Revisions for *Sets and Classes,* 2007.03.02 ed.

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Some corrections and changes.

1 General

• A list of symbols should be provided.

• The relation symbolized by $\in$ and called **containment** on p. 34 would be better called **membership** (as on p. 19).

2 Significant changes

• p. 10, ¶ 1.1.2. The latter part of this paragraph needs to be rethought. *Set* and *class* are not the most 'generally applicable' collective nouns; they are the most abstract. For us, *set* will be the name of something whose members are other sets.

• P. 11: expand ¶ 1.2.2: say more about this 'correspondence' between $\cup$ and $\vee$.

• P. 21, ¶ 2.2.4, after the list: the comment 'depending on the axioms' makes the truth-value of $a \in a$ sound arbitrary. The Foundation Axiom will say that the sentence is false; but ¶ 8.3.5 shows that this axiom can be understood merely as a *definition* of the sets that we choose to study.

• ¶ 2.2.5: The sets that we study can be called *pure* sets (Moschovakis) or *hereditary* sets (Kunen).

• P. 22, ¶ 2.2.9: The alternative formulation of $\mathbf{V}$ should also use the 'official' language: so make it $\{x\colon x \in x \Rightarrow x \in x \Rightarrow x \in x\}$.

• P. 25: Not all of our theorems will have formal deductions even in principle: In ¶ 8.3.5, it will be noted that **WF** is a model of ZF. This conclusion can be formulated as an infinite list of sentences in the official language, each with a formal proof. We conclude that ZF is consistent: this can be formulated (as Gödel showed) as a single sentence of the official language; but it has no formal proof.

• Exercises might be added to Ch. 2.

• P. 32, bottom: the definition of **field** is not really needed.

• P. 34, ¶ 3.5.6 (ii): the reference should be to ¶ 3.5.5.

• P. 35, ¶ 3.5.7: a **least** element is also a **minimum** element.

• P. 36, ¶ 3.6.2: Instead of 'virtual class' here, we might speak of a **family** of classes. Then a class $\boldsymbol{C}$ and a binary relation $\boldsymbol{R}$ determine the families $\{x\boldsymbol{R} \colon x \in \boldsymbol{C}\}$ and $\{\boldsymbol{R}x \colon x \in \boldsymbol{C}\}$. In particular, if $\boldsymbol{E} \subseteq \boldsymbol{C} \times \boldsymbol{C}$ and is an equivalence-relation on $\boldsymbol{C}$, then $\boldsymbol{C}/\boldsymbol{E}$ is the family $\{x\boldsymbol{E} \colon x \in \boldsymbol{C}\}$.

• P. 37, ¶ 3.6.4: item (v) should be item (ii); also, in (vi), the notation π_0 and π_1 that will be used in ¶ 5.4.2 (Recursion with Parameter) can be introduced.

• P. 46: replace ¶ 4.2.4 with:

1. Suppose $(\boldsymbol{C}, \boldsymbol{F}, i)$ is a recursive structure. Then $\boldsymbol{C}$ can be denoted suggestively by

$$\{i, \boldsymbol{F}(i), \boldsymbol{F}(\boldsymbol{F}(i)), \dots\}.$$

Some possibilities are depicted in Fig. 4.1. Note well that possibly $\boldsymbol{F}$ is not injective, and possibly $i \in \boldsymbol{F}[\boldsymbol{C}]$. (However, these possibilities seem to be mutually exclusive.)

{par:rec}

2. Suppose again $(\boldsymbol{C}, \boldsymbol{F}, i)$ is a recursive structure, and $(\boldsymbol{D}, \boldsymbol{G}, j)$ is another iterative structure (not necessarily recursive). There *may be* a function $\boldsymbol{H}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that

(i) $\boldsymbol{H}(i) = j$,

(ii) $a \in \boldsymbol{C} \Rightarrow \boldsymbol{H}(\boldsymbol{F}(a)) = \boldsymbol{G}(\boldsymbol{H}(a))$, that is, $\boldsymbol{H} \circ \boldsymbol{F} = \boldsymbol{G} \circ \boldsymbol{H}$ on $\boldsymbol{C}$.

The first rule says what $\boldsymbol{H}(i)$ is; the second says how to obtain $\boldsymbol{H}(\boldsymbol{F}(a))$ from $\boldsymbol{H}(a)$. By induction, $\boldsymbol{H}$ is uniquely determined by these rules: see Corollary 5 below. In this case, we say that $\boldsymbol{H}$ is **recursively defined** by the given rules.

