FOUNDATIONS OF ANALYSIS

THE ARITHMETIC OF WHOLE, RATIONAL, IRRATIONAL AND COMPLEX NUMBERS

A Supplement to Text-Books on the Differential and Integral Calculus

BY

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CHAPTER I

NATURAL NUMBERS

§ 1

Axioms

We assume the following to be given:

A set (i.e. totality) of objects called natural numbers, possessing the properties—called axioms—to be listed below.

Before formulating the axioms we make some remarks about the symbols $=$ and $\neq$ which will be used.

Unless otherwise specified, small italic letters will stand for natural numbers throughout this book.

If x is given and y is given, then

either x and y are the same number; this may be written

$$x = y$$

($=$ to be read "equals");

or x and y are not the same number; this may be written

$$x \neq y$$

($\neq$ to be read "is not equal to").

Accordingly, the following are true on purely logical grounds:

1) $$x = x$$

for every x.

2) If

$$x = y$$

then

$$y = x.$$

3) If

$$x = y, \quad y = z$$

then

$$x = z.$$

Thus a statement such as

$$a = b = c = d,$$

which on the face of it means merely that

$$a = b, \quad b = c, \quad c = d,$$

contains the additional information that, say,

$$a = c, \quad a = d, \quad b = d.$$

(Similarly in the later chapters.)

Now, we assume that the set of all natural numbers has the following properties:

Axiom 1: *1 is a natural number.*

That is, our set is not empty; it contains an object called 1 (read "one").

Axiom 2: *For each x there exists exactly one natural number, called the successor of x, which will be denoted by x'.*

In the case of complicated natural numbers x, we will enclose in parentheses the number whose successor is to be written down, since otherwise ambiguities might arise. We will do the same, throughout this book, in the case of $x + y$, xy, $x - y$, $-x$, x^y, etc.

Thus, if

$$x = y$$

then

$$x' = y'.$$

Axiom 3: *We always have*

$$x' \neq 1.$$

That is, there exists no number whose successor is 1.

Axiom 4: *If*

$$x' = y'$$

then

$$x = y.$$

That is, for any given number there exists either no number or exactly one number whose successor is the given number.

Axiom 5 (Axiom of Induction): *Let there be given a set $\mathfrak{M}$ of natural numbers, with the following properties:*

I) *1 belongs to $\mathfrak{M}$.*

II) *If x belongs to $\mathfrak{M}$ then so does x'.*

Then $\mathfrak{M}$ contains all the natural numbers.

§ 2

Addition

Theorem 1: *If*

$$x \neq y$$

then

$$x' \neq y'.$$

Proof: Otherwise, we would have

$$x' = y'$$

and hence, by Axiom 4,

$$x = y.$$

Theorem 2: $$x' \neq x.$$

Proof: Let $\mathfrak{M}$ be the set of all x for which this holds true.

I) By Axiom 1 and Axiom 3,

$$1' \neq 1;$$

therefore 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x' \neq x,$$

and hence by Theorem 1,

$$(x')' \neq x',$$

so that x' belongs to $\mathfrak{M}$.

By Axiom 5, $\mathfrak{M}$ therefore contains all the natural numbers, i.e. we have for each x that

$$x' \neq x.$$

Theorem 3: *If*

$$x \neq 1,$$

then there exists one (hence, by Axiom 4, exactly one) *u such that*

$$x = u'.$$

Proof: Let $\mathfrak{M}$ be the set consisting of the number 1 and of all those x for which there exists such a u. (For any such x, we have of necessity that

$$x \neq 1$$

by Axiom 3.)

I) 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then, with u denoting the number x, we have

$$x' = u',$$

so that x' belongs to $\mathfrak{M}$.

By Axiom 5, $\mathfrak{M}$ therefore contains all the natural numbers; thus for each

$$x \neq 1$$

there exists a u such that

$$x = u'.$$

Theorem 4, and at the same time **Definition 1:** *To every pair of numbers x, y, we may assign in exactly one way a natural number, called $x + y$* (+ to be read "plus"), *such that*

1) $x + 1 = x'$ *for every x,*

2) $x + y' = (x + y)'$ *for every x and every y.*

$x + y$ is called the sum of x and y, or the number obtained by addition of y to x.

Proof: A) First we will show that for each fixed x there is at most one possibility of defining $x + y$ for all y in such a way that

$$x + 1 = x'$$

and

$$x + y' = (x + y)' \qquad \text{for every } y.$$

Let a_y and b_y be defined for all y and be such that

$$a_1 = x', \qquad b_1 = x',$$
$$a_{y'} = (a_y)', \quad b_{y'} = (b_y)' \quad \text{for every } y.$$

Let $\mathfrak{M}$ be the set of all y for which

$$a_y = b_y.$$

I)
$$a_1 = x' = b_1;$$

hence 1 belongs to $\mathfrak{M}$.

