CHAINS OF THEORIES AND COMPANIONABILITY

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Abstract. The theory of fields that are equipped with a countably infinite family of commuting derivations is not companionable; but if the axiom is added whereby the characteristic of the fields is zero, then the resulting theory is companionable. Each of these two theories is the union of a chain of companionable theories. In the case of characteristic zero, the model-companions of the theories in the chain form another chain, whose union is therefore the model-companion of the union of the original chain. However, in a signature with predicates, in all finite numbers of arguments, for linear dependence of vectors, the two-sorted theory of vector-spaces with their scalar-fields is companionable, and it is the union of a chain of companionable theories, but the model-companions of the theories in the chain are mutually inconsistent. Finally, the union of a chain of non-companionable theories may be companionable.

A **theory** in a given signature is a set of sentences, in the first-order logic of that signature, that is closed under logical implication. We shall consider chains $(T_m\colon m \in \omega)$ of theories: this means

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \qquad (*)$$

The signature of T_m will be $\mathscr{S}_m$, so automatically $\mathscr{S}_0 \subseteq \mathscr{S}_1 \subseteq \mathscr{S}_2 \subseteq \cdots$

In one motivating example, $\mathscr{S}_m$ is $\{0, 1, -, +, \cdot, \partial_0, \dots, \partial_{m-1}\}$, the signature of fields with m additional singulary operation-symbols; and T_m is m-DF, the theory of fields (of any characteristic) with m commuting derivations. In this example, each T_{m+1} is a **conservative extension** of T_m, that is, $T_{m+1} \supseteq T_m$ and every sentence in T_{m+1} of signature $\mathscr{S}_m$ is already in T_m. We establish this by showing that every model of T_m expands to a model of T_{m+1}. (This condition is sufficient, but not necessary [3, §2.6, exer. 8, p. 66].) If $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, then $(K, \partial_0, \dots, \partial_m) \models (m+1)$-DF, where ∂_m is the 0-derivation.

The union of the theories m-DF can be denoted by ω-DF: it is the theory of fields with ω-many commuting derivations. Each of the theories m-DF has a *model-companion,* called m-DCF [15]; but we shall show (as Theorem 3 below) that ω-DF has no model-companion. Let us recall that a **model-companion** of a theory T is a theory T^* in the same signature such that (1) $T_\forall = T^*{}_\forall$, that is, every model of one of the theories embeds in a model of the other, and (2) T^* is **model-complete,** that is, $T^* \cup \operatorname{diag}(\mathfrak{M})$ axiomatizes a complete theory for all models $\mathfrak{M}$ of T^*. Here $\operatorname{diag}(\mathfrak{M})$ is the quantifier-free theory of $\mathfrak{M}$ with parameters: equivalently, $\operatorname{diag}(\mathfrak{M})$ is the theory of all structures in which $\mathfrak{M}_M$ embeds. (These notions, with historical references, are reviewed further in [15].) A theory has at most one model-companion, by an argument with interwoven elementary chains.

Let m-DF$_0$ be m-DF with the additional requirement that the field have characteristic 0. Then m-DF$_0$ has a model-companion, called m-DCF$_0$ [10]. It is known (and will be proved anew below) that m-DCF$_0 \subseteq (m+1)$-DCF$_0$. It will follow then that the union ω-DF$_0$ of the m-DF$_0$ has a model-companion, which is the union of the m-DCF$_0$. This is by the following general result, a version of which

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has been observed also by Alice Medvedev [11, 12]. Again, the theories T_k are as in $(*)$ above.

Theorem 1. *Suppose each theory T_k has a model-companion ${T_k}^*$, and*

$$ {T_0}^* \subseteq {T_1}^* \subseteq {T_2}^* \subseteq \cdots \tag{†}$$

Then the theory $\bigcup_{k\in\omega} T_k$ has a model-companion, namely $\bigcup_{k\in\omega} {T_k}^*$.

Proof. Write U for $\bigcup_{k\in\omega} T_k$, and U^* for $\bigcup_{k\in\omega} {T_k}^*$. Suppose $\mathfrak{A} \models U$, and Γ is a finite subset of $U^* \cup \operatorname{diag}(\mathfrak{A})$. Then Γ is a subset of ${T_k}^* \cup \operatorname{diag}(\mathfrak{A} \upharpoonright \mathscr{S}_k)$ for some k in ω, and also $\mathfrak{A} \upharpoonright \mathscr{S}_k \models T_k$. Since $({T_k}^*)_\forall \subseteq T_k$, the structure $\mathfrak{A} \upharpoonright \mathscr{S}_k$ must embed in a model of ${T_k}^*$; and this model will be a model of Γ. We conclude that Γ is consistent. Therefore $U^* \cup \operatorname{diag}(\mathfrak{A})$ is consistent. Thus ${U^*}_\forall \subseteq U$. By symmetry $U_\forall \subseteq U^*$.

Similarly, if $\mathfrak{B} \models U^*$, then ${T_k}^* \cup \operatorname{diag}(\mathfrak{B} \upharpoonright \mathscr{S}_k)$ axiomatizes a complete theory in each case, and therefore $U^* \cup \operatorname{diag}(\mathfrak{B})$ is complete. □

The foregoing proof does not require that the signatures $\mathscr{S}_k$ form a chain, but needs only that every finite subset of $\bigcup_{k\in\omega} \mathscr{S}_k$ be included in some $\mathscr{S}_k$. This is the setting for Medvedev's [12, Prop. 2.4, p. 6], which then has the same proof as the foregoing. Also in Medvedev's setting, each ${T_{k+1}}^*$ is a conservative extension of ${T_k}^*$; but only the weaker assumption ${T_k}^* \subseteq {T_{k+1}}^*$ is needed in the proof.

Medvedev notes that many properties that the theories T_k might have are 'local' and are therefore preserved in $\bigcup_{k\in\omega} T_k$: examples are completeness, elimination of quantifiers, stability, and simplicity. In her main application, $\mathscr{S}_n$ is the signature of fields with singulary operation-symbols $\sigma_{m/n!}$, where $m \in \mathbb{Z}$; and T_n is the theory of fields on which the $\sigma_{m/n!}$ are automorphisms such that

$$\sigma_{k/n!} \circ \sigma_{m/n!} = \sigma_{(k+m)/n!}.$$

Then T_n includes the theory S_n of fields with the single automorphism $\sigma_{1/n!}$. Using [16, §1] (which is based on [3, ch. 5]), we may observe at this point that reduction of models of T_n to models of S_n is actually an equivalence of the categories $\operatorname{Mod}^{\subseteq}(T_n)$ and $\operatorname{Mod}^{\subseteq}(S_n)$, whose objects are models of the indicated theories, and whose morphisms are embeddings. We thus have at hand a (rather simple) instance of the hypothesis of the following theorem.

