving Theorem 6, we effectively showed in Figure 3.6

$$BC \parallel GH \;\; \& \;\; CE \parallel HL \implies BE \parallel GL, \tag{3.10}$$

provided also

$$CE \parallel AB. \tag{3.11}$$

Then by Thales's Theorem itself, since sameness of ratio is transitive, (3.10) holds, without need for (3.11). This is Desargues's Theorem: if the straight lines through corresponding vertices of two triangles concur, and two pairs of corresponding sides of the triangles are parallel, then the third pair must be parallel, as in Figure 3.7. We have proved this from Book

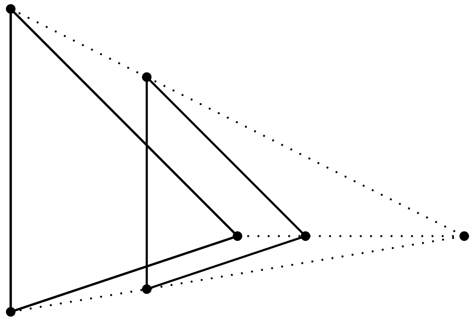


Figure 3.7: Desargues's Theorem

I of Euclid, without the Archimedean assumption of Book V, though *with* Common Notion 5 for areas.

Desargues's Theorem has other cases in the Euclidean plane. When we add to this plane the "line at infinity," thus obtaining the projective plane, then the three pairs of parallel lines in the theorem intersect on the new line. But then any line of the projective plane can serve as a line at infinity added to a Euclidean plane. A way to show this is by the kind of coordinatization that we are in the process of developing.

3.3 Locus problems

Fixing a unit length, Descartes [21, p. 5] defines the product ab of lengths a and b as another length, given by the rule that we may express as

$$1 : a : : b : ab.$$

Denoting Decartes's product thus, by juxtaposition alone, while continuing to denote with a dot the area of a rectangle with given dimensions, by Theorem 6 we have

$$ab \cdot 1 = a \cdot b.$$

In particular, Cartesian multiplication is commutative, and it distributes over addition, since

$$a \cdot b = b \cdot a, \qquad a \cdot (b + c) = a \cdot b + a \cdot c,$$

and from Common Notion 5 we have

$$d \cdot 1 = e \cdot 1 \implies d = e.$$

That Cartesian multiplication is associative can be seen from the related operation of composition of ratios, given by the rule

$$(a : b) \mathbin{\&} (b : c) :: a : c.$$

One may prefer to use the sign $=$ of equality here, rather than the sign $::$ of sameness of ratio, if one judges it not to be immediate that the compound ratio $(a : b) \mathbin{\&} (b : c)$ depends not merely on the ratios $a : b$ and $b : c$, but on their given representations in terms of a, b, and c. Nonetheless, it does depend only on the ratios, by Corollary 2. Then composition of ratios is immediately associative. We have generally

$$(a : b) \mathbin{\&} (c : d) :: e : d,$$

provided $a : b :: e : c$; and such e can be found, by the method of *Elements* I.44 and 45. Then

$$(a : 1) \mathbin{\&} (b : 1) :: ab : 1,$$

and from this we can derive $a(bc) = (ab)c$. Also

$$(a : 1) \mathbin{\&} (1 : a) :: 1 : 1,$$

so multiplication is invertible.

One can define the sum of two ratios as one defines the sum of fractions in school, by finding a common denominator. In particular,

$$(a : 1) + (b : 1) = (a + b) : 1,$$

where now it does seem appropriate to start using the sign of equality. Now both lengths and ratios compose fields, in fact ordered fields, which are isomorphic under $x \mapsto x : 1$. Descartes shows this implicitly, in order to solve ancient problems. One may object that we have not introduced additive inverses, whether of lengths or of ratios. We can do this by assigning to each class of parallel straight lines a direction, so that the signed length of BA is the additive inverse of the signed length of AB.

Descartes [21, p. 40, n. 59] alludes to a passage in the *Collection* where Pappus [70, pp. 346–53] describes three kinds of geometry problem: **plane,** as being solved by means of straight lines and circles, which lie in a plane; **solid,** as requiring also the use of conic sections, which are sections of a solid figure, the cone; and **linear,** as involving more complicated *lines,* that is, curves. An example of a linear problem then would be the quadrature or squaring of the circle, achieved by means of the **quadratrix** or "tetragonizer" (*τετραγωνίζουσα*), which Pappus [70, pp. 336–47] defines as being traced in a square, such as $ABGD$ in Figure 3.8, by the intersection of two straight lines, one horizontal and moving from the top edge BG to the bottom edge AD, the other swinging about the lower left corner A from the left edge AB to the bottom edge AD. If there is a point H where the quadratrix meets the lower edge of the square, then, as Pappus shows,

$$BD : AB : : AB : AH,$$

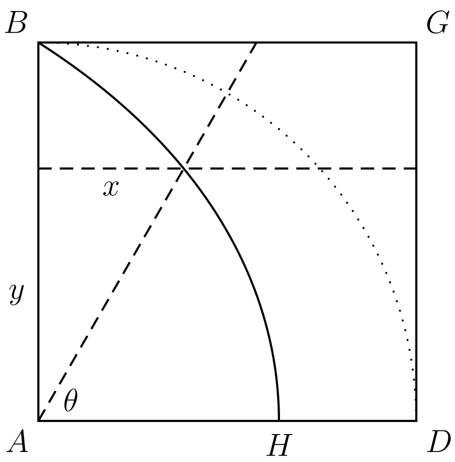


Figure 3.8: The quadratrix

where BD is the circular arc centered at A. In modern terms, with variables as in the figure, AB being taken as a unit,

$$\frac{\theta}{y} = \frac{\pi}{2}, \qquad \tan\theta = \frac{y}{x},$$

so

$$x = \frac{2}{\pi} \cdot \frac{\theta}{\tan\theta}.$$

As θ vanishes, x goes to $2/\pi$. This then is the length of AH. Pappus points out that we have no way to construct the quadratrix without knowing where the point H is in the first place. He attributes this criticism to one Sporus, about whom we apparently have no source but Pappus himself [6, p. 285, n. 78].

