PARABOLAS (*DRAFT*)

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We work with a cone with distinguished axial triangle. The cone is cut by a plane whose intersection with the plane of the base of the cone is perpendicular to the base of the axial triangle (extended as necessary). The intersection of the cutting plane with the conic surface is a **conic section.** The intersection of the cutting plane with the axial triangle is the **diameter** of the section; the diameter meets the section at a **vertex.** Straight lines in the plane of the section and parallel to its intersection with the base of the cone are **ordinate** lines. An ordinate line from the section to the diameter may be called an **ordinate;** it cuts off from the diameter (on the side of the vertex) an **abscissa.**

We assume further that the diameter is parallel to a side of the axial triangle. The intersection of the cutting plane and the conic surface is the conic section called a **parabola** (παραβολή *application*), for reasons to be established presently.

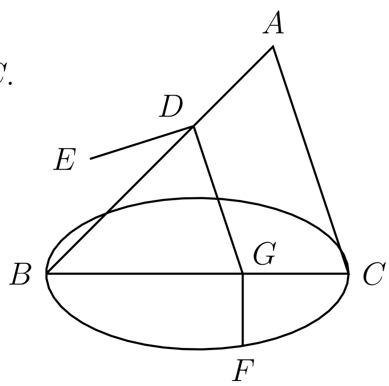
1. Propositions 11, 20

The **upright side** of the parabola is a certain straight line that

(i) in position is perpendicular to the plane of the parabola at the vertex of the parabola,
(ii) in length has to the segment of the side of the axial triangle between the vertex of the parabola and the apex of the triangle the same ratio that the square on the base of the axial triangle has to the rectangle on the other two sides.

Suppose in particular that the axial triangle is ABC, with apex A. Let the vertex of the parabola be D on AB. Then the upright side is DE, where

$$DE : DA :: BC^2 : BA \cdot AC.$$



Let an arbitrary point on the parabola be chosen, different from the vertex. Without loss of generality, let it be F, in the base of the cone. Let the ordinate FG be dropped to the axis. Then

$$FG^2 = BG \cdot GC.$$

But

$$BC : BA :: GC : DA,$$
$$BC : AC :: BG : DG,$$

and therefore, compounding these ratios, we have

$$BC^2 : BA \cdot AC :: GC \cdot BG : DA \cdot DG,$$

that is,

$$BC^2 : BA \cdot AC :: FG^2 : DA \cdot DG,$$

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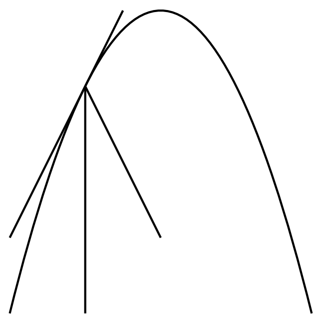
so that

$$DE : DA :: FG^2 : DA \cdot DG,$$
$$DE \cdot DG : DA \cdot DG :: FG^2 : DA \cdot DG,$$
$$DE \cdot DG = FG^2.$$

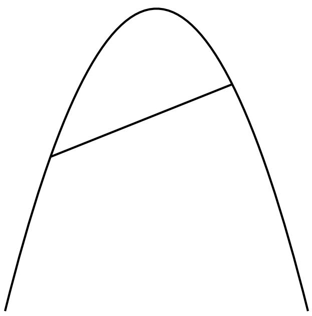
In particular, squares on ordinates are to each other in the ratio of the corresponding abscissas.

2. Proposition 17, 18, 19

Straight lines drawn ordinatewise at the vertex fall outside the section; the section need not be parabola. For, suppose a straight line drawn from the vertex passes within the section, but is not the diameter. Apollonius seems to slip here by assuming that this straight line must meet the section. If it does meet the section, then a chord is formed that is not bisected by the diameter: therefore that chord is not drawn ordinatewise.



Suppose this straight line does not meet the section. Then it must meet the chord of the section that is in the base of the cone. This chord is ordinate, and therefore the other straight line is not ordinate.



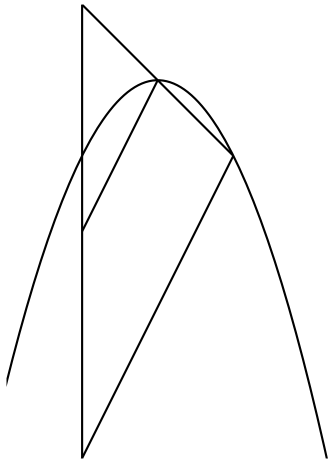
Straight lines within a section that are parallel to a chord meet the section in both directions; again, the section need not be a parabola. For, the chord cuts off a piece of the section. If the straight line is within the region bounded by the chord and this cut-off piece, then it must meet the section, as it cannot meet the chord. Suppose the straight line is on the other side of the chord. From one end of the chord, draw another chord to the other unbounded piece of the section. The original straight line meets this chord either inside or outside the section; in either case, it must meet the section. By symmetry, it meets the section in the other direction too.

Hence, in particular, ordinate lines within the section must meet the section in both directions.

3. Propositions 22, 24, 26

Every chord of the parabola meets the diameter (extended as necessary). For, from the ends of the chord, let ordinates be dropped. If these are equal, then there is only one corresponding abscissa, and the chord must consist of these two ordinates. If the ordinates are unequal, then the chord is not parallel to the diameter, so these two straight lines must meet.

Similarly, a straight line outside the parabola, but meeting it at a point, must meet the diameter. For, drop the ordinate from the meeting point, and drop another with a greater abscissa. (We can do this by 19.) The latter ordinate is longer; extended to meet the original straight line (as it must), it becomes longer still. Therefore the original straight line is not parallel to the diameter; so the two straight lines meet.



Hence a straight line parallel to the diameter meets the parabola at most once (and then it must pass within the parabola). And it meets at least once: For, if the line is within the section, the claim is obvious. Suppose it is outside the section. From a point A of the straight line, drop a straight line AB ordinatewise to the diameter. Within the parabola, find an abscissa CD that, with the upright side, makes a rectangle equal to AB^2. Let the corresponding ordinate (on the same side of the diameter as AB) be DE. Then AB and DE are equal and parallel; so AE is parallel to the diameter and is an extension of the original straight line.

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