Theorem 2. *Suppose (I, J) is a bi-interpretation of theories S and T such that I is an equivalence of the categories $\operatorname{Mod}^{\subseteq}(S)$ and $\operatorname{Mod}^{\subseteq}(T)$. If S has the model-companion S^*, and $S \subseteq S^*$, then T also has a model-companion, which is the theory of those models $\mathfrak{B}$ of T such that $J(\mathfrak{B}) \models S^*$.*

Proof. The class of models $\mathfrak{B}$ of T such that $J(\mathfrak{B}) \models S^*$ is elementary. Let T^* be its theory. Then $T \subseteq T^*$. Suppose $\mathfrak{B} \models T$. Then $J(\mathfrak{B}) \models S$, so $J(\mathfrak{B})$ embeds in a model $\mathfrak{A}$ of S^*. Consequently $I(J(\mathfrak{B}))$ embeds in $I(\mathfrak{A})$. Also $I(\mathfrak{A}) \models T^*$, since $\mathfrak{A} \cong J(I(\mathfrak{A}))$. Since also $\mathfrak{B} \cong I(J(\mathfrak{B}))$, we conclude that $\mathfrak{B}$ embeds in a model of T^*. Finally, T^* is model-complete. Indeed, suppose now $\mathfrak{B}$ and $\mathfrak{C}$ are models of T^* such that $\mathfrak{B} \subseteq \mathfrak{C}$. An embedding of $J(\mathfrak{B})$ in $J(\mathfrak{C})$ is induced, and these structures are models of S^*, so the embedding is elementary. Therefore the induced embedding of $I(J(\mathfrak{B}))$ in $I(J(\mathfrak{C}))$ is also elementary. By the equivalence of the categories, $\mathfrak{B} \preccurlyeq \mathfrak{C}$. □

In the present situation, the theory S_n has a model-companion [9, 1]; let us denote this by ACFA_n. By the theorem then, T_n has a model-companion ${T_n}^*$, which is axiomatized by $T_n \cup \mathrm{ACFA}_n$. We have $\mathrm{ACFA}_n \subseteq {T_{n+1}}^*$ by [1, 1.12, Cor. 1, p. 3013]. By Theorem 1 then, $\bigcup_{n\in\omega} T_n$ has a model-companion, which is the union of the ${T_n}^*$. Medvedev calls this union $\mathbb{Q}$ACFA; she shows for example that it preserves the simplicity of the ACFA_n, as noted above, though it does not preserve their supersimplicity.

The following is similar to the result that the theory of fields with a derivation *and* an automorphism (of the field-structure only) has no model-companion [14]. The obstruction lies in positive characteristics p, where all derivatives of elements with p-th roots must be 0.

Theorem 3. *The theory* ω-DF *has no model-companion.*

Proof. We use that an $\forall\exists$ theory T has a model-companion if and only if the class of its *existentially closed* models is elementary, and in this case the model-companion is the theory of this class [2]. (A model $\mathfrak{A}$ of T is an **existentially closed** model, provided that if $\mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preccurlyeq_1 \mathfrak{B}$, that is, all quantifier-free formulas over A that are soluble in $\mathfrak{B}$ are soluble in $\mathfrak{A}$.) For each n in ω, the theory ω-DF has an existentially closed model $\mathfrak{A}_n$, whose underlying field includes $\mathbb{F}_p(\alpha)$, where α is transcendental; and in this model,

$$\partial_k \alpha = \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then α has no p-th root in $\mathfrak{A}_n$. Therefore, in a non-principal ultraproduct of the $\mathfrak{A}_n$, α has no p-th root, although $\partial_n \alpha = 0$ for all n in ω, so that α does have a p-th root in some extension. Thus the ultraproduct is not an existentially closed model of ω-DF. Therefore the class of existentially closed models of ω-DF is not elementary. □

It follows then by Theorem 1 that m-DCF $\not\subseteq$ $(m+1)$-DCF for at least one m. In fact this is so for all m, since

$$m\text{-DCF} \vdash p = 0 \to \forall x \, \Big(\bigwedge_{i<m} \partial_i x = 0 \to \exists y \; y^p = x \Big)$$

(where $p = 0$ means $\overbrace{1 + \cdots + 1}^{p} = 0$, which is true precisely in models having characteristic p), but $(m+1)$-DCF does not entail this sentence, since

$$(m+1)\text{-DCF} \vdash \exists x \, \big(\bigwedge_{i<m} \partial_i x = 0 \land \partial_m x \neq 0 \big).$$

However, this observation by itself is not enough to establish the last theorem. For, by the results of [16], it is possible for each T_k to have a model-companion ${T_k}^*$, while $\bigcup_{k\in\omega} T_k$ has a model-companion that is not $\bigcup_{k\in\omega} {T_k}^*$. We may even require T_{k+1} to be a conservative extension of T_k.

Indeed, if $k > 0$, then in the notation of [16], VS_k is the theory of vector-spaces with their scalar-fields in the signature $\{+, -, \mathbf{0}, \circ, 0, 1, *, P^k\}$, where $\circ$ is multiplication of scalars, and $*$ is the action of the scalar-field on the vector-space, and P^k is k-ary linear dependence. In particular, P^2 may written also as $\|$. Then VS_k has a model-companion, ${\mathrm{VS}_k}^*$, which is the theory of k-dimensional vector-spaces over algebraically closed fields [16, Thm 2.3]. Let $\mathrm{VS}_\omega = \bigcup_{1\leqslant k<\omega} \mathrm{VS}_k$. (This was called VS_∞ in [16].) This theory has the model-companion ${\mathrm{VS}_\omega}^*$, which is the theory of infinite-dimensional vector-spaces over algebraically closed fields [16, Thm 2.4]. In particular ${\mathrm{VS}_\omega}^*$ is not the union of the ${\mathrm{VS}_k}^*$, because these are mutually inconsistent. We now turn this into a result about chains:

Theorem 4. *If* $1 \leqslant n < \omega$, *let* T_n *be the theory axiomatized by* $\mathrm{VS}_1 \cup \cdots \cup \mathrm{VS}_n$. *Then* T_n *has a model-companion* ${T_n}^*$, *which is axiomatized by* $T_n \cup {\mathrm{VS}_n}^*$. *Also* T_{n+1} *is a conservative extension of* T_n. *However, the model-companion* ${\mathrm{VS}_\omega}^*$ *of the union* VS_ω *of the chain* $(T_n \colon 1 \leqslant n < \omega)$ *is not the union of the* ${T_n}^*$.