II) If y belongs to $\mathfrak{M}$, then

$$a_y = b_y,$$

hence by Axiom 2,

$$(a_y)' = (b_y)',$$

therefore

$$a_{y'} = (a_y)' = (b_y)' = b_{y'},$$

so that y' belongs to $\mathfrak{M}$.

Hence $\mathfrak{M}$ is the set of all natural numbers; i.e. for every y we have

$$a_y = b_y.$$

B) Now we will show that for each x it is actually possible to define $x + y$ for all y in such a way that

$$x + 1 = x'$$

and

$$x + y' = (x + y)' \qquad \text{for every } y.$$

Let $\mathfrak{M}$ be the set of all x for which this is possible (in exactly one way, by A)).

I) For

$$x = 1,$$

the number

$$x + y = y'$$

is as required, since

$$x + 1 = 1' = x',$$
$$x + y' = (y')' = (x + y)'.$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let x belong to $\mathfrak{M}$, so that there exists an $x + y$ for all y. Then the number

$$x' + y = (x + y)'$$

is the required number for x', since

$$x' + 1 = (x + 1)' = (x')'$$

and

$$x' + y' = (x + y')' = ((x + y)')' = (x' + y)'.$$

Hence x' belongs to $\mathfrak{M}$.

Therefore $\mathfrak{M}$ contains all x.

Theorem 5 (Associative Law of Addition):

$$(x + y) + z = x + (y + z).$$

Proof: Fix x and y, and denote by $\mathfrak{M}$ the set of all z for which the assertion of the theorem holds.

I) $(x + y) + 1 = (x + y)' = x + y' = x + (y + 1)$;

thus 1 belongs to $\mathfrak{M}$.

II) Let z belong to $\mathfrak{M}$. Then

$$(x + y) + z = x + (y + z),$$

hence

$$(x + y) + z' = ((x + y) + z)' = (x + (y + z))' = x + (y + z)' = x + (y + z'),$$

so that z' belongs to $\mathfrak{M}$.

Therefore the assertion holds for all z.

Theorem 6 (Commutative Law of Addition):

$$x + y = y + x.$$

Proof: Fix y, and let $\mathfrak{M}$ be the set of all x for which the assertion holds.

I) We have

$$y + 1 = y',$$

and furthermore, by the construction in the proof of Theorem 4,

$$1 + y = y',$$

so that

$$1 + y = y + 1$$

and 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x + y = y + x,$$

therefore

$$(x + y)' = (y + x)' = y + x'.$$

By the construction in the proof of Theorem 4, we have

$$x' + y = (x + y)',$$

hence

$$x' + y = y + x',$$

so that x' belongs to $\mathfrak{M}$.

The assertion therefore holds for all x.

Theorem 7: $$y \neq x + y.$$

Proof: Fix x, and let $\mathfrak{M}$ be the set of all y for which the assertion holds.

I) $$1 \neq x',$$
$$1 \neq x + 1;$$

1 belongs to $\mathfrak{M}$.

II) If y belongs to $\mathfrak{M}$, then

$$y \neq x + y,$$

hence

$$y' \neq (x + y)',$$
$$y' \neq x + y',$$

so that y' belongs to $\mathfrak{M}$.

Therefore the assertion holds for all y.

Theorem 8: *If*

$$y \neq z$$

then

$$x + y \neq x + z.$$

Proof: Consider a fixed y and a fixed z such that

$$y \neq z,$$

and let $\mathfrak{M}$ be the set of all x for which

$$x + y \neq x + z.$$

I)

$$y' \neq z',$$

$$1 + y \neq 1 + z;$$

hence 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x + y \neq x + z,$$

hence

$$(x + y)' \neq (x + z)',$$

$$x' + y \neq x' + z,$$

so that x' belongs to $\mathfrak{M}$.

Therefore the assertion holds always.

Theorem 9: *For given x and y, exactly one of the following must be the case:*

1)
$$x = y.$$

2) *There exists a u* (exactly one, by Theorem 8) *such that*

$$x = y + u.$$

3) *There exists a v* (exactly one, by Theorem 8) *such that*

$$y = x + v.$$

Proof: A) By Theorem 7, cases 1) and 2) are incompatible. Similarly, 1) and 3) are incompatible. The incompatibility of 2) and 3) also follows from Theorem 7; for otherwise, we would have

$$x = y + u = (x + v) + u = x + (v + u) = (v + u) + x.$$

Therefore we can have at most one of the cases 1), 2) and 3).