{par:rec2}

3. Note first another possible kind of recursive definition: $(\boldsymbol{C}, \boldsymbol{F}, i)$ is recursive, $\boldsymbol{E} \subseteq \boldsymbol{D}$, and $\boldsymbol{G} : \boldsymbol{D} \to \boldsymbol{D}$, then perhaps there is a sub-class $\boldsymbol{R}$ of $\boldsymbol{C} \times \boldsymbol{D}$ such that (in the notation of ¶ 3.6.2)

(i) $i\boldsymbol{R} = \boldsymbol{E}$,

(ii) $a \in \boldsymbol{C} \Rightarrow \boldsymbol{F}(a)\boldsymbol{R} = \boldsymbol{G}[\,a\boldsymbol{R}\,]$.

Then $\boldsymbol{R}$ too is uniquely determined by these rules, so it too is **recursively defined:**

{thm:rec-uni}

4 Theorem. *Suppose $(\boldsymbol{C}, \boldsymbol{F}, i)$ is recursive, $\boldsymbol{G} : \boldsymbol{D} \to \boldsymbol{D}$, and $\boldsymbol{E} \subseteq \boldsymbol{D}$. Then there is at most one relation $\boldsymbol{R}$ as in ¶ 3.*

Proof. Suppose $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ are two such relations. Let

$$\boldsymbol{C}_1 = \{x \colon x \in \boldsymbol{C} \mathbin{\&} x\boldsymbol{R}_0 = x\boldsymbol{R}_1\}.$$

Since $i\boldsymbol{R}_0 = \boldsymbol{E} = i\boldsymbol{R}_1$, we have $i \in \boldsymbol{C}_1$. Suppose $a \in \boldsymbol{C}_1$, so $a\boldsymbol{R}_0 = a\boldsymbol{R}_1$. Then

$$\boldsymbol{F}(a)\boldsymbol{R}_0 = \boldsymbol{G}[\,a\boldsymbol{R}_0\,] = \boldsymbol{G}[\,a\boldsymbol{R}_1\,] = \boldsymbol{F}(a)\boldsymbol{R}_1,$$

so $\boldsymbol{F}(a) \in \boldsymbol{C}_1$. By induction (and Lemma 4.2.3), $\boldsymbol{C}_1 = \boldsymbol{C}$. Since $\operatorname{dom}(\boldsymbol{R}_0) \subseteq \boldsymbol{C}$ and $\operatorname{dom}(\boldsymbol{R}_1) \subseteq \boldsymbol{C}$, we conclude that $\boldsymbol{R}_0 = \boldsymbol{R}_1$. □

{cor:rec-uni}

5 Corollary. *Suppose $(\boldsymbol{C}, \boldsymbol{F}, i)$ is recursive, and $(\boldsymbol{D}, \boldsymbol{G}, j)$ is iterative. Then there is at most one function $\boldsymbol{H}$ as in ¶ 3.*

Proof. The function $\boldsymbol{H}$ (if it exists) is a relation, namely a sub-class $\boldsymbol{R}$ of $\boldsymbol{C} \times \boldsymbol{D}$. Let $\boldsymbol{E} = \{\boldsymbol{H}(i)\}$. Then

(i) $i\boldsymbol{R} = \{\boldsymbol{H}(i)\} = \boldsymbol{E}$;

(ii) $a \in \boldsymbol{C} \Rightarrow \boldsymbol{F}(a)\boldsymbol{R} = \{\boldsymbol{H}(\boldsymbol{F}(a))\} = \{\boldsymbol{G}(\boldsymbol{H}(a))\} = \boldsymbol{G}[\,\{\boldsymbol{H}(a)\}\,] = \boldsymbol{G}[\,a\boldsymbol{R}\,]$.

By the theorem, $\boldsymbol{R}$ is unique, so $\boldsymbol{H}$ is unique. □

• P. 48, ¶ 4.3.5: The Recursion Theorem should be given more generally:

{thm:rec}

6 Theorem (Recursion)**.** *Suppose $(\boldsymbol{C}, \boldsymbol{F}, i)$ is an arithmetic structure, $\boldsymbol{G} \colon \boldsymbol{D} \to \boldsymbol{D}$, and $\boldsymbol{E} \subseteq \boldsymbol{D}$. Then there is (uniquely, by Theorem 4) a sub-class $\boldsymbol{R}$ of $\boldsymbol{C} \times \boldsymbol{D}$ such that*

(i) $i\boldsymbol{R} = \boldsymbol{E}$*,*

(ii) $a \in \boldsymbol{C} \Rightarrow \boldsymbol{F}(a)\boldsymbol{R} = \boldsymbol{G}[\,a\boldsymbol{R}\,]$*.*