Proof. Every vector-space can be considered as a model of every VS_k and hence of every T_k. In particular, T_{n+1} is a conservative extension of T_n. If the theories ${T_n}^*$ are as claimed, then they are mutually inconsistent, and so ${\mathrm{VS}_\omega}^*$ is not their union. It remains to show that there are theories ${T_n}^*$ as claimed. We already know this when $n = 1$. For the other cases, if $1 \leqslant k < n$, we define the relations P^k in models of VS_n of dimension at least n.

Let $\mathrm{VS}_n^{\mathrm{m}}$ the theory of such models: that is, $\mathrm{VS}_n^{\mathrm{m}}$ is axiomatized by VS_n and the requirement that the space have dimension at least n. The relation P^1 is defined in models of $\mathrm{VS}_n^{\mathrm{m}}$ (and indeed in models of VS_n) by the quantifier-free formula $\boldsymbol{x} = \boldsymbol{0}$. If $n > 2$, then there are existential formulas that, in each model of $\mathrm{VS}_n^{\mathrm{m}}$, define the relation $\parallel$ and its complement [16, §2, p. 431]. More generally, if $1 \leqslant k < n - 1$, then, using existential formulas, we can define P^{k+1} and its complement in models of $T_k \cup \mathrm{VS}_n^{\mathrm{m}}$ or just $\mathrm{VS}_k \cup \mathrm{VS}_n^{\mathrm{m}}$. Indeed, $\neg P^{k+1}\boldsymbol{x}_0 \cdots \boldsymbol{x}_k$ is equivalent to $\exists(\boldsymbol{x}_{k+1}, \dots, \boldsymbol{x}_{n-1})\ \neg P^n \boldsymbol{x}_0 \cdots \boldsymbol{x}_{n-1}$, and $P^{k+1}\boldsymbol{x}_0 \cdots \boldsymbol{x}_k$ is equivalent to

$$\exists(\boldsymbol{x}_{k+1}, \dots, \boldsymbol{x}_n)\ \Bigg(P^k \boldsymbol{x}_1 \cdots \boldsymbol{x}_k \vee \Bigg(\neg P^n \boldsymbol{x}_1 \cdots \boldsymbol{x}_n \wedge \bigwedge_{j=k+1}^{n} P^n \boldsymbol{x}_0 \cdots \boldsymbol{x}_{j-1}\boldsymbol{x}_{j+1} \cdots \boldsymbol{x}_n\Bigg)\Bigg).$$

For, in a space of dimension at least n, if $(\boldsymbol{a}_0, \dots, \boldsymbol{a}_k)$ is linearly dependent, but $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k)$ is not, this means precisely that $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ is independent for some $(\boldsymbol{a}_{k+1}, \dots, \boldsymbol{a}_n)$, but $\boldsymbol{a}_0$ is a *unique* linear combination of $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$, and in fact of $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{j-1}, \boldsymbol{a}_{j+1}, \dots \boldsymbol{a}_n)$ whenever $k+1 \leqslant j \leqslant n$, and (therefore) of $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k)$.

By [16, Lem 1.1, 1.2], if $1 \leqslant k < n - 1$, we now have that reduction from models of $T_{k+1} \cup \mathrm{VS}_n^{\mathrm{m}}$ to models of $T_k \cup \mathrm{VS}_n^{\mathrm{m}}$ is an equivalence of the categories $\mathrm{Mod}^{\subseteq}(T_{k+1} \cup \mathrm{VS}_n^{\mathrm{m}})$ and $\mathrm{Mod}^{\subseteq}(T_k \cup \mathrm{VS}_n^{\mathrm{m}})$. Combining these results for all k, we have that reduction from models of $T_{n-1} \cup \mathrm{VS}_n^{\mathrm{m}}$ to models of $\mathrm{VS}_n^{\mathrm{m}}$ is an equivalence of the categories $\mathrm{Mod}^{\subseteq}(T_{n-1} \cup \mathrm{VS}_n^{\mathrm{m}})$ and $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_n^{\mathrm{m}})$. Since $\mathrm{VS}_n \subseteq \mathrm{VS}_n^{\mathrm{m}}$ and every model of VS_n embeds in a model of $\mathrm{VS}_n^{\mathrm{m}}$, the two theories have the same model-companion, namely ${\mathrm{VS}_n}^*$. Similarly, T_n and $T_{n-1} \cup \mathrm{VS}_n^{\mathrm{m}}$ have the same model-companion; and by Theorem 2, this is axiomatized by $T_n \cup {\mathrm{VS}_n}^*$. □

A one-sorted version of the last theorem can be developed as follows. Let $\mathrm{VS}_n^{\mathrm{r}}$ comprise the sentences of $\mathrm{VS}_n^{\mathrm{m}}$ having one-sorted signature $\{\boldsymbol{0}, -, +, P^n\}$ of the sort of vectors alone. It is not obvious that all models of $\mathrm{VS}_n^{\mathrm{m}}$ can be furnished with scalar-fields to make them models of $\mathrm{VS}_n^{\mathrm{r}}$ again; but this will be the case. By [16, Thm 1.1], it is the case when $n = 2$: reduction of models of $\mathrm{VS}_2^{\mathrm{m}}$ to models of $\mathrm{VS}_2^{\mathrm{r}}$ is an equivalence of the categories $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_2^{\mathrm{m}})$ and $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_2^{\mathrm{r}})$. This reduction is therefore **conservative,** by the definition of [16, p. 426]. It is said further at [16, p. 431] that reduction from $\mathrm{VS}_n^{\mathrm{m}}$ to $\mathrm{VS}_n^{\mathrm{r}}$ is conservative when $n > 2$; but the details are not spelled out. However, the claim can be established as follows. Immediately, reduction from $\mathrm{VS}_2 \cup \mathrm{VS}_n^{\mathrm{m}}$ to $\mathrm{VS}_2^{\mathrm{r}} \cup \mathrm{VS}_n^{\mathrm{r}}$ is conservative. In particular, models of the latter set of sentences really are vector-spaces without their scalar-fields. It is noted in effect in the proof of Theorem 4 that reduction from $\mathrm{VS}_2 \cup \mathrm{VS}_n^{\mathrm{m}}$ to $\mathrm{VS}_n^{\mathrm{m}}$ is conservative. Furthermore, in models of the latter theory, the defining of parallelism and its complement is done with existential formulas *in the signature of vectors alone.* Therefore reduction from $\mathrm{VS}_2^{\mathrm{r}} \cup \mathrm{VS}_n^{\mathrm{r}}$ to $\mathrm{VS}_n^{\mathrm{r}}$ is conservative. We now have the following commutative diagram of reduction-functors, three of them being conservative, that is, being equivalences of categories.

$$\begin{array}{ccc} \mathrm{Mod}^{\subseteq}(\mathrm{VS}_2 \cup \mathrm{VS}_n^{\mathrm{m}}) & \longrightarrow & \mathrm{Mod}^{\subseteq}(\mathrm{VS}_n^{\mathrm{m}}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}^{\subseteq}(\mathrm{VS}_2^{\mathrm{r}} \cup \mathrm{VS}_n^{\mathrm{r}}) & \longrightarrow & \mathrm{Mod}^{\subseteq}(\mathrm{VS}_n^{\mathrm{r}}) \end{array}$$

Therefore the remaining reduction, from $\mathrm{VS}_n^{\mathrm{m}}$ to $\mathrm{VS}_n^{\mathrm{r}}$, must be conservative.