B) Let x be fixed, and let $\mathfrak{M}$ be the set of all y for which one (hence by A), exactly one) of the cases 1), 2) and 3) obtains.

I) For $y = 1$, we have by Theorem 3 that either

$$x = 1 = y \qquad \text{(case 1))}$$

or

$$x = u' = 1 + u = y + u \qquad \text{(case 2))}.$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let y belong to $\mathfrak{M}$. Then

either (case 1) for y)

$$x = y,$$

hence

$$y' = y + 1 = x + 1 \qquad \text{(case 3) for } y');$$

or (case 2) for y)

$$x = y + u,$$

hence if

$$u = 1,$$

then

$$x = y + 1 = y' \qquad \text{(case 1) for } y');$$

but if

$$u \neq 1,$$

then, by Theorem 3,

$$u = w' = 1 + w,$$

$$x = y + (1 + w) = (y + 1) + w = y' + w$$

$$\text{(case 2) for } y');$$

or (case 3) for y)

$$y = x + v,$$

hence

$$y' = (x + v)' = x + v'$$

$$\text{(case 3) for } y').$$

In any case, y' belongs to $\mathfrak{M}$.

Therefore we always have one of the cases 1), 2) and 3).

§ 3

Ordering

Definition 2: *If*

$$x = y + u$$

then

$$x > y.$$

($>$ to be read "is greater than.")

Definition 3: *If*

$$y = x + v$$

then

$$x < y.$$

($<$ to be read "is less than.")

Theorem 10: *For any given x, y, we have exactly one of the cases*

$$x = y, \quad x > y, \quad x < y.$$

Proof: Theorem 9, Definition 2 and Definition 3.

Theorem 11: *If*

$$x > y$$

then

$$y < x.$$

Proof: Each of these means that

$$x = y + u$$

for some suitable u.

Theorem 12: *If*

$$x < y$$

then

$$y > x.$$

Proof: Each of these means that

$$y = x + v$$

for some suitable v.

Definition 4:

$$x \geqq y$$

means

$$x > y \quad \textit{or} \quad x = y.$$

($\geqq$ to be read "is greater than or equal to.")

Definition 5:

$$x \leqq y$$

means

$$x < y \quad or \quad x = y.$$

($\leqq$ to be read "is less than or equal to.")

Theorem 13: *If*

$$x \geqq y$$

then

$$y \leqq x.$$

Proof: Theorem 11.

Theorem 14: *If*

$$x \leqq y$$

then

$$y \geqq x.$$

Proof: Theorem 12.

Theorem 15 (Transitivity of Ordering): *If*

$$x < y, \quad y < z,$$

then

$$x < z.$$

Preliminary Remark: Thus if

$$x > y, \quad y > z,$$

then

$$x > z,$$

since

$$z < y, \quad y < x,$$
$$z < x;$$

but in what follows I will not even bother to write down such statements, which are obtained trivially by simply reading the formulas backwards.

Proof: With suitable v, w, we have

$$y = x + v, \quad z = y + w,$$

hence

$$z = (x + v) + w = x + (v + w),$$
$$x < z.$$

Theorem 16: *If*

$$x \leqq y, \quad y < z \quad or \quad x < y, \quad y \leqq z,$$

then

$$x < z.$$

Proof: Obvious if an equality sign holds in the hypothesis: otherwise, Theorem 15 does it.

Theorem 17: *If*

$$x \leqq y, \quad y \leqq z,$$

then

$$x \leqq z.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise, Theorem 16 does it.

A notation such as

$$a < b \leqq c < d$$

is justified on the basis of Theorems 15 and 17. While its immediate meaning is

$$a < b, \quad b \leqq c, \quad c < d,$$

it also implies, according to these theorems, that, say

$$a < c, \quad a < d, \quad b < d.$$

(Similarly in the later chapters.)

Theorem 18: $$x + y > x.$$

Proof: $$x + y = x + y.$$

Theorem 19: *If*

$$x > y, \text{ or } x = y, \text{ or } x < y,$$

then

$$x + z > y + z, \text{ or } x + z = y + z, \text{ or } x + z < y + z,$$

respectively.

