NUMBER-THEORY EXERCISES, II.I

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Exercise 1. Let S be the set of Pythagorean triples (a, b, c) such that $\gcd(a, b, c) = 1$; a, b, and c are positive; and $a < b$. Order S by the rule

$$(a, b, c) < (d, e, f) \iff (a + b < d + e) \vee (a + b = d + e \wedge b < e).$$

Find the first few elements of S with respect to this ordering.

Exercise 2. Solve $x^2 + 4y^2 = z^2$.

Exercise 3. Solve $x^4 + y^2 = z^2$.

Exercise 4.

(a) Show that $f(x, y) = 0$ is soluble if and only if $f(3x + 2y, 4x + 3y) = 0$ is soluble.

(b) Find necessary and sufficient conditions on a, b, c, and d such that an arbitrary Diophantine equation $f(x, y) = 0$ is soluble if and only if $f(ax + by, cx + dy) = 0$ is soluble.

Exercise 5.

(a) Find the expansion of $\sqrt{d}$ as a continued fraction for various d, including 7.

(b) Solve the Pell equation $x^2 - dy^2 = 1$ for these d.

Exercise 6.

(a) Show $[a_0; a_1, \dots, a_k, a_{k+1}, \dots, a_n] = [a_0; a_1, \dots, a_k, [a_{k+1}, \dots, a_n]]$.

(b) Show $[a_0; a_1, \dots, a_k, a_{k+1}, \dots] = [a_0; a_1, \dots, a_k, [a_{k+1}, \dots]]$.

(c) Compute $[\overline{2; 1}]$ (which is $[2; 1, \overline{2, 1}]$) in terms of radicals.

(d) Show that $[a_0; a_1, \dots, a_k, \overline{a_{k+1}, \dots, a_n}]$ is always the root of a quadratic polynomial.

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