Now there is a version of Theorem 4 where T_n is axiomatized by $\mathrm{VS}_2^{\mathrm{r}} \cup \cdots \cup \mathrm{VS}_n^{\mathrm{r}}$. Indeed, by Theorem 2, T_n has a model-companion, which is the theory (in the same signature) of n-dimensional vector-spaces over algebraically closed fields; and the union of the T_n has a model-companion, which is the theory of infinite-dimensional vector-spaces over algebraically closed fields; but this theory is not the union of the model-companions of the T_n.

The implication $A \Rightarrow B$ in the following is used implicitly at [1, 1.12, p. 3013] to establish the result used above, that if (K, σ) is a model of ACFA, then so is (K, σ^m), assuming $m \geqslant 1$.

Theorem 5. *Assuming as usual $T_0 \subseteq T_1$, where each T_k has signature $\mathscr{S}_k$, we consider the following conditions.*

A. For every model $\mathfrak{A}$ of T_1 and model $\mathfrak{B}$ of T_0 such that

$$\mathfrak{A} \upharpoonright \mathscr{S}_0 \subseteq \mathfrak{B}, \tag{‡}$$

there is a model $\mathfrak{C}$ of T_1 such that

$$\mathfrak{A} \subseteq \mathfrak{C}, \qquad \mathfrak{B} \subseteq \mathfrak{C} \upharpoonright \mathscr{S}_0. \tag{§}$$

B. The reduct to $\mathscr{S}_0$ of every existentially closed model of T_1 is an existentially closed model of T_0.

C. T_0 has the Amalgamation Property: *if one model embeds in two others, then those two in turn embed in a fourth model, compatibly with the original embeddings.*

D. T_1 is $\forall\exists$ (so that every model embeds in an existentially closed model).

We have the two implications

$$A \implies B, \qquad B \;\&\; C \;\&\; D \implies A,$$

but there is no implication among the four conditions that does not follow from these. This is true, even if T_1 is required to be a conservative extension of T_0.

Proof. Suppose A holds. Let $\mathfrak{A}$ be an existentially closed model of T_1, and let $\mathfrak{B}$ be an arbitrary model of T_0 such that (‡) holds. By hypothesis, there is a model $\mathfrak{C}$ of T_1 such that (§) holds. Then $\mathfrak{A} \preccurlyeq_1 \mathfrak{C}$, and therefore $\mathfrak{A} \upharpoonright \mathscr{S}_0 \preccurlyeq_1 \mathfrak{C} \upharpoonright \mathscr{S}_0$, and *a fortiori* $\mathfrak{A} \upharpoonright \mathscr{S}_0 \preccurlyeq_1 \mathfrak{B}$. Therefore $\mathfrak{A} \upharpoonright \mathscr{S}_0$ must be an existentially closed model of T_0. Thus B holds.

Suppose conversely B holds, along with C and D. Let $\mathfrak{A} \models T_1$ and $\mathfrak{B} \models T_0$ such that (‡) holds. We establish the consistency of $T_1 \cup \mathrm{diag}(\mathfrak{A}) \cup \mathrm{diag}(\mathfrak{B})$. It is enough to show the consistency of

$$T_1 \cup \mathrm{diag}(\mathfrak{A}) \cup \{\exists \boldsymbol{x}\ \varphi(\boldsymbol{x})\}, \tag{¶}$$

where φ is an arbitrary quantifier-free formula of $\mathscr{S}_0(A)$ that is soluble in $\mathfrak{B}$. By D, there is an existentially closed model $\mathfrak{C}$ of T_1 that extends $\mathfrak{A}$. By B then, $\mathfrak{C} \upharpoonright \mathscr{S}_0$ is an existentially closed model of T_0 that extends $\mathfrak{A} \upharpoonright \mathscr{S}_0$. By C, both $\mathfrak{B}$ and $\mathfrak{C} \upharpoonright \mathscr{S}_0$ embed over $\mathfrak{A} \upharpoonright \mathscr{S}_0$ in a model of T_0. In particular, φ will be soluble in this model. Therefore φ is already soluble in $\mathfrak{C} \upharpoonright \mathscr{S}_0$ itself. Thus $\mathfrak{C}$ is a model of (¶). Therefore A holds.

The foregoing arguments eliminate the five possibilities marked X on the table below, where 0 means false, and 1, true. We give examples of each of the remaining

	1	X	2	3	4	X	5	6	7	X	8	9	10	X	X	11
A	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
B	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
C	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
D	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

cases, numbered according to the table. In each example, T_0 will be the reduct of T_1 to $\mathscr{S}_0$. We shall denote by $\mathscr{S}_{\mathrm{f}}$ the signature $\{+, \cdot, -, 0, 1\}$ of fields; and by $\mathscr{S}_{\mathrm{vs}}$, the signature $\{+, -, \mathbf{0}, \circ, 0, 1, *\}$ of vector-spaces as two-sorted structures.

1. We first give an example in which none of the four lettered conditions hold. Let $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}} \cup \{a, b\}$ and $\mathscr{S}_1 = \mathscr{S}_0 \cup \{c\}$. Let T_1 be the theory of fields of characteristic p with distinguished elements a, b, and c such that $\{a, c\}$ or $\{b, c\}$ is p-independent, and if $\{b, c\}$ is p-independent, then so is $\{b, c, d\}$ for some d. Then T_0 is the theory of fields of characteristic p in which, for some c, $\{a, c\}$ or $\{b, c\}$ is p-independent, and if $\{b, c\}$ is p-independent, then so is $\{b, c, d\}$ for some d. The negations of the four lettered conditions are established as follows. Throughout, a, b, c, and d will be algebraically independent over $\mathbb{F}_p$.

$\neg A$. We have

$$(\mathbb{F}_p(a, b^{1/p}, c), a, b, c) \models T_1, \qquad (\mathbb{F}_p(a, b^{1/p}, c^{1/p}), a, b) \models T_0,$$

but if $(\mathbb{F}_p(a, b^{1/p}, c), a, b, c)$ is a substructure of a model (K, a, b, c) of T_1, then K cannot contain $c^{1/p}$.

$\neg B$. T_0 has no existentially closed models, since an element of a model that is p-independent from a or b will always have a p-th root in some extension. Similarly, no model of T_1 in which $\{a, c\}$ is not p-independent is existentially closed. But T_1 does have existentially closed models, which are just the separably closed fields of characteristic p with p-basis $\{a, c\}$ and with an additional element b.