Proof: 1) If

$$x > y$$

then

$$x = y + u,$$
$$x + z = (y + u) + z = (u + y) + z = u + (y + z) = (y + z) + u,$$
$$x + z > y + z.$$

2) If

$$x = y$$

then clearly

$$x + z = y + z.$$

3) If

$$x < y$$

then

$$y > x,$$

hence, by 1),

$$y + z > x + z,$$
$$x + z < y + z.$$

Theorem 20: *If*

$$x + z > y + z, \text{ or } x + z = y + z, \text{ or } x + z < y + z,$$

then

$$x > y, \text{ or } x = y, \text{ or } x < y, \text{ respectively.}$$

Proof: Follows from Theorem 19, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 21: *If*

$$x > y, \quad z > u,$$

then

$$x + z > y + u.$$

Proof: By Theorem 19, we have

$$x + z > y + z$$

and

$$y + z = z + y > u + y = y + u,$$

hence

$$x + z > y + u.$$

Theorem 22: *If*

$$x \geqq y, \; z > u \text{ or } x > y, \; z \geqq u,$$

then

$$x + z > y + u.$$

Proof: Follows from Theorem 19 if an equality sign holds in the hypothesis, otherwise from Theorem 21.

Theorem 23: *If*

$$x \geqq y, \; z \geqq u,$$

then

$$x + z \geqq y + u.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise Theorem 22 does it.

Theorem 24: $x \geqq 1.$

Proof: Either

$$x = 1$$

or

$$x = u' = u + 1 > 1.$$

Theorem 25: *If*

$$y > x$$

then

$$y \geqq x + 1.$$

Proof:

$$y = x + u,$$

$$u \geqq 1,$$

hence

$$y \geqq x + 1.$$

Theorem 26: *If*

$$y < x + 1$$

then

$$y \leqq x.$$

Proof: Otherwise we would have

$$y > x$$

and therefore, by Theorem 25,

$$y \geqq x + 1.$$

Theorem 27: *In every non-empty set of natural numbers there is a least one* (i.e. one which is less than any other number of the set).

Proof: Let $\mathfrak{N}$ be the given set, and let $\mathfrak{M}$ be the set of all x which are $\leqq$ every number of $\mathfrak{N}$.

By Theorem 24, the set $\mathfrak{M}$ contains the number 1. Not every x belongs to $\mathfrak{M}$; in fact, for each y of $\mathfrak{N}$ the number $y + 1$ does not belong to $\mathfrak{M}$, since

$$y + 1 > y.$$

Therefore there is an m in $\mathfrak{M}$ such that $m + 1$ does not belong to $\mathfrak{M}$; for otherwise, every natural number would have to belong to $\mathfrak{M}$, by Axiom 5.

Of this m I now assert that it is $\leqq$ every n of $\mathfrak{N}$, and that it belongs to $\mathfrak{N}$. The former we already know. The latter is established by an indirect argument, as follows: If m did not belong to $\mathfrak{N}$, then for each n of $\mathfrak{N}$ we would have

$$m < n,$$

hence, by Theorem 25,

$$m + 1 \leqq n;$$

thus $m + 1$ would belong to $\mathfrak{M}$, contradicting the statement above by which m was introduced.

§ 4

Multiplication

Theorem 28 and at the same time **Definition 6:** *To every pair of numbers x, y, we may assign in exactly one way a natural number, called* $x \cdot y$ ($\cdot$ to be read "times"; however, the dot is usually omitted), *such that*

1) $x \cdot 1 = x$ *for every* x,
2) $x \cdot y' = x \cdot y + x$ *for every* x *and every* y.

$x \cdot y$ *is called the product of* x *and* y, *or the number obtained from multiplication of* x *by* y.

Proof (*mutatis mutandis*, word for word the same as that of Theorem 4): A) We will first show that for each fixed x there is at most one possibility of defining xy for all y in such a way that

$$x \cdot 1 = x$$

and

$$xy' = xy + x \text{ for every } y.$$

Let a_y and b_y be defined for all y and be such that

$$a_1 = x, \quad b_1 = x,$$
$$a_{y'} = a_y + x, \quad b_{y'} = b_y + x \text{ for every } y.$$

Let $\mathfrak{M}$ be the set of all y for which

$$a_y = b_y.$$

I)
$$a_1 = x = b_1;$$

hence 1 belongs to $\mathfrak{M}$.

II) If y belongs to $\mathfrak{M}$, then

$$a_y = b_y,$$

hence

$$a_{y'} = a_y + x = b_y + x = b_{y'},$$

so that y' belongs to $\mathfrak{M}$.

Hence $\mathfrak{M}$ is the set of all natural numbers; i.e. for every y we have

$$a_y = b_y.$$

B) Now we will show that for each x, it is actually possible to define xy for all y in such a way that

$$x \cdot 1 = x$$

and

$$xy' = xy + x \text{ for every } y.$$

Let $\mathfrak{M}$ be the set of all x for which this is possible (in exactly one way, by A)).