A solid problem that Pappus describes [70, pp. 486–9] is the **four-line locus problem:** find the locus of points such that the rectangle whose dimensions are the distances to two given straight lines bears a given ratio to the rectangle whose dimensions are the distances to two more given straight lines. According to Pappus, theorems of Apollonius were needed to solve this problem; but it is not clear whether Pappus thinks Apollonius actually did work out a full solution. By the last three propositions, namely 54–6, of Book III of the *Conics* of Apollonius, it is implied that the conic sections are three-line loci, that is, solutions to the four-line locus problem when two of the lines are identical. Taliaferro [3, pp. 267–75] works out the details and derives the theorem that the conic sections are four-line loci.

Descartes works out a full solution to the four-line locus problem [21, pp. 59–80]. He also solves a particular *five*-line locus problem, where four of the straight lines—say ℓ_0, ℓ_1, ℓ_2, and ℓ_3—are parallel to one another, each a distance a from the previous, while the fifth line—ℓ_4—is perpendicular to them [21, pp. 83–4]. What is the locus of points such that the product of their distances to ℓ_0, ℓ_1, and ℓ_3 is equal to the product of a with the distances to ℓ_2 and ℓ_4? One can write down an *equation* for the locus, and Descartes does. Letting distances from ℓ_4 and ℓ_2 be x and y respectively, Descartes obtains

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy. \tag{3.12}$$

This may allow us to plot points on the desired locus, obtaining the bold solid curve in Figure 3.9; but we could already do that. The equation is thus not a solution to the locus problem, since it does not tell us what the locus *is.* But Descartes shows that the locus is traced by the intersection of a moving parabola with a straight line passing through one fixed point and one point that moves with the parabola, as suggested by the dashed lines in Figure 3.9. The parabola has axis sliding along ℓ_2, and the latus rectum of the parabola is a, so the

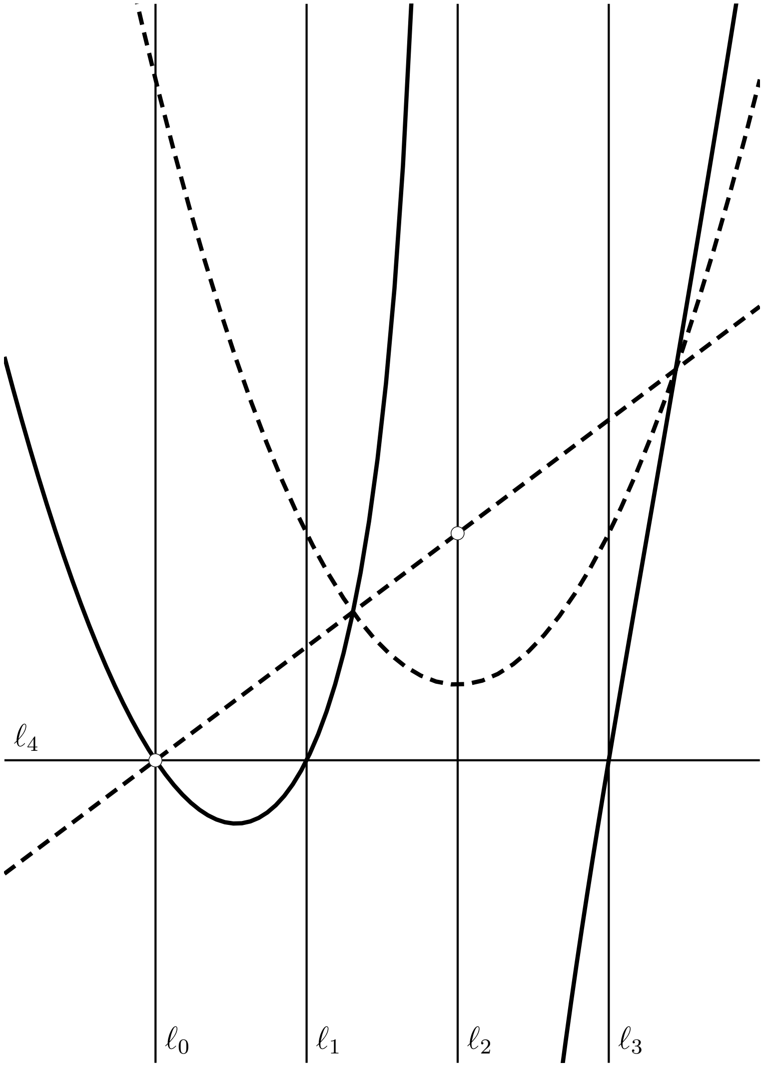


Figure 3.9: Solution of a five-line locus problem

parabola is given by $ax = y^2$ when its vertex is on ℓ_4. The straight line passes through the intersection of ℓ_0 and ℓ_4 and through the point on the axis of the parabola whose distance from the vertex is a.

We shall be looking at latera recta again later; meanwhile, one may consult my article "Abscissas and Ordinates" [57], to learn more than one ever imagined wanting to know about the terminology.

Descartes's solution of a five-line locus problem is apparently one that Pappus would recognize as such. Thus Descartes's algebraic methods would seem to represent an advance, and not just a different way of doing mathematics. As Descartes knows [21, p. 22, n. 34], Pappus [71, pp. 600-3] could *formulate* the $2n$- and $(2n + 1)$-line locus problems for arbitrary n. If $n > 3$, the ratio of the product of n segments with the product of n segments can be understood as the ratio compounded of the respective ratios of segment to segment. Given $2n$ lengths $a_1, \ldots, a_n, b_1, \ldots, b_n$, we can understand the ratio of the product of the a_k to the product of the b_k as the composite ratio

$$(a_1 : b_1) \mathbin{\&} \cdots \mathbin{\&} (a_n : b_n).$$

Pappus recognizes this. Descartes expresses the solution of the $2n$-line locus problem as an nth-degree polynomial equation in x and y, where y is the distance from a point of the locus to one of the given straight lines, and x is the distance from a given point on that line to the foot of the perpendicular from the point of the locus. Today we call the line the **x-axis,** and the perpendicular through the given point on it the **y-axis;** but Descartes does not seem to have done this expressly.

Descartes does effectively allow oblique axes. The original locus problems literally involve not distances to the given lines, but lengths of straight lines drawn at given angles to the given lines.