$\neg C$. T_0 does not have the Amalgamation Property, since $(\mathbb{F}_p(a, b^{1/p}, c), a, b)$ and $(\mathbb{F}_p(a^{1/p}, b, c, d), a, b)$ are models that do not embed in the same model over the common substructure $(\mathbb{F}_p(a, b, c), a, b)$, which is a model of T_0.

$\neg D$. T_1 is not $\forall\exists$, since, as we have already noted, models in which $\{a, c\}$ is not p-independent do not embed in existentially closed models.

2. For an example of the column headed by 2 in the table, we let $\mathscr{S}_0$ and $\mathscr{S}_1$ be as in 1; but now T_1 is the theory of fields of characteristic p with distinguished elements a, b, and c such that $\{a, c, d\}$ or $\{b, c, d\}$ is p-independent for some d. This ensures that T_1 has no existentially closed models, so B holds vacuously; but the other three conditions still fail.

3. T_0 and T_1 are the same theory, so A and B hold trivially; and this theory is the theory of vector-spaces of dimension at least 2, in the signature $\mathscr{S}_{\mathrm{vs}}$, so the theory neither has the Amalgamation Property, nor is $\forall\exists$.

4. T_1 is DF_p with the additional requirement that the field have p-dimension at least 2; and $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}}$, so T_0 is the theory of fields of characteristic p with p-dimension at least 2. The latter theory has the Amalgamation Property; but the other conditions fail. Indeed, let $(\mathbb{F}_p(a, b), D)$ be the model of T_1 in which $Da = 1$ and $Db = 0$: then the field $\mathbb{F}_p(a, b)$ embeds in $\mathbb{F}_p(a^{1/p}, b)$, which is a model of T_0, but D does not extend to this field. Also, T_0 has no existentially closed models; but T_1 does, and indeed it has a model-companion, namely DCF_p. Also T_1 is not $\forall\exists$, since T_0 is not: there is a chain of models of the latter, whose union is not a

model, and we can make the structures in the chain into models of T_1 by adding the zero derivation.

5. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}}$, and $\mathscr{S}_1 = \mathscr{S}_0 \cup \{a\}$. T_1 is the theory of fields of characteristic p with distinguished element a, which is p-independent from another element; so T_0 is (as in 4) the theory of fields of characteristic p with p-dimension at least 2. Then we already have that C holds. But A fails: just let $\mathfrak{A}$ be $(\mathbb{F}_p(a,b),a)$, and let $\mathfrak{B}$ be $\mathbb{F}_p(a^{1/p},b)$. Also T_1 has no existentially closed models, so B holds trivially, but T_1 is not $\forall\exists$.

6. T_0 and T_1 are the same, namely the theory of fields of characteristic p of positive p-dimension, in the signature of fields, so this theory has the Amalgamation Property, but is not $\forall\exists$.

7. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{vs}}$, $\mathscr{S}_1 = \mathscr{S}_0 \cup \{\|, \boldsymbol{a}, \boldsymbol{b}\}$, and T_1 is axiomatized by $\mathrm{VS}_2 \cup \{\boldsymbol{a} \nparallel \boldsymbol{b}\}$, so it is $\forall\exists$. Then T_0 is the theory of vector-spaces of dimension at least 2. As in Theorem 4 above, T_1 has a model-companion, namely the theory of vector-spaces over algebraically closed fields with basis $\{\boldsymbol{a}, \boldsymbol{b}\}$. But T_0 has no existentially closed models, since for all independent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in some model, the equation

$$x * \boldsymbol{a} + y * \boldsymbol{b} = \boldsymbol{0} \tag{$\|$}$$

is always soluble in some extension. Thus B fails. Then T_0 also does not have the Amalgamation Property, since the solutions of ($\|$) may satisfy $2x^2 = y^2$ in one extension, but $3x^2 = y^2$ in another. Similarly, A fails, since the reduct to $\mathscr{S}_0$ of a model of T_1 may embed in a model of T_0 in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

8. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{vs}} \cup \{\|\}$, $\mathscr{S}_1 = \mathscr{S}_0 \cup \{\boldsymbol{a}, \boldsymbol{b}\}$, and T_1 is axiomatized by VS_2 together with

$$\forall x\ \forall y\ (x * \boldsymbol{a} + y * \boldsymbol{b} = \boldsymbol{0} \to 2x^2 = y^2). \tag{$**$}$$

Then T_0 is the theory of vector-spaces such that either the dimension is at least 2, or the scalar field contains $\sqrt{2}$. As in 7, T_0 does not have the Amalgamation Property. The theory T_1 is $\forall\exists$. It also has the model $(\mathbb{Q} * \boldsymbol{a} \oplus \mathbb{Q} * \boldsymbol{b}, \boldsymbol{a}, \boldsymbol{b})$, and $\mathbb{Q} * \boldsymbol{a} \oplus \mathbb{Q} * \boldsymbol{b}$ embeds in the model $\mathbb{Q}(\sqrt{2}, \sqrt{3}) * \boldsymbol{a}$ of T_0 when we let $\boldsymbol{b} = \sqrt{3} * \boldsymbol{a}$; but then the latter space embeds in no space in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are as required by ($**$). So A fails. Finally, T_1 has a model-companion, axiomatized by ${\mathrm{VS}_2}^*$ together with

$$\exists x\ \exists y\ (x * \boldsymbol{a} + y * \boldsymbol{b} = \boldsymbol{0} \land 2x^2 = y^2 \land x \neq 0);$$

and T_0 has a model-companion, which is just ${\mathrm{VS}_2}^*$; so B holds.

9. T_0 and T_1 are both VS_1.

10. $T_1 = \mathrm{DF}_p$, and T_0 is the reduct to $\mathscr{S}_{\mathrm{f}}$, namely field-theory in characteristic p.

