Linear Algebra Problems

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Sources include

- Cemal Koç, *Topics in Linear Algebra*
- Steven Roman, *Advanced Linear Algebra* (2d ed.)

Problem 1. $X \mapsto \tau_X$, where $\tau_A(Y) = AY$, is an isomorphism from M to $\mathcal{L}(K^n, K^n)$.

Problem 2. If V is a finite-dimensional vector space, τ is an endomorphism of V (in other words, a linear operator on V), and τ and τ^2 have the same rank, show that the kernel and image of τ have trivial intersection.

Problem 3. Supposing $\{\boldsymbol{a}_k : k \in n\}$ is a basis of V, define τ_{ij} in $\mathcal{L}(V)$ by

$$\tau_{ij}(\boldsymbol{a}_k) = \begin{cases} \boldsymbol{a}_k & \text{if } k \neq i, \\ \boldsymbol{a}_i + \boldsymbol{a}_j & \text{if } k = i. \end{cases}$$

(a) Under what conditions on the scalar field is each τ_{ij} invertible?
(b) Under what conditions on the scalar field do the τ_{ij} compose a basis of $\mathcal{L}(V)$?

Problem 4. Prove concerning determinants:
(a) $\det(\dots, \boldsymbol{v}, \dots, \boldsymbol{v}, \dots) = 0$,
(b) $\det A \cdot \mathrm{I} = A \operatorname{adj}(A)$,
(c) $\det A^{\mathrm{t}} = \det A$.

Problem 5. If an abelian group has a multiplication (meaning a binary operation distributing in both senses over addition), and this has both a left and a right identity, show that they are equal.

Problem 6. For which 2×2 matrices is the minimal polynomial the same as the characteristic polynomial?

Problem 7. If A is a real matrix and $A^5+A^3+A = 0 = A^5-A^3-A$, what is the minimal polynomial of A?

Problem 8. Prove that the characteristic polynomial of every real 2×2 symmetric matrix is the product of linear factors.

Problem 9. Show that AB and BA have the same eigenvalues.

Problem 10. If A is diagonalizable, show the same for every polynomial function $f(A)$ of A.

Problem 11. If A is **nilpotent,** that is, some power of A is the zero matrix, show A is not diagonalizable.

Problem 12. Find the Jordan normal form of every 5×5 nilpotent matrix.

Problem 13. Find the Jordan normal form of every $n \times n$ complex matrix of which some positive power is I.