Cebir I (MAT 531)

David Pierce

27 Aralık 2013

Problem 1. Prove or disprove:

a) $(\mathbb{Q}^+, \cdot)$ is a free abelian group.

b) $(\mathbb{Q}, +)$ is a free abelian group.

c) There are exactly 9 nonisomorphic abelian groups of order 1000.

d) All groups of order 143 are abelian.

e) All groups of order 143 are cyclic.

f) There is a simple group of order 385.

Solution.

a) $(\mathbb{Q}^+, \cdot)$ is a free abelian group, since if $(p_i \colon i \in \omega)$ is the list of primes, the map

$$(x_i \colon i \in \omega) \mapsto \prod_{i \in \omega} {p_i}^{x_i}$$

from $\sum_{i \in \omega} \mathbb{Z}$ to $\mathbb{Q}^+$ is an isomorphism.

b) $(\mathbb{Q}, +)$ is not a free abelian group. Indeed, suppose h is an embedding of a free abelian group $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ in $\mathbb{Q}$. If $h(1, 0, \dots) = a$, then $2b = a$ for some b in $\mathbb{Q}$, but b is not in the image of h.

c) There are exactly 9 nonisomorphic abelian groups of order 1000. For, $1000 = 2^3 \cdot 5^3$, so (by the Chinese Remainder Theorem) an abelian group of order 1000 is the direct sum of an abelian group of order 2^3 and an abelian group of order 5^3. there are three possibilities for each of these (by the classification of finitely generated abelian groups), namely

$$\mathbb{Z}_{p^3}, \qquad \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}, \qquad \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p.$$

d) All groups of order 143 are abelian. For, $143 = 11 \cdot 13$, and $13 \not\equiv 1 \pmod{11}$, so the Sylow subgroups are normal subgroups, and the group is isomorphic to $\mathbb{Z}_{11} \oplus \mathbb{Z}_{13}$.

e) All groups of order 143 are cyclic, since as we have just shown, they are isomorphic to $\mathbb{Z}_{11} \oplus \mathbb{Z}_{13}$, which is isomorphic to $\mathbb{Z}_{143}$ by the Chinese Remainder Theorem.

f) There is no simple group of order 385. For, $385 = 5 \cdot 7 \cdot 11$. Let n be the number of Sylow 7-subgroups. Then

$$n \mid 55, \qquad n \equiv 1 \pmod{7}.$$

From the first condition, $n \in \{1, 5, 11, 55\}$. Therefore $n = 1$. Thus there is a unique Sylow 7-subgroup, which is a proper normal subgroup.

Problem 2. Describe the following groups as products of cyclic groups.

a) $\langle a, b \mid a^7, b^5, a^4ba^6b^4 \rangle$

b) $\langle a, b \mid a^{11}, b^5, a^4ba^{10}b^4 \rangle$

Solution.

a) Let $G = \langle a, b \mid a^7, b^5, a^4ba^6b^4 \rangle$. Here $a^4 = bab^{-1}$. Thus all elements of G are of the form $a^i b^j$, and moreover

$$G = \langle a \rangle \rtimes \langle b \rangle.$$

Here $\langle a \rangle$ is either isomorphic to $\mathbb{Z}_7$ or trivial, and $\langle b \rangle$ is either isomorphic to $\mathbb{Z}_5$ or trivial. Since the map $x \mapsto 4x$ is an automorphism of $\mathbb{Z}_7$ of order 3, and $3 \nmid 5$, $\langle a \rangle$ will be trivial. Indeed, using $ba^kb^{-1} = a^{4^k}$, we have

$$a = b^3ab^{-3}, \quad a^4 = b^4ab^{-4}, \quad a^2 = a^{16} = b^5ab^{-5} = a.$$

Hence $a = \mathrm{e}$, so $G = \langle b \mid b^5 \rangle$, which is isomorphic to $\mathbb{Z}_5$.

b) Now let $G = \langle a, b \mid a^{11}, b^5, a^4ba^{10}b^4 \rangle$. Here again $a^4 = bab^{-1}$, but now $x \mapsto 4x$ is an automorphism of $\mathbb{Z}_{11}$ of order 5. So $G \cong \mathbb{Z}_{11} \rtimes \mathbb{Z}_5$. (Strictly we use van Dyck's Theorem here, according to which there is an epimorphism from G to $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ taking a to $(1, 0)$ and b to $(0, 1)$. Since G has at most as many elements as $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$, the epimorphism must be an isomorphism.)