11. T_0 and T_1 are both field-theory. □

Now let $\omega\text{-DCF}_0 = \bigcup_{m \in \omega} m\text{-DCF}_0$. As noted above, by Theorem 1, we shall have that ω-DCF_0 is the model-companion of ω-DF_0, once we know

$$m\text{-DCF}_0 \subseteq (m+1)\text{-DCF}_0. \tag{$\dagger\dagger$}$$

In fact this inclusion is Tracey McGrail's [10, Proposition 5.1.5], which is based ultimately (through her Lemma 2.2.17 & Theorem 2.2.18) on E. R. Kolchin's [5, Lemma III.5, p. 137, & Lemma IV.2, p. 167]. The same inclusion ($\dagger\dagger$) is used in Omar León Sánchez's geometric characterization [7, 8, Theorem 3.4] of the models of $(m+1)$-DCF_0. In fact it is León Sánchez who has referred us to McGrail's result; but in his paper, the result is implicitly derived from [7, Corollary. 2.5], which is Kolchin's [6, Ch. 0, §4, Corollary 1, p. 11]. In our terms, Kolchin's result is that, if $(L, \partial_0, \dots, \partial_{m-1}) \models m\text{-DCF}_0$, and L has a sub-ring R that is closed under the ∂_i (where $i < m$), and ∂_m is a derivation from R to L that commutes with the other ∂_i, then $\partial_m = \tilde{\partial}_m \restriction R$ for some $\tilde{\partial}_m$ such that $(L, \partial_0, \dots, \partial_{m-1}, \tilde{\partial}_m) \models (m+1)\text{-DCF}_0$. In particular, Condition A of Theorem 5 holds when T_0 is m-DF_0

and T_1 is $(m+1)$-DF_0. Then Condition B holds, and therefore (††). In short, [7, Theorem 3.4] is an implicit application of Theorem 5. We can establish Kolchin's result "geometrically," using points rather than differential polynomials, as follows.

Lemma (Kolchin). *Condition A of Theorem 5 holds when T_0 is m-DF_0 and T_1 is $(m+1)$-DF_0.*

Proof. Suppose $(L, \partial_0, \dots, \partial_{m-1})$ is a model of m-DF_0, and L has a subfield K that is closed under the ∂_i (where $i < m$), and there is also a derivation ∂_m on K such that $(K, \partial_0 \restriction K, \dots, \partial_{m-1} \restriction K, \partial_m)$ is a model of $(m+1)$-DF_0. We shall include $(L, \partial_0, \dots, \partial_{m-1})$ in another model of m-DF_0, namely a model that expands to a model of $(m+1)$-DF_0 that includes $(K, \partial_0, \dots, \partial_m)$.

If $K = L$, we are done. Suppose $a \in L \smallsetminus K$. We shall define a differential field $(K\langle a\rangle, \tilde\partial_0, \dots, \tilde\partial_m)$, where $a \in K\langle a\rangle$, and for each i in m,

$$\tilde\partial_i \restriction K\langle a\rangle \cap L = \partial_i \restriction K\langle a\rangle \cap L, \qquad (‡‡)$$

and $\tilde\partial_m \restriction K = \partial_m$. Then we shall be able to repeat the process, in case $L \nsubseteq K\langle a\rangle$: we can work with an element of $L \smallsetminus K\langle a\rangle$ as we did with a. Ultimately we shall obtain the desired model of $(m+1)$-DF_0 with reduct that includes $(L, \partial_0, \dots, \partial_{m-1})$.

Considering ω^{m+1} as the set of $(m+1)$-tuples of natural numbers, we shall have

$$K\langle a\rangle = K(a^\sigma : \sigma \in \omega^{m+1}),$$

where

$$a^\sigma = \tilde\partial_0{}^{\sigma(0)} \cdots \tilde\partial_m{}^{\sigma(m)} a. \qquad (§§)$$

In particular then, by (‡‡), we must have

$$\sigma(m) = 0 \implies a^\sigma = \partial_0{}^{\sigma(0)} \cdots \partial_{m-1}{}^{\sigma(m-1)} a.$$

Using this rule, we make the definition

$$K_1 = K(a^\sigma : \sigma(m) = 0).$$

We may assume that the derivations $\tilde\partial_i$ have been defined so far that

$$i < m \implies \tilde\partial_i \restriction K_1 = \partial_i \restriction K_1, \qquad \tilde\partial_m \restriction K = \partial_m \restriction K. \qquad (¶¶)$$

Then (§§) holds when $\sigma(m) < 1$.

Now suppose that, for some positive j in ω, we have been able to define the field $K(a^\sigma : \sigma(m) < j)$, and for each i in m, we have been able to define $\tilde\partial_i$ as a derivation on this field, and we have been able to define $\tilde\partial_m$ as a derivation from $K(a^\sigma : \sigma(m) < j-1)$ to $K(a^\sigma : \sigma(m) < j)$, so that (¶¶) holds, and (§§) holds when $\sigma(m) < j$. We want to define the a^σ such that $\sigma(m) = j$, and we want to be able to extend the derivations $\tilde\partial_i$ appropriately.

If $i < m+1$, then, as in [15, §4.1], we let $\boldsymbol{i}$ denote the characteristic function of $\{i\}$ on $m+1$: that is, $\boldsymbol{i}$ will be the element of ω^{m+1} that takes the value 1 at i and 0 elsewhere. Considered as a product structure, ω^{m+1} inherits from ω the binary operations $-$ and $+$. For each i in $m+1$, we have a derivation $\tilde\partial_i$ from $K(a^\sigma : (\sigma + \boldsymbol{i})(m) < j)$ to $K(a^\sigma : \sigma(m) < j)$ such that (¶¶) holds, and also, if $\sigma(m) < j$, then

$$\sigma(i) > 0 \implies \tilde\partial_i a^{\sigma - \boldsymbol{i}} = a^\sigma. \qquad (***)$$

We now define the a^σ, where $\sigma(m) = j$, so that, first of all, we can extend $\tilde\partial_m$ so that (***) holds when $\sigma(m) = j$ and $i = m$; but we must also ensure that (***) can hold also when $\sigma(m) = j$ and $i < m$. To do this, we shall have to make an inductive hypothesis, which is vacuously satisfied when $j = 1$. We shall also proceed recursively again. More precisely, we shall refine the recursion that we are already engaged in.

We well-order the elements σ of ω^{m+1} by the linear ordering $\lhd$ determined by the left-lexicographic ordering of the $(m+1)$-tuples

$$\bigl(\sigma(m), \sigma(0)+\dots+\sigma(m-1), \sigma(0), \sigma(1), \dots, \sigma(m-2)\bigr).$$

Then $(\omega^{m+1}, \lhd)$ has the order-type of the ordinal ω^2. This is a difference from the linear ordering defined in [15, §4.1] and elsewhere. However, for all σ and τ in ω^{m+1}, and all i in $m+1$, we still have

$$\sigma \lhd \tau \implies \sigma + \boldsymbol{i} \lhd \tau + \boldsymbol{i}.$$

We have assumed that, when $\tau = (0, \dots, 0, j)$, we have the field $K(a^\sigma \colon \sigma \lhd \tau)$, together with, for each i in $m+1$, a derivation $\tilde{\partial}_i$ from $K(a^\xi \colon \xi + \boldsymbol{i} \lhd \tau)$ to $K(a^\xi \colon \xi \lhd \tau)$ such that (¶¶) holds, and also, if $\sigma \lhd \tau$, then ($***$) holds. We have noted that we *can* have all of this when $\tau = (0, \dots, 0, 1)$. Suppose we have all of this for *some* τ in ω^{m+1} such that $(0, \dots, 0, 1) \trianglelefteqslant \tau$, that is, $\tau(m) > 0$. We want to define the extension $K(a^\sigma \colon \sigma \trianglelefteqslant \tau)$ of $K(a^\sigma \colon \sigma \lhd \tau)$ so that we can extend the $\tilde{\partial}_i$ appropriately. For defining a^τ, there are two cases to consider. We use the rules for derivations gathered, for example, in [14, Fact 1.1].