3.4 The nine-point conic

The nine-point conic is the solution of a locus problem. The solution had been known earlier; but apparently the solution was first remarked on in 1892 by Maxime Bôcher [11], who says,

> It does not seem to have been noticed that a few well-known facts, when properly stated, yield the following direct generalization of the famous nine point circle theorem:—
>
> *Given a triangle ABC and a point P in its plane, a conic can be drawn through the following nine points:*
>
> *(1) The middle points of the sides of the triangle;*
>
> *(2) The middle points of the lines joining P to the vertices of the triangle;*
>
> *(3) The points where these last named lines cut the sides of the triangle.*
>
> The conic possessing these properties is simply the locus of the centre of the conics passing through the four points A, B, C, P (cf. Salmon's Conic Sections, p. 153, Ex. 3, and p. 302, Ex. 15).

Bôcher's references fit the sixth and tenth editions of Salmon's *Treatise on Conic Sections,* dated 1879 and 1896 respectively [65, 66]; but in the "Third Edition, revised and enlarged," dated 1855 [64], the references would be p. 137, Ex. 4, and p. 284, Art. 339.

Following Salmon, we start with a general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{3.13}$$

of the second degree. We do not assume that x and y are measured at right angles to one another: we allow oblique axes. We do assume $ab \neq 0$. If we wish, we can eliminate the xy term by redrawing the x-axis, and changing its scale, so that the point now designed by (x, y) is the one that was called $(x + hy/a, y)$ before. The curve that was defined by (3.13) is now defined by

$$ax^2 + \left(b - \frac{h^2}{a}\right) y^2 + 2gx + 2\left(f - \frac{gh}{a}\right) y + c = 0. \quad (3.14)$$

If the linear terms in (3.13) are absent, then they are absent from (3.14) as well.

The equation (3.14) defines an ellipse or hyperbola, no matter what the angle is between the axes, or what their relative scales are. This is perhaps not, strictly speaking, high-school knowledge. One may know from school that an equation

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

defines a certain curve called ellipse or hyperbola, depending on whether the upper or lower sign is taken. But one assumes that x and y are measured orthogonally. One does not learn why the curves are named as they are [57], and one does not learn that, if the appropriate oblique axes are chosen, then the curve has an equation of the same form. But this is just what Book I of the *Conics* of Apollonius is devoted to showing [3].

Going back to the original axes, and the curve defined by (3.13), we translate the axes, so that the new origin is the point formerly called (x', y'). The curve is now given by

$$ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0,$$

where

$$g' = ax' + hy' + g, \qquad f' = by' + hx' + f, \tag{3.15}$$

and we are not interested in c'. The curve given by (3.13) has center at (x', y') just in case $(g', f') = (0, 0)$.

For a complete quadrangle in which one pair (at least) of opposite sides are not parallel, those sides determine a coordinate system. In this system, suppose the vertices of the complete quadrilateral are $(\lambda, 0)$ and $(\lambda', 0)$ on the x-axis and $(0, \mu)$ and $(0, \mu')$ on the y-axis. Let the conic given by (3.13) pass through these four points. Setting $y = 0$, we obtain the equation

$$ax^2 + 2gx + c = 0,$$

which must have roots λ and λ', so that the equation is

$$a\bigl(x^2 - (\lambda + \lambda')x + \lambda\lambda'\bigr) = 0.$$

From this we obtain

$$2g = -a(\lambda + \lambda'), \qquad c = a\lambda\lambda'.$$

Likewise, setting $x = 0$ in (3.13) yields the equation

$$by^2 + 2fy + c = 0,$$

which must be

$$b\bigl(y^2 - (\mu + \mu')y + \mu\mu'\bigr) = 0,$$

from which we obtain

$$2f = -b(\mu + \mu'), \qquad c = b\mu\mu'.$$

From the two expressions for c, we have

$$a\lambda\lambda' = b\mu\mu'.$$

In (3.13), we are free to let $a = \mu\mu'$. Then $b = \lambda\lambda'$, and (3.13) becomes

$$\begin{aligned}\mu\mu' x^2 + 2hxy + \lambda\lambda' y^2 \\ - \mu\mu'(\lambda + \lambda)x - \lambda\lambda'(\mu + \mu')y + \lambda\lambda'\mu\mu' = 0.\end{aligned} \quad (3.16)$$

By the computations for (3.15), the center of the conic in (3.16) satisfies

$$\begin{aligned}2\mu\mu' x + 2hy - \mu\mu'(\lambda + \lambda') &= 0,\\ 2\lambda\lambda' y + 2hx - \lambda\lambda'(\mu + \mu') &= 0.\end{aligned}$$

Eliminating h yields

$$2\mu\mu' x^2 - \mu\mu'(\lambda + \lambda')x = 2\lambda\lambda' y^2 - \lambda\lambda'(\mu + \mu')y. \quad (3.17)$$

This is the equation of an ellipse or hyperbola passing through the origin. We can also write the equation as

$$\mu\mu' x\left(x - \frac{\lambda + \lambda'}{2}\right) = \lambda\lambda' y\left(y - \frac{\mu + \mu'}{2}\right); \quad (3.18)$$

this shows that the conic passes through the midpoints of the sides of the complete quadrangle that lie along the axes. By symmetry, the conic passes through the midpoints of all six sides of the quadrangle, and through its three diagonal points—if these exist, that is, if none of the three pairs of opposite sides of the quadrangle are parallel.

I suggested earlier that for Descartes to solve a five-line locus problem, it was not enough to find the equation (3.12); he had to describe the solution geometrically. We do have a thorough geometric understanding of solutions of second-degree equations like (3.17) and (3.18) or even (3.13). Alternatively, there is a modern "synthetic" approach [20, 8.71, p. 118], that is, an approach based not on field axioms, but on geometric axioms: here axioms for a projective plane, but with a line at infinity designated, so that midpoints of segments of other lines can be defined.

I do not know how directly the nine-point conic can be derived from the work of Apollonius. Here I shall just want to look briefly at an example of how Apollonius uses areas in a way not easily made algebraic. David Hilbert shows how algebra is *possible,* without *a priori* assumptions about areas; but it is not clear how much is gained.

