Lineer Cebir (MAT 114)

Sınav *Çözümleri*

David Pierce, MSGSÜ

31 Mart 2017

Sınav, aşağıdaki sözlerle başladı:

> Çözüm yöntemlerinizi düşünerek seçin. Çözümlerinizi net bir şekilde yazın. Mümkünse cevaplarınızı kontrol edin. İyi çalışmalar dilerim!

Özel olarak Problem 2 ve Problem 4'ün cevapları kontrol edilebilir, dolayısıyla yanlış cevaplar sıfır puan alabilir. Her problem, 8 puandır.

Problem 1. $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$ *ve* $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$ *ise* $\det(AB)$ *hesaplayın.*

Çözüm. $\det(AB) = \det A \det B = (5!)^2 = (120)^2 = 14400.$

Not. Determinant göndermesi çarpımsal olduğundan AB çarpımını bulmak gerekmez.

Problem 2. $\left\{\begin{array}{rcrcrcrcrcr} x & + & 6y & + & z & - & t & - & u & = & 0 \\ -x & - & 6y & & & + & 5t & + & 2u & = & 1 \\ & & & & -z & - & 4t & & & = & 1 \\ -2x & - & 12y & & & + & 10t & + & 2u & = & -2 \end{array}\right\}$ *sistemi veriliyor.*

(a) Sistemi çözün.
(b) Karşılık gelen homojen sistemin temel çözümlerini verin.

Çözüm. (a)

$$\begin{pmatrix} 1 & 6 & 1 & -1 & -1 & 0 \\ -1 & -6 & 0 & 5 & 2 & 1 \\ 0 & 0 & -1 & -4 & 0 & 1 \\ -2 & -12 & 0 & 10 & 2 & -2 \end{pmatrix} \xrightarrow[2R_1+R_4]{R_1+R_2} \begin{pmatrix} 1 & 6 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & -1 & -4 & 0 & 1 \\ 0 & 0 & 2 & 8 & 0 & -2 \end{pmatrix} \xrightarrow[-2R_2+R_4]{R_2+R_3}$$

$$\begin{pmatrix} 1 & 6 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -2 & -4 \end{pmatrix} \xrightarrow{2R_3+R_4} \begin{pmatrix} 1 & 6 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_3+R_1]{-R_3+R_2}$$

$$\begin{pmatrix} 1 & 6 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-R_2+R_1} \begin{pmatrix} 1 & 6 & 0 & -5 & 0 & 3 \\ 0 & 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

dolayısıyla sistemin çözümleri,

$$\begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} -6y+5t+3 \\ y \\ -4t-1 \\ t \\ 2 \end{pmatrix} = y \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$$

(b) $\begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ve $\begin{pmatrix} 5 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}$.

Not. Cevaplar kontrol edilebilir: $A = \begin{pmatrix} 1 & 6 & 1 & -1 & -1 \\ -1 & -6 & 0 & 5 & 2 \\ 0 & 0 & -1 & -4 & 0 \\ -2 & -12 & 0 & 10 & 2 \end{pmatrix}$ ise

$$A \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad A \begin{pmatrix} 5 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad A \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix}$$

Problem 3. $A = \begin{pmatrix} 0 & 5 & -1 \\ 18 & 2 & 9 \\ 26 & 3 & 13 \end{pmatrix}$ *ise* A^{-1}, $\det A$, *ve* $\operatorname{Ek} A$ *bulun.*

Çözüm.

$$\operatorname{Ek} A = \begin{pmatrix} \det\begin{pmatrix} 2 & 9 \\ 3 & 13 \end{pmatrix} & -\det\begin{pmatrix} 5 & -1 \\ 3 & 13 \end{pmatrix} & \det\begin{pmatrix} 5 & -1 \\ 2 & 9 \end{pmatrix} \\ -\det\begin{pmatrix} 18 & 9 \\ 26 & 13 \end{pmatrix} & \det\begin{pmatrix} 0 & -1 \\ 26 & 13 \end{pmatrix} & -\det\begin{pmatrix} 0 & -1 \\ 18 & 9 \end{pmatrix} \\ \det\begin{pmatrix} 18 & 2 \\ 26 & 3 \end{pmatrix} & -\det\begin{pmatrix} 0 & 5 \\ 26 & 3 \end{pmatrix} & \det\begin{pmatrix} 0 & 5 \\ 18 & 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -1 & -68 & 47 \\ 0 & 26 & -18 \\ 2 & 130 & -90 \end{pmatrix}.$$