1. If $a^{\tau - \boldsymbol{m}}$ is algebraic over $K(a^\xi \colon \xi \lhd \tau - \boldsymbol{m})$, then the derivative $\tilde{\partial}_m a^{\tau-\boldsymbol{m}}$ is determined as an element of $K(a^\xi \colon \xi \lhd \tau)$; we let a^τ be this element.
2. If $a^{\tau - \boldsymbol{m}}$ is not algebraic over $K(a^\xi \colon \xi \lhd \tau - \boldsymbol{m})$, then we let a^τ be transcendental over $L(a^\xi \colon \xi \lhd \tau)$. We are then free to define $\tilde{\partial}_m a^{\tau-\boldsymbol{m}}$ as a^τ. (We require a^τ to be transcendental over $L(a^\xi \colon \xi \lhd \tau)$, and not just over $K(a^\xi \colon \xi \lhd \tau)$, so that we can establish (‡‡) later.)

We now check that, when $i < m$ and $\tau(i) > 0$, we can define $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ as a^τ. Here we make the inductive hypothesis mentioned above, namely that the foregoing two-part definition of a^τ was already used to define $a^{\tau-\boldsymbol{i}}$. Again we consider two cases.

1. Suppose $a^{\tau-\boldsymbol{i}}$ is algebraic over $K(a^\xi \colon \xi \lhd \tau - \boldsymbol{i})$. Then $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ is determined as an element of $K(a^\xi \colon \xi \lhd \tau)$. Thus the value of the bracket $[\tilde{\partial}_i, \tilde{\partial}_m]$ at $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ is determined: indeed, we have

$$[\tilde{\partial}_i, \tilde{\partial}_m] a^{\tau-\boldsymbol{i}-\boldsymbol{m}} = \tilde{\partial}_i \tilde{\partial}_m a^{\tau-\boldsymbol{i}-\boldsymbol{m}} - \tilde{\partial}_m \tilde{\partial}_i a^{\tau-\boldsymbol{i}-\boldsymbol{m}} = \tilde{\partial}_i a^{\tau-\boldsymbol{i}} - a^\tau.$$

   By inductive hypothesis, since $a^{\tau-\boldsymbol{i}}$ is algebraic over $K(a^\xi \colon \xi \lhd \tau - \boldsymbol{i})$, also $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ must be algebraic over $K(a^\xi \colon \xi \lhd \tau - \boldsymbol{i} - \boldsymbol{m})$. Since the bracket is 0 on this field, it must be 0 at $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ as well [15, Lem. 4.2].
2. If $a^{\tau-\boldsymbol{i}}$ is transcendental over $K(a^\xi \colon \xi \lhd \tau - \boldsymbol{i})$, then, since we are given $\tilde{\partial}_i$ as a derivation whose domain is this field, we are free to define $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ as a^τ.

Thus we have obtained $K(a^\xi \colon \xi \trianglelefteqslant \tau)$ as desired. By induction, we obtain the differential field $(K(a^\sigma \colon \sigma \in \omega^{m+1}), \tilde{\partial}_0, \dots, \tilde{\partial}_m)$ such that (§§) and (¶¶) hold.

It remains to check that (‡‡) holds. It is enough to show

$$K\langle a \rangle \cap L \subseteq K_1. \qquad (\dagger\dagger\dagger)$$

(We have the reverse inclusion.) Suppose $\tau \in \omega^{m+1}$ and $\tau(m) > 0$. By the definition of a^τ,

$$a^\tau \in K(a^\sigma \colon \sigma \lhd \tau)^{\mathrm{alg}} \implies a^\tau \in K(a^\sigma \colon \sigma \lhd \tau), \qquad (\ddagger\ddagger\ddagger)$$

$$a^\tau \notin K(a^\sigma \colon \sigma \lhd \tau)^{\mathrm{alg}} \implies a^\tau \notin L(a^\sigma \colon \sigma \lhd \tau)^{\mathrm{alg}}. \qquad (§§§)$$

Suppose $b \in K\langle a \rangle \cap L$. Since $b \in K\langle a \rangle$, we have, for some τ in ω^{m+1}, that b is a rational function over K_1 of those a^σ such that $\boldsymbol{m} \trianglelefteqslant \sigma \trianglelefteqslant \tau$. But then, by (‡‡‡), we do not need any a^σ that is algebraic over $K(a^\xi \colon \xi \lhd \sigma)$, since it actually belongs to this field. When we throw out all such a^σ, then, by (§§§), those that remain are algebraically independent over L. Thus we have

$$b \in K_1(a^{\sigma_0}, \dots, a^{\sigma_{n-1}}) \cap L$$

for some σ_j in ω^{m+1} such that $(a^{\sigma_0}, \dots, a^{\sigma_{n-1}})$ is algebraically independent over L. Therefore we may assume $n = 0$, and $b \in K_1$. Thus (†††) holds, and we have the differential field $(K\langle a\rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_m)$ fully as desired.

We have to be able to repeat this contruction, in case $L \not\subseteq K\langle a\rangle$. If $b \in L \smallsetminus K\langle a\rangle$, we have to be able to construct $K\langle a, b\rangle$, and so on. Let $L\langle a\rangle$ be the compositum of $K\langle a\rangle$ and L. Since m-DF_0 has the Amalgamation Property, we can extend the $\tilde{\partial}_i$, where $i < m$, to commutating derivations on the field $L\langle a\rangle$ that extend the original ∂_i on L. Thus we have a model $(L\langle a\rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$ of m-DF_0 and a model $(K\langle a\rangle, \tilde{\partial}_0 \restriction K\langle a\rangle, \dots, \tilde{\partial}_{m-1} \restriction K\langle a\rangle, \tilde{\partial}_m)$ of $(m+1)$-DF_0 that include, respectively, the models that we started with. Now we can continue as before, ultimately extending the domain of $\tilde{\partial}_m$ to include all of L. At limit stages of this process, we take unions, which is no problem, since m-DF_0 and $(m+1)$-DF_0 are $\forall\exists$. □

Theorem 6. *The theory* ω-DF_0 *has a model-companion, which is* ω-DCF_0. *This theory admits full elimination of quantifiers, is complete, and is properly stable.*