4 Lengths and Areas

4.1 Algebra

The points of an unbounded straight line are elements of an ordered abelian group with respect to the obvious notion of addition, once an origin and a direction have been selected. If we set up two straight lines at right angles to one another, letting their intersection point be the origin, then, after fixing also a unit length, we obtain Hilbert's definition of multiplication as in Figure 4.1, the two oblique lines being parallel.

Not having accepted the Eudoxan definition of proportion, or Thales's Theorem, Hilbert can still show that multiplication is commutative and associative. He does this by means of what he calls Pascal's Theorem, although it was referred to in the Introduction above as Pappus's Theorem: if the vertices of a

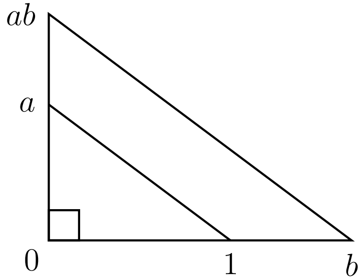


Figure 4.1: Hilbert's multiplication

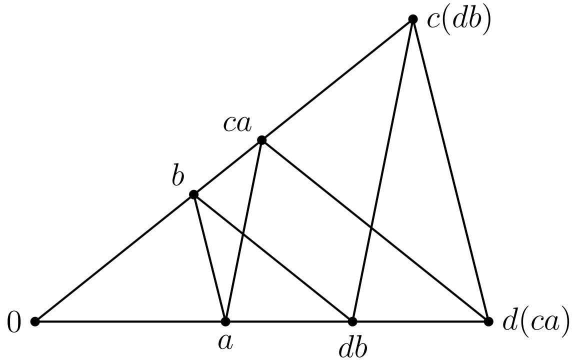


Figure 4.2: Pappus's Theorem

hexagon lie alternately on two straight lines in the projective plane, then the intersection points of the three pairs of opposite sides lie on a straight line. Pascal announced the generalization in which the original two straight lines can be an arbitrary conic section [14, 69].

When we give Pappus's Theorem in the Euclidean plane, the simplest case occurs as in Figure 4.2, labelled for proving commutativity and associativity of multiplication (in case the angle at 0 is right). In general, if there are two pairs of parallel opposite sides of the hexagon that is woven like a spider's web or cat's cradle across the angle, then the third pair of opposite sides are parallel as well. By the numbering in Hultsch's edition of Pappus's *Collection* [51], which is apparently the numbering made originally by Commandinus [52, pp. 62–3, 77], the result is Proposition 134 in Book VII; it is also number VIII of Pappus's lemmas for the now-lost *Porisms* of Euclid.

Lemma VIII seems to have been sadly forgotten. The case of Pappus's Theorem where the three pairs of opposite sides

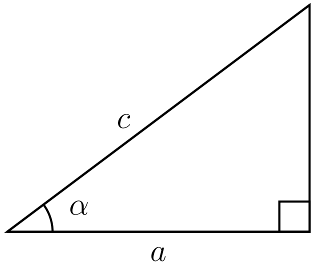


Figure 4.3: Hilbert's trigonometry

all intersect is Propositions 138 and 139, or Lemmas XII and XIII: these cover the cases when the straight lines on which the vertices of the hexagon lie are parallel and not, respectively. Kline cites only Proposition 139 as giving Pappus's Theorem [46, p. 128]. In his summary of most of Pappus's lemmas for Euclid's *Porisms,* Heath [40, p. 419–24] lists Propositions 138, 139, 141, and 143 as constituting Pappus's Theorem. The last two are converses of the first two, as Pappus states them. It is not clear that Pappus recognizes a single theme behind the several of his lemmas that constitute the theorem named for him. He omits the case where exactly one pair of opposite sides are parallel.

Heath omits to mention Proposition 134 at all. This is a strange oversight for an important theorem. Pappus proves it by means of areas, using Euclid's I.39, as we proved Theorem 4. Without using areas, Hilbert gives two elaborate proofs, one of which, in the situation of Figure 4.3, uses the notation

$$a = \alpha c$$

for what today might be written as $a = c \cos \alpha$.

For proving associativity and commutativity of multiplication, Hartshorne has a more streamlined approach, in *Geom-*

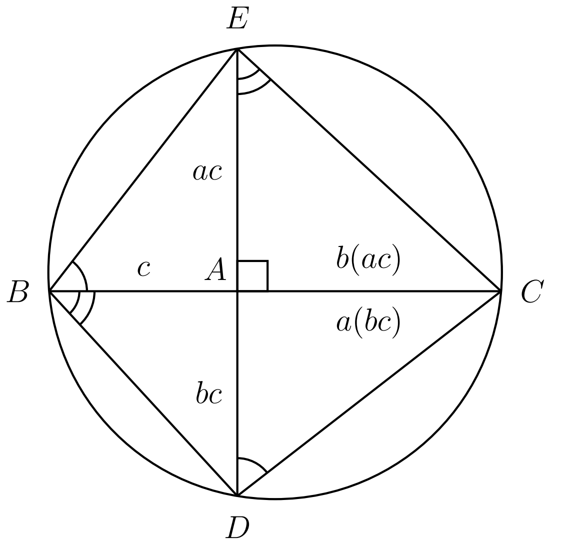


Figure 4.4: Associativity and commutativity

etry: Euclid and Beyond [37, pp. 170–2]. In Figure 4.4, AC has the two lengths indicated, so these are the same; that is,

$$a(bc) = b(ac). \tag{4.1}$$

Letting $c = 1$ gives commutativity; then this with (4.1) gives associativity. Distributivity follows from Figure 4.5, where

$$ab + ac = a(b + c).$$

The advantage of defining multiplication in terms of lengths alone (and right angles, and parallelism, but not parallelograms or other bounded regions of the plane) is that it allows all straight-sided regions to be linearly ordered by size, without assuming *a priori* that the whole region is greater than the part.

