

# MSGSÜ, MAT 221

## Final Sınavı çözümleri

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Notlandıran için çözümlerinizin nasıl okunacağı açık olmalı.  
İngilizceyi veya Türkçeyi kullanabilirsiniz.

**Problem 1.** Let  $p$  be prime and  $0 < k < p$ . We know  $k! \cdot (p - k)! \mid p!$ . Show that

$$p \mid \frac{p!}{k! \cdot (p - k)!}.$$

*Solution.* We are given that  $p! = a \cdot k! \cdot (p - k)!$  for some  $a$ . We want to show  $p \mid a$ . We know  $p \mid p!$ , that is,

$$p \mid a \cdot k! \cdot (p - k)!.$$

But  $p \nmid k!$  and  $p \nmid (p - k)!$ . Therefore, by Euclid's Lemma,  $p \mid a$ .

**Problem 2.** Assume  $a$  is an odd integer. For all natural numbers  $k$ , show

$$a^{2^k} \equiv 1 \pmod{2^{k+2}}.$$

*Solution.* We use induction.

1. When  $k = 1$ , the claim is  $a^2 \equiv 1 \pmod{8}$  for all odd  $a$ . This claim is true, since  $(\pm 1)^2 \equiv 1$  and  $(\pm 3) \equiv 1 \pmod{8}$ .
2. Suppose the claim is true when  $k = \ell$ . We want to show  $2^{\ell+3} \mid a^{2^{\ell+1}} - 1$ . We have

$$a^{2^{\ell+1}} - 1 = \left(a^{2^\ell}\right)^2 - 1 = (a^{2^\ell} + 1)(a^{2^\ell} - 1).$$

By inductive hypothesis,  $2^{\ell+2} \mid a^{2^\ell} - 1$ . Since  $a$  is odd,  $2 \mid a^{2^\ell} + 1$ . Therefore  $2 \cdot 2^{\ell+2} \mid a^{2^{\ell+1}} - 1$ .

**Problem 3.** Find the least positive integer  $x$  such that

$$x \equiv 10 \pmod{20} \quad \& \quad x \equiv 3 \pmod{23}.$$

*Solution.* First apply the Euclidean algorithm to 20 and 23:

$$\begin{aligned} 23 &= 20 + 3, & 1 &= 3 - 2 \\ 20 &= 3 \cdot 6 + 2, & &= 3 - (20 - 3 \cdot 6) = 3 \cdot 7 - 20 \\ 3 &= 2 \cdot 1 + 1, & &= (23 - 20) \cdot 7 - 20 = 23 \cdot 7 - 20 \cdot 8. \end{aligned}$$

Then

$$\begin{aligned} x &\equiv 10 \cdot 23 \cdot 7 - 3 \cdot 20 \cdot 8 = 1610 - 480 \\ &= 1130 = 460 \cdot 2 + 210 \equiv 210 \pmod{460}. \end{aligned}$$

Thus  $x = 210$ .

**Problem 4.** Find the least positive integer  $x$  such that

$$7^{10\,000\,002} \equiv x \pmod{1375}.$$

*Solution.*  $1375 = 5 \cdot 275 = 5^2 \cdot 55 = 5^3 \cdot 11$ , so

$$\phi(1375) = 5^3 \cdot 11 \cdot \frac{4}{5} \cdot \frac{10}{11} = 5^3 \cdot 2^3 = 1000.$$

By Euler's Theorem,  $7^{1000} \equiv 1 \pmod{1375}$ . Since also

$$10\,000\,002 \equiv 2 \pmod{1000},$$

we have  $7^{10\,000\,002} \equiv 7^2 \pmod{1375}$ , and thus  $x = 49$ .