Proof. By Kolchin's lemma, and Theorem 5, Condition B of the theorem holds when T_0 is m-DF_0 and T_1 is $(m+1)$-DF_0: this means (††). Since m is arbitrary, it follows by Theorem 1 that ω-DCF_0 is the model-companion of ω-DF_0. Since the m-DCF_0 have the properties of quantifier-elimination, completeness, and stability [10], the observations of Medvedev noted earlier allow us to conclude that ω-DCF_0 also has these properties. Although each m-DCF_0 is actually ω-stable, ω-DCF_0 is not even superstable, since if A is a set of constants (in the sense that all of their derivatives are 0), then as σ ranges over A^ω, the sets $\{\partial_m x = \sigma(m) \colon m \in \omega\}$ belong to distinct complete types. □

Let us note by the way that, in Kolchin's lemma, we cannot establish Condition A of Theorem 5 in the stronger form in which the structure $\mathfrak{C}$ is required to be a mere *expansion* to $\mathscr{S}_1$ of $\mathfrak{B}$:

Proposition. *If* $m > 0$, *there is a model* $\mathfrak{K}$ *of* $(m+1)$-DF_0 *with a reduct that is included in a model* $\mathfrak{L}$ *of* m-DF_0, *while* $\mathfrak{L}$ *does not expand to a model of* $(m+1)$-DF_0 *that includes* $\mathfrak{K}$.

Proof. We generalize the example of [4] repeated in [13, Ex. 1.2, p. 927]. Suppose K is a pure transcendental extension $\mathbb{Q}(a^\sigma \colon \sigma \in \omega^{m+1})$ of $\mathbb{Q}$. We make this into a model of $(m+1)$-DF_0 by requiring $\partial_i a^\sigma = a^{\sigma + \boldsymbol{i}}$ in each case. Let L be the pure transcendental extension $K(b^\tau \colon \tau \in \omega^{m-1})$ of K. We make this into a model of m-DF_0 by extending the ∂_i so that, if $i < m-1$, we have $\partial_i b^\tau = b^{\tau + \boldsymbol{i}}$, while $\partial_{m-1} b^\tau$ is the element $a^{(\tau,0,0)}$ of K. Note that indeed if $i < m-1$, then

$$[\partial_i, \partial_{m-1}] b^\tau = \partial_i a^{(\tau,0,0)} - \partial_{m-1} b^{\tau + \boldsymbol{i}} = 0.$$

Suppose, if possible, ∂_m extends to L as well so as to commute with the other ∂_i. Then for any τ in ω^{m-1} we have $\partial_m b^\tau = f(b^\xi \colon \xi \in \omega^{m-1})$ for some polynomial f over K. But then, writing $\partial_\eta f$ for the derivative of f with respect to the variable indexed by η, we have, as by [14, Fact 1.1(0)],

$$\begin{aligned} a^{(\tau,0,1)} &= \partial_m\partial_{m-1}b^\tau \\ &= \partial_{m-1}\partial_m b^\tau \\ &= \partial_{m-1}(f(b^\xi\colon \xi\in\omega^{m-1})) \\ &= \sum_{\eta\in\omega^{m-1}} \partial_\eta f(b^\xi\colon \xi\in\omega^{m-1})\cdot a^{(\eta,0,0)} + f^{\partial_{m-1}}(b^\xi\colon \xi\in\omega^{m-1}), \end{aligned}$$

where the sum has only finitely many nonzero terms. The polynomial expression $f^{\partial_{m-1}}(b^\xi\colon \xi\in\omega^{m-1})$ cannot have $a^{(\tau,0,1)}$ as a constant term, since this is not $\partial_{m-1}x$ for any x in K. Thus we have obtained an algebraic relation among the b^σ and a^τ; but there can be no such relation. □

Finally, the union of a chain of non-companionable theories may be companionable:

Theorem 7. *In the signature $\{f\}\cup\{c_k\colon k\in\omega\}$, where f is a singular operation-symbol and the c_k are constant-symbols, let T_0 be axiomatized by the sentences*

$$\forall x\ \forall y\ (fx=fy\to x=y)$$

and, for each k in ω,

$$\forall x\ (f^{k+1}x\neq x),\qquad \forall x\ (fx=c_k\to x=c_{k+1}),\qquad fc_{k+2}=c_{k+1}\to fc_{k+1}=c_k.$$

For each n in ω, let T_{n+1} be axiomatized by

$$T_n\cup\{fc_{n+1}=c_n\}.$$

Then

(1) *each T_n is universally axiomatized, and* a fortiori $\forall\exists$, *so it does have existentially closed models;*
(2) *each T_n has the Amalgamation Property;*
(3) *every existentially closed model of T_{n+1} is an existentially closed model of T_n;*
(4) *no T_n is companionable;*
(5) *$\bigcup_{n\in\omega} T_n$ is companionable.*

Proof. Let $\mathfrak{A}_m$ be the model of T_0 with universe $\omega\times\omega$ such that

$$f^{\mathfrak{A}_m}(k,\ell)=(k,\ell+1),\qquad\qquad {c_k}^{\mathfrak{A}_m}=\begin{cases}(k-m,0), & \text{if } k>m,\\ (0,m-k), & \text{if } k\leqslant m.\end{cases}$$

Let $\mathfrak{A}_\omega$ be the model of T_0 with universe $\mathbb{Z}$ such that

$$f^{\mathfrak{A}_\omega}k=k+1,\qquad\qquad {c_k}^{\mathfrak{A}_\omega}=-k.$$

Then $\mathfrak{A}_m$ is a model of each T_k such that $k\leqslant m$; and $\mathfrak{A}_\omega$ is a model of each T_k. Moreover, each model of T_k consists of a copy of some $\mathfrak{A}_\beta$ such that $k\leqslant\beta\leqslant\omega$, along with some (or no) disjoint copies of ω and $\mathbb{Z}$ in which f is interpreted as $x\mapsto x+1$. Conversely, every structure of this form is a model of T_k. The β such that $\mathfrak{A}_\beta$ embeds in a given model of T_k is uniquely determined by that model. Consequently T_k has the Amalgamation Property. Also, a model of T_k is an existentially closed model if and only if includes no copies of ω (outside the embedded $\mathfrak{A}_\beta$): This establishes that every existentially closed model of T_{k+1} is an existentially closed model of T_k.

The existentially closed models of T_k are those models that omit the type $\{\forall y\ fy\neq x\}\cup\{x\neq c_j\colon j\in\omega\}$. In particular, $\mathfrak{A}_m$ is an existentially closed model of T_k, if $k\leqslant m$; but $\mathfrak{A}_m$ is elementarily equivalent to a structure that realizes the given type. Thus T_k is not companionable.

Finally, the model-companion of $\bigcup_{k\in\omega} T_k$ is axiomatized by this theory, together with $\forall x\ \exists y\ fy = x$. □

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