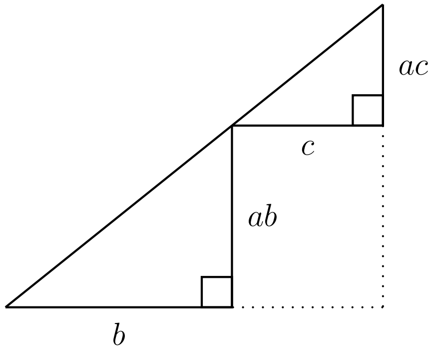


Figure 4.5: Distributivity

Euclid provides for an ordering in *Elements* I.44 and 45, which show that every straight-sided region is equal to a rectangle on a given base. Finding the rectangle involves I.43. Showing the rectangle unique requires the converse of I.43, which in turn requires Common Notion 5 for areas. The triangle may be equal to the rectangle on the same base with half the height; but there are three choices of base, and so three rectangles result that are equal to the triangle. When they are all made equal to rectangles on the same base, why should they have the same height? Common Notion 5 is one reason; but Hilbert doesn't need it.

The triangle is equal to the rectangle whose base is half the perimeter of the triangle and whose height is the radius of the inscribed circle: see Figure 4.6, where the triangle ABC is equal to the sum of $AG \cdot GD$, $BG \cdot GD$, and $CE \cdot ED$, but $ED = GD$. However, if ABC is cut into two triangles, and rectangles equal to the two triangles are found, added together, and made equal to a rectangle whose base is again half the perimeter of ABC, we need to know that the height is equal to GD.

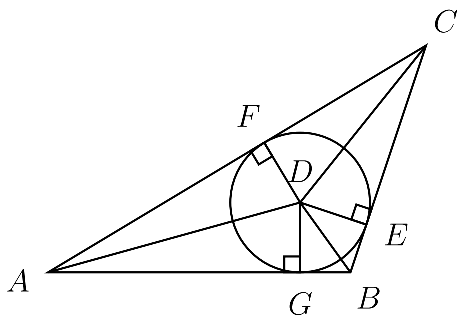


Figure 4.6: Circle inscribed in triangle

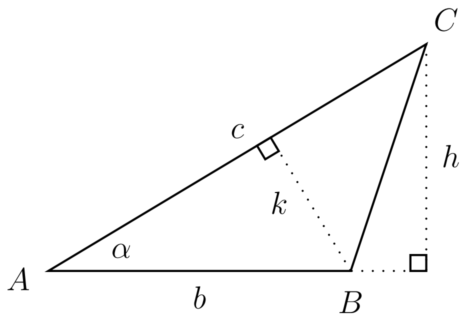


Figure 4.7: Area of triangle in two ways

In the notation of Figure 4.7, we have

$$h = c \sin \alpha, \qquad\qquad k = b \sin \alpha,$$

where $\sin \alpha$ stands for the appropriate length, and the multiplication is Hilbert's; and then

$$bh = bc \sin \alpha = cb \sin \alpha = ck.$$

Thus we can define the area of ABC unambiguously as the *length* $bh/2$. With this definition, when the triangle is divided into two parts, or indeed into many parts, as Hilbert shows, the area of the whole is the sum of the areas of the parts.

Moreover, if two areas become equal when equal areas are added, then the two original areas are themselves equal: this is Euclid's Common Notion 3, but Hilbert makes it a theorem. One might classify this theorem with the one that almost every human being learns in childhood, but almost nobody ever recognizes *as* a theorem: no matter how you count a finite set, you always get the same number. It *can* be valuable to become clear about the basics, as I have argued concerning the little-recognized distinction between induction and recursion [55].

4.2 Apollonius

We look at a proof by Apollonius, in order to consider Descartes's idea, quoted in the Introduction, that ancient mathematicians had a secret method.

We first set the stage; this is done in more detail in [57]. A **cone** is determined by a **base,** which is a circle, and a **vertex,** not in the plane of the base, but not necessarily hovering right over the center of the base either: the cone may be oblique. The surface of the cone is traced out by the straight lines that pass through the vertex and the circumference of the base. A diameter of the base is also the base of an **axial triangle,** whose apex is the vertex of the cone. If a chord of the base of the cone cuts the base of the axial triangle at right angles, then a plane containing the chord and parallel to a side of the axial triangle cuts the surface of the cone in a **parabola.** The cutting plane cuts the axial triangle in a straight line that is called a **diameter** of the parabola because the line bisects the chords of the parabola that are parallel to the base of the cone. Half of such a chord is an **ordinate;** it cuts off from the diameter the corresponding **abscissa,** the other endpoint of this being the **vertex** of the parabola. There is some bounded straight line, the **latus rectum,** such that, when the square on any ordinate is made equal to a rectangle on the abscissa, the other side of the rectangle is precisely the latus rectum: this is Proposition I.11 of Apollonius, and it is the reason for the term *parabola,* meaning application.

The tangent to the parabola at the vertex is parallel to the ordinates. We are going to show that every straight line parallel to the diameter is another diameter, with a corresponding latus rectum; and latera recta are to one another as the squares on the straight lines, each of which is drawn drawn tangent to the parabola from vertex to other diameter.

In Cartesian terms, we may start with a diameter that is an **axis** in the sense of being at right angles to its ordinates. If the latus rectum is ℓ, then the parabola can be given by

$$\ell y = x^2. \tag{4.2}$$

As in Figure 4.8, the tangent to the parabola at $(a, a^2/\ell)$ cuts the y-axis at $-a^2/\ell$: this can be shown with calculus, but it is Proposition I.35 of Apollonius. The tangent then is

$$\ell y = 2ax - a^2.$$

We shall take this and $x = a$ as new axes, say x'- and y'-axes. If d is the distance between the new origin and the intersection of the x'-axis with the y-axis, then, since the x- and y-axes are orthogonal, by the Pythagorean Theorem (*Elements* I.47) we have

$$d^2 = a^2 + \left(\frac{2a^2}{\ell}\right)^2 = \frac{a^2}{\ell^2}(\ell^2 + 4a^2).$$