Kontrol ederiz:

$$\operatorname{Ek} A \cdot A = \begin{pmatrix} -1 & -68 & 47 \\ 0 & 26 & -18 \\ 2 & 130 & -90 \end{pmatrix} \begin{pmatrix} 0 & 5 & -1 \\ 18 & 2 & 9 \\ 26 & 3 & 13 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

dolayısıyla

$$\det A = -2, \qquad A^{-1} = \frac{1}{\det A} \operatorname{Ek} A = \begin{pmatrix} 1/2 & 34 & -47/2 \\ 0 & -13 & 9 \\ -1 & -65 & 45 \end{pmatrix}.$$

Not. Diğer yöntem daha uzundur:

$$\left(\begin{array}{cccccc} 0 & 5 & -1 & 1 & 0 & 0 \\ 18 & 2 & 9 & 0 & 1 & 0 \\ 26 & 3 & 13 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccccc} 18 & 2 & 9 & 0 & 1 & 0 \\ 0 & 5 & -1 & 1 & 0 & 0 \\ 26 & 3 & 13 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{18}R_1}$$

$$\left(\begin{array}{cccccc} 1 & 1/9 & 1/2 & 0 & 1/18 & 0 \\ 0 & 5 & -1 & 1 & 0 & 0 \\ 26 & 3 & 13 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-26R_1+R_3} \left(\begin{array}{cccccc} 1 & 1/9 & 1/2 & 0 & 1/18 & 0 \\ 0 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1/9 & 0 & 0 & -13/9 & 1 \end{array}\right) \xrightarrow{9R_3}$$

$$\left(\begin{array}{cccccc} 1 & 1/9 & 1/2 & 0 & 1/18 & 0 \\ 0 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -13 & 9 \end{array}\right) \xrightarrow{-5R_3+R_2} \left(\begin{array}{cccccc} 1 & 1/9 & 1/2 & 0 & 1/18 & 0 \\ 0 & 0 & -1 & 1 & 65 & -45 \\ 0 & 1 & 0 & 0 & -13 & 9 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{cccccc} 1 & 1/9 & 1/2 & 0 & 1/18 & 0 \\ 0 & 1 & 0 & 0 & -13 & 9 \\ 0 & 0 & -1 & 1 & 65 & -45 \end{array}\right) \xrightarrow{-R_3} \left(\begin{array}{cccccc} 1 & 1/9 & 1/2 & 0 & 1/18 & 0 \\ 0 & 1 & 0 & 0 & -13 & 9 \\ 0 & 0 & 1 & -1 & -65 & 45 \end{array}\right) \xrightarrow{-\frac{1}{2}R_3+R_1}$$

$$\left(\begin{array}{cccccc} 1 & 1/9 & 0 & 1/2 & 293/9 & -45/2 \\ 0 & 1 & 0 & 0 & -13 & 9 \\ 0 & 0 & 1 & -1 & -65 & 45 \end{array}\right) \xrightarrow{-\frac{1}{9}R_2+R_1} \left(\begin{array}{cccccc} 1 & 0 & 0 & 1/2 & 34 & -47/2 \\ 0 & 1 & 0 & 0 & -13 & 9 \\ 0 & 0 & 1 & -1 & -65 & 45 \end{array}\right),$$

$$A^{-1} = \begin{pmatrix} 1/2 & 34 & -47/2 \\ 0 & -13 & 9 \\ -1 & -65 & 45 \end{pmatrix}, \qquad \det A = (-1)^3 \cdot \frac{1}{9} \cdot 18 = -2,$$

$$\operatorname{Ek} A = \det A \cdot A^{-1} = \begin{pmatrix} -1 & -68 & 47 \\ 0 & 26 & -18 \\ 2 & 130 & 90 \end{pmatrix}.$$

Problem 4. *a'nın ve b'nin hangi değerleri için* $\left\{\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 1 \\ -2x & + & ay & + & 2z & = & b \\ & & y & + & (a-1)z & = & 0 \end{array}\right\}$ *sisteminin*

(a) tek bir çözümü vardır?

(b) birden fazla çözümü vardır?

(c) hiç çözümü yoktur?

Çözüm.

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ -2 & a & 2 & b \\ 0 & 1 & a-1 & 0 \end{pmatrix} \xrightarrow{2R_1+R_2} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & a+4 & 0 & b+2 \\ 0 & 1 & a-1 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & a-1 & 0 \\ 0 & a+4 & 0 & b+2 \end{pmatrix} \xrightarrow{-(a+4)R_2+R_3} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & a-1 & 0 \\ 0 & 0 & -(a+4)(a-1) & b+2 \end{pmatrix}$$

dolayısıyla
(a) $a \notin \{-4, 1\}$ ise tek bir çözümü vardır.
(b) $a \in \{-4, -1\}$ ve $b = -2$ ise birden fazla çözümü vardır.
(c) $a \in \{-4, -1\}$ ve $b \neq -2$ ise hiç çözümü yoktur.