I) For

$$x = 1,$$

the number

$$xy = y$$

is as required, since

$$x \cdot 1 = 1 = x,$$

$$xy' = y' = y + 1 = xy + x.$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let x belong to $\mathfrak{M}$, so that there exists an xy for all y. Then the number

$$x'y = xy + y$$

is the required number for x', since

$$x' \cdot 1 = x \cdot 1 + 1 = x + 1 = x'$$

and

$$x'y' = xy' + y' = (xy + x) + y' = xy + (x + y') = xy + (x + y)'$$
$$= xy + (x' + y) = xy + (y + x') = (xy + y) + x' = x'y + x'.$$

Hence x' belongs to $\mathfrak{M}$.

Therefore $\mathfrak{M}$ contains all x.

Theorem 29 (Commutative Law of Multiplication):

$$xy = yx.$$

Proof: Fix y, and let $\mathfrak{M}$ be the set of all x for which the assertion holds.

I) We have

$$y \cdot 1 = y,$$

and furthermore, by the construction in the proof of Theorem 28,

$$1 \cdot y = y,$$

hence

$$1 \cdot y = y \cdot 1,$$

so that 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$xy = yx,$$

hence

$$xy + y = yx + y = yx'.$$

By the construction in the proof of Theorem 28, we have

$$x'y = xy + y,$$

hence

$$x'y = yx',$$

so that x' belongs to $\mathfrak{M}$.

The assertion therefore holds for all x.

Theorem 30 (Distributive Law):

$$x(y + z) = xy + xz.$$

Preliminary Remark: The formula

$$(y + z)x = yx + zx$$

which results from Theorem 30 and Theorem 29, and similar analogues later on, need not be specifically formulated as theorems, nor even be set down.

Proof: Fix x and y, and let $\mathfrak{M}$ be the set of all z for which the assertion holds true.

I) $$x(y + 1) = xy' = xy + x = xy + x \cdot 1;$$

1 belongs to $\mathfrak{M}$.

II) If z belongs to $\mathfrak{M}$, then

$$x(y + z) = xy + xz,$$

hence

$$x(y + z') = x((y + z)') = x(y + z) + x = (xy + xz) + x$$
$$= xy + (xz + x) = xy + xz',$$

so that z' belongs to $\mathfrak{M}$.

Therefore, the assertion always holds.

Theorem 31 (Associative Law of Multiplication):

$$(xy)z = x(yz).$$

Proof: Fix x and y, and let $\mathfrak{M}$ be the set of all z for which the assertion holds true.

I) $$(xy) \cdot 1 = xy = x(y \cdot 1);$$

hence 1 belongs to $\mathfrak{M}$.

II) Let z belong to $\mathfrak{M}$. Then

$$(xy)z = x(yz),$$

and therefore, using Theorem 30,

$$(xy)z' = (xy)z + xy = x(yz) + xy = x(yz + y) = x(yz'),$$

so that z' belongs to $\mathfrak{M}$.

Therefore $\mathfrak{M}$ contains all natural numbers.

Theorem 32: *If*

$$x > y, \text{ or } x = y, \text{ or } x < y,$$

then

$$xz > yz, \text{ or } xz = yz, \text{ or } xz < yz, \text{ respectively.}$$

Proof: 1) If

$$x > y$$

then

$$x = y + u,$$
$$xz = (y + u)z = yz + uz > yz.$$

2) If

$$x = y$$

then clearly

$$xz = yz.$$

3) If

$$x < y$$

then

$$y > x,$$

hence by 1),

$$yz > xz,$$
$$xz < yz.$$

Theorem 33: *If*

$$xz > yz, \text{ or } xz = yz, \text{ or } xz < yz,$$

then

$$x > y, \text{ or } x = y, \text{ or } x < y, \text{ respectively.}$$

Proof: Follows from Theorem 32, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 34: *If*

$$x > y, \; z > u,$$

then

$$xz > yu.$$

Proof: By Theorem 32, we have

$$xz > yz$$

and

$$yz = zy > uy = yu,$$

hence

$$xz > yu.$$

Theorem 35: *If*

$$x \geqq y,\ z > u \ \textit{or}\ x > y,\ z \geqq u,$$

then

$$xz > yu.$$

Proof: Follows from Theorem 32 if an equality sign holds in the hypothesis; otherwise from Theorem 34.

Theorem 36: *If*

$$x \geqq y,\ z \geqq u,$$

then

$$xz \geqq yu.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise Theorem 35 does it.