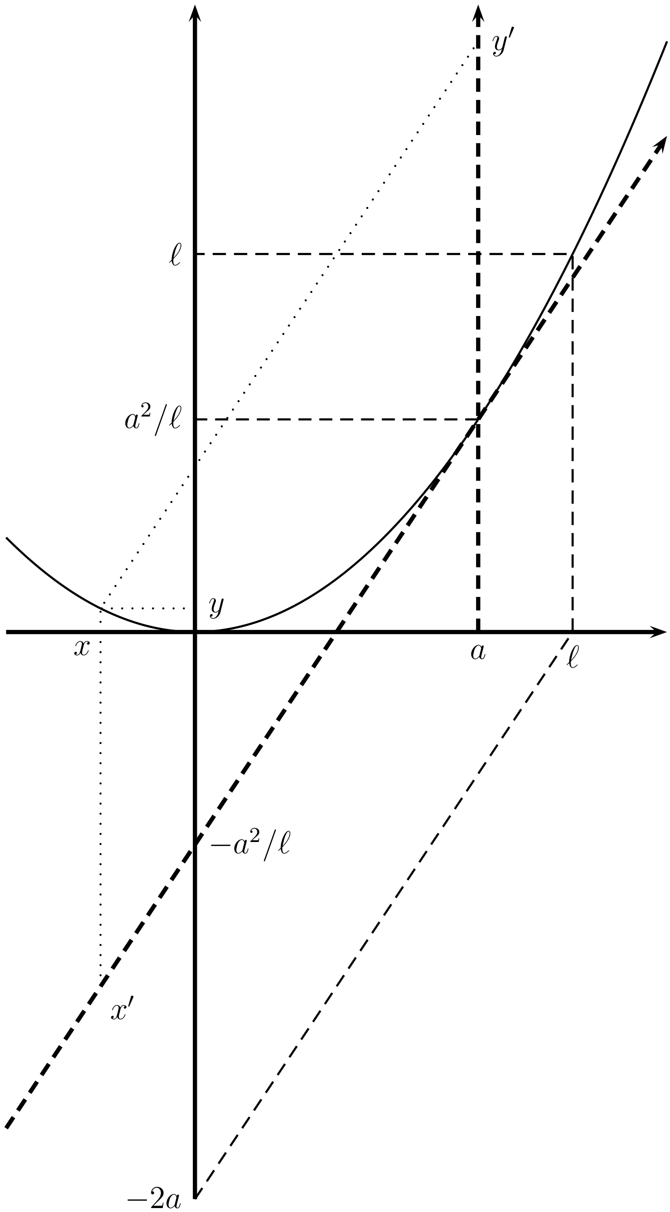


Figure 4.8: Change of coordinates

Let $b = \sqrt{\ell^2 + 4a^2}$, so $d = ba/\ell$. Then

$$\ell x' = \frac{\ell d}{a}(x - a) = bx - ba, \qquad \ell y' = \ell y - 2ax + a^2,$$

so

$$bx = \ell x' + ba, \qquad \begin{aligned} \ell y &= \ell y' + \frac{2a}{b}(\ell x' + ba) - a^2 \\ &= \ell y' + \frac{2\ell a}{b} x' + a^2. \end{aligned}$$

Plugging into (4.2) yields

$$\begin{gathered} \ell y' + \frac{2\ell a}{b} x' + a^2 = \left(\frac{\ell}{b} x' + a\right)^2, \\ b^2 y' = \ell (x')^2. \end{gathered}$$

In particular, the new latus rectum is b^2/ℓ, which is as claimed, since $b^2/\ell^2 = d^2/a^2$.

For his own proof of this, Apollonius uses a lemma, Proposition I.42 [2, p. 128–31]. In Figure 4.9, we have a parabola ΓΔB with diameter BΘ. Here ΔZ and ΓΘ are ordinates, and BH is parallel to these, so it is tangent to the parabola at B. The straight line AΓ is tangent to the parabola at Γ, which means, by I.35,

$$\mathrm{AB} = \mathrm{B\Theta}. \tag{4.3}$$

Then

$$\mathrm{A\Gamma\Theta} = \mathrm{H\Theta}, \tag{4.4}$$

the latter being the parallelogram with those opposite vertices. The straight line ΔE is drawn parallel to ΓA. Then triangles AΓΘ and EΔZ are similar, so their ratio is that of the squares on their bases, those bases being the ordinates mentioned. Then

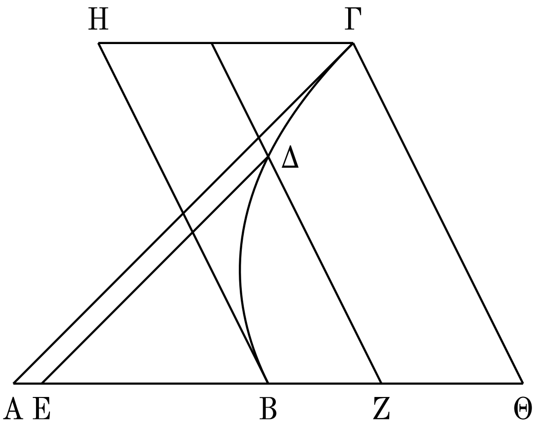


Figure 4.9: Proposition I.42 of Apollonius

the abscissas BΘ and BZ are in that ratio, and hence the parallelograms HΘ and HZ are in that ratio. By (4.4) then, AΓΘ has the same ratio to HZ that it does to EΔZ. Therefore

$$\mathrm{E\Delta Z} = \mathrm{HZ}.$$

The relative positions of Δ and Γ on the parabola are irrelevant to the argument: this will matter for the next theorem.

In Figure 4.10 now, KΔB is a parabola with diameter BM, and ΓΔ is tangent to the parabola, and through Δ, parallel to the diameter, straight line ΔN is drawn and extended to Z so that ZB is parallel to the ordinate ΔΞ. A length H is taken such that

$$\mathrm{E\Delta} : \mathrm{\Delta Z} : : \mathrm{H} : 2\mathrm{\Gamma\Delta}. \tag{4.5}$$

Through a random point K on the parabola, KΛ is drawn parallel to the tangent ΓΔ. We shall show

$$\mathrm{K\Lambda}^2 = \mathrm{H} \cdot \mathrm{\Delta\Lambda}, \tag{4.6}$$

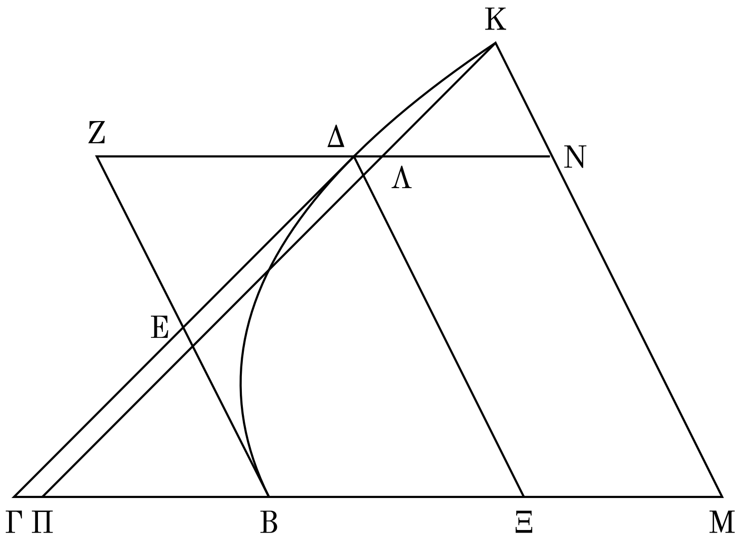


Figure 4.10: Proposition I.49 of Apollonius

so that ΔN serves as a new diameter of the parabola, with corresponding latus rectum H. This is Proposition I.49 of Apollonius.

Since, as before, $\Gamma\mathrm{B} = \mathrm{B}\Xi$, we have

$$\mathrm{EZ}\Delta = \mathrm{E}\Gamma\mathrm{B}. \tag{4.7}$$

Let ordinate KNM be drawn. Adding to either side of (4.7) the pentagon ΔEBMN, we have

$$\mathrm{ZM} = \Delta\Gamma\mathrm{MN}.$$

Let KΛ be extended to Π. By the lemma that we proved above, $\mathrm{K}\Pi\mathrm{M} = \mathrm{ZM}$. Thus

$$\mathrm{K}\Pi\mathrm{M} = \Delta\Gamma\mathrm{MN}.$$

Subtracting the trapezoid ΛΠΜΝ gives

$$\mathrm{K\Lambda N} = \Lambda\Gamma.$$

From this, by Theorem 5 above, we have

$$\mathrm{K}\Lambda \cdot \Lambda\mathrm{N} = 2\Lambda\Delta \cdot \Delta\Gamma. \tag{4.8}$$

We now compute

$$\begin{aligned}
\mathrm{K}\Lambda^2 : \mathrm{K}\Lambda \cdot \Lambda\mathrm{N} &: : \mathrm{K}\Lambda : \Lambda\mathrm{N} \\
&: : \mathrm{E}\Delta : \Delta\mathrm{Z} \\
&: : \mathrm{H} : 2\Gamma\Delta && \text{[by (4.5)]} \\
&: : \mathrm{H} \cdot \Lambda\Delta : 2\Lambda\Delta \cdot \Gamma\Delta \\
&: : \mathrm{H} \cdot \Lambda\Delta : \mathrm{K}\Lambda \cdot \Lambda\mathrm{N}, && \text{[by (4.8)]}
\end{aligned}$$

which yields (4.6). We have assumed K to be on the other side of ΔN from BM. The argument can be adapted to the other case. Then, as a corollary, we have that ΔN bisects all chords parallel to ΔΓ. In fact Apollonius proves this independently, in Proposition I.46.

Could Apollonius have created the proof of I.49 for pedagogical or ideological reasons, after verifying the theorem itself by Cartesian methods, such we we employed? I have not found any reason to think so. Before Apollonius, it seems that only the *right* cone was studied, and the only sections considered were made by planes that were orthogonal to straight lines in the surface of the cone [1, p. xxiv]. Whether an ellipse, parabola, or hyperbola was obtained depended on the angle at the vertex of the cone. The recognition that the cone can be oblique, and every section can be obtained from every cone, seems to be due to Apollonius. That our Cartesian argument started with orthogonal axes corresponds to starting with a right cone. On the other hand, this feature was not essential to the argument; we did not really need the parameter b.

5 Unity

In addition to *Elements* I.26, which is A.S.A. and A.A.S. as discussed earlier, Proclus [62, 157.11, 250.20, 299.4] attributes to Thales three more of Euclid's propositions, depicted in Figure 5.1:

1) the diameter bisects the circle, as in the remark on, or addendum to, the definition of diameter given at the head of the *Elements*;
2) the base angles of an isosceles triangle are equal (I.5);
3) vertical angles are equal (I.15).

Kant alludes to the second of these theorems in the Preface to the B Edition of the *Critique of Pure Reason,* in a purple passage of praise for the person who discovered the theorem [44, B x–xi, p. 107–8]:

> **Mathematics** has, from the earliest times to which the history of human reason reaches, in that admirable people the

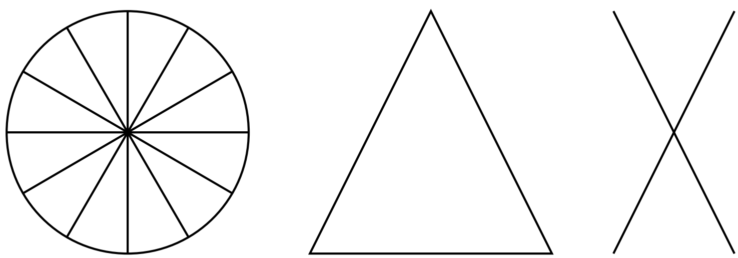


Figure 5.1: Symmetries

> Greeks, traveled the secure path of a science. Yet it must not be thought that it was as easy for it as for logic—in which reason has to do only with itself—to find that royal path, or rather itself to open it up; rather, I believe that mathematics was left groping about for a long time (chiefly among the Egyptians), and that its transformation is to be ascribed to a **revolution,** brought about by the happy inspiration of a single man in an attempt from which the road to be taken onward could no longer be missed, and the secure course of a science was entered on and prescribed for all time and to an infinite extent. The history of this revolution in the way of thinking—which was far more important than the discovery of the way around the famous Cape—and of the lucky one who brought it about, has not been preserved for us. But the legend handed down to us by Diogenes Laertius—who names the reputed inventor of the smallest elements of geometrical demonstration, even of those that, according to common judgment, stand in no need of proof—proves that the memory of the alteration wrought by the discovery of this new path in its earliest footsteps must have seemed exceedingly important to mathematicians, and was thereby rendered unforgettable. A new light broke upon the person who demonstrated the isosceles triangle (whether he was called "Thales" or had some other name).

The boldface is Kant's. The editors cite a letter in which Kant confirms the allusion to *Elements* I.5.

In apparent disagreement with Kant, I would suggest that revolutions in thought need not persist; they must be made afresh by each new thinker. The student need not realize her or his potential for new thought, no matter how open the royal path may seem to the teacher.

Kant refers to the discovery of the southern route around Africa: does he mean the discovery by Bartolomeu Dias in

1488, or the discovery by the Phoenicians, sailing the other directions, two thousand years earlier, described by Herodotus [42, IV.42, pp. 297–9]? The account of Herodotus is made plausible by his disbelief that, in sailing west around the Cape of Good Hope, the Phoenicians could have found the sun on their right. Their route was not maintained, which is why Dias can be hailed as its discoverer.

Likewise may routes to mathematical understanding not be maintained. Much of ancient mathematics has been lost to us, in the slow catastrophe alluded to by the title of *The Forgotten Revolution* [63, p. 8]. Here Lucio Russo points out that the last of the eight books of Apollonius on conic sections no longer exists, while Books V–VII survive only in Arabic translation, and we have only Books I–IV in the original Greek. Presumably this is because the later books of Apollonius were found too difficult by anybody who could afford to have copies made.

We may be able to recover the achievement of Thales, if Thales be his name. There is no reason in principle why we cannot understand him as well as we understand anybody; but time and loss present great obstacles. Kant continues with his own interpretation of Thales's thought:

> For he found that what he had to do was not to trace what he saw in this figure, or even trace its mere concept, and read off, as it were, from the properties of the figure; but rather that he had to produce the latter from what he himself thought into the object and presented (through construction) according to *a priori* concepts, and that in order to know something securely *a priori* he had to ascribe to the thing nothing except what followed necessarily from what he himself had put into it in accordance with its concept.

Kant is right, if he means that the equality of the base angles of an isosceles triangle does not follow merely from a figure of an isosceles triangle. The figure itself represents only one instance of the general claim. One has to recognize that the figure is constructed according to a *principle,* which in this case can be understood as symmetry. The three shapes in Figure 5.1 embody a symmetry that justifies the corresponding theorems, even though Euclid does not appeal to this symmetry in his proofs.

As we discussed, Thales is also thought to have discovered that the angle in a semicircle is right, and he may have done this by recognizing that any two diameters of a circle are diagonals of an equiangular quadrilateral. Such quadrilaterals are rectangles, but this is not so fundamental an observation as the equality of the base angles of an isosceles triangle; in fact the "observation" can be disputed, as it was by Lobachevski [48].

Thales is held to be the founder of the Ionian school of philosophy. In *Before Philosophy* [33, p. 251], the Frankforts say of the Ionian school that its members

> proceeded, with preposterous boldness, on an entirely unproved assumption. They held that the universe is an intelligible whole. In other words, they presumed that a single order underlies the chaos of our perceptions and, furthermore, that we are able to comprehend that order.

For Thales, the order of the world was apparently to be explained through the medium of water. However, this information is third-hand at best. Aristotle says in *De Caelo* [8, II.13, pp. 430],

> By these considerations some have been led to assert that the earth below us is infinite, saying, with Xenophanes of Colophon, that it has 'pushed its roots to infinity',—in or-

der to save the trouble of seeking for the cause . . . Others say the earth rests upon water. This, indeed, is the oldest theory that has been preserved, and is attributed to Thales of Miletus. It was supposed to stay still because it floated like wood and other similar substances, which are so constituted as to rest upon water but not upon air. As if the same account had not to be given of the water which carries the earth as of the earth itself!

Aristotle has only second-hand information on Thales, who seems not to have written any books. It may be that Aristotle does not understand what questions Thales was trying to answer. Aristotle has his own questions, and criticizes Thales for not answering them.

Aristotle may do a little better by Thales in the *Metaphysics* [9, 983^{a}24, pp. 693–5], where he says,

> Of the first philosophers, then, most thought the principles which were of the nature of matter were the only principles of all things . . . Yet they do not all agree as to the number and the nature of these principles. Thales, the founder of this type of philosophy, says the principle is water (for which reason he declared that the earth rests on water), getting the notion perhaps from seeing that the nutriment of all things is moist, and that heat itself is generated from the moist and kept alive by it (and that from which they come to be is a principle of all things). He got his notion from this fact, and from the fact that the seeds [*τὰ σπέρματα*] of all things have a moist nature, and that water is the origin of the nature of moist things.

In *The Idea of Nature* [16, pp. 31–2], Collingwood has a poetic interpretation of this:

> The point to be noticed here is not what Aristotle says but what it presupposes, namely that Thales conceived the world

> of nature as an organism: in fact, as an animal . . . he may possibly have conceived the earth as grazing, so to speak, on the water in which it floats, thus repairing its own tissues and the tissues of everything in it by taking in water from this ocean and transforming it, by processes akin to respiration and digestion, into the various parts of its own body . . . This animal lived in the medium out of which it was made, as a cow lives in a meadow. But now the question arose, How did the cow get there? . . . The world was not born, it was made; made by the only maker that dare frame its fearful symmetry: God.

Collingwood is presumably alluding to the sixth and final stanza of William Blake's poem "The Tyger" [10]:

> Tyger Tyger burning bright,
> In the forests of the night,
> What immortal hand or eye,
> Dare frame thy fearful symmetry?

Like most animals, the tiger exhibits bilateral symmetry, the kind of symmetry shown in Figure 5.1, though this may not be what Blake is referring to: his illustration for the poem shows a tiger from the side, not the front. For Euclid, *συμμετρία* is what we now call commensurability; symmetry may also be balance and harmony in a non-mathematical sense [59]. But this would seem to be what Thales saw, or sought, in the world, and it is akin to his recognition of the unity underlying all isosceles triangles, a unity whereby the equality of the base angles of each of them can be established once for all. What made Thales a philosopher made him a mathematician.

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