Proof. Let $\boldsymbol{B}$ be the sub-class

$$\Big\{x \colon \forall y\ \Big(y \in x \Rightarrow \exists z\ \big(z \in \boldsymbol{E} \mathbin{\&} y = (i, z)\big) \vee \\ \vee \exists u\ \exists v\ \big((u, v) \in \boldsymbol{C} \times \boldsymbol{D} \cap x \mathbin{\&} y = (\boldsymbol{F}(u), \boldsymbol{G}(v))\big)\Big)\Big\}$$

of $\mathcal{P}(\boldsymbol{C} \times \boldsymbol{D})$, and let $\boldsymbol{R} = \bigcup \boldsymbol{B}$, so $\boldsymbol{R} \subseteq \boldsymbol{C} \times \boldsymbol{D}$. If $a \in \boldsymbol{E}$, then $\{(i, a)\}$ is a set (by the Pairing Axiom), so it belongs to $\boldsymbol{B}$, and hence $i\ \boldsymbol{R}\ a$. Suppose $b\ \boldsymbol{R}\ c$. Then $(b, c) \in d$ for some d in $\boldsymbol{B}$, so $d \cup \{(\boldsymbol{F}(b), \boldsymbol{G}(c))\}$ is a set (by the Weak Union Axiom), and this set belongs to $\boldsymbol{B}$, so $\boldsymbol{F}(b)\ \boldsymbol{R}\ \boldsymbol{G}(c)$. We now have the following characterization of $\boldsymbol{R}$:

$$a\ \boldsymbol{R}\ b \Leftrightarrow ((a = i \mathbin{\&} b \in \boldsymbol{E}) \vee \exists u\ \exists v\ (u\ \boldsymbol{R}\ v \mathbin{\&} a = \boldsymbol{F}(u) \mathbin{\&} b = \boldsymbol{G}(v))).$$

Since $i \notin \boldsymbol{F}[\,\boldsymbol{C}\,]$, we have $i\ \boldsymbol{R}\ a \Leftrightarrow a \in \boldsymbol{E}$, so $i\boldsymbol{R} = \boldsymbol{E}$. Since $\boldsymbol{F}$ is injective, if $a \in \boldsymbol{C}$, we have $\boldsymbol{F}(a)\ \boldsymbol{R}\ b \Leftrightarrow b \in \boldsymbol{G}[\,a\boldsymbol{R}\,]$, so $\boldsymbol{F}(a)\boldsymbol{R} = \boldsymbol{G}[\,a\boldsymbol{R}\,]$. □

{cor:rec}

7 Corollary. *Suppose $(\boldsymbol{C}, \boldsymbol{F}, i)$ is an arithmetic structure and $(\boldsymbol{D}, \boldsymbol{G}, j)$ is an iterative structure. Then there is (uniquely, by Corollary 5) a function $\boldsymbol{H}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that* *(i)* $\boldsymbol{H}(i) = j$,

(ii) $a \in \boldsymbol{C} \Rightarrow \boldsymbol{H}(\boldsymbol{F}(a)) = \boldsymbol{G}(\boldsymbol{H}(a))$, *that is,* $\boldsymbol{H} \circ \boldsymbol{F} = \boldsymbol{G} \circ \boldsymbol{H}$.

Proof. Exercise. □

• P. 51, end of § 4.4: 'We seem to have this if $i = \varnothing$**, and** $\boldsymbol{F}$ and $\boldsymbol{G}$ **are both** $x \mapsto x \cup \{x\}$.'

• P. 52, proof of 4.5.4, part (i): 'Then $b \neq \alpha$, since α is well-ordered by containment, and such orderings are by definition strict'.

• P. 53, ¶ 4.5.8 should begin: 'The structure $(\mathbf{ON}, x \mapsto x', \varnothing)$ satisfies the hypothesis of Theorem 4.4.3 with respect to the ordering $\in$; hence there is a class $\{\varnothing, \varnothing', \varnothing''\} \ldots$'.

• ¶¶ 5.2.1–2 can be replaced with the following:

8. We shall define the binary operation of *addition* on $\mathbb{N}$ so that

(i) $m + 0 = m$,

(ii) $m + n^+ = (m + n)^+$.

These rules tell how to add 0, and they tell how to add n^+, provided one can add n. The rules *will* determine a unique operation, by a variant of the Recursion Theorem (¶ 6). Moreover, suppose $(\boldsymbol{A}, \boldsymbol{S}, i)$ is a recursive structure. (This will be so throughout this section.) Then we shall be able to define addition on $\boldsymbol{A}$ by the rules

(i) $a + i = a$,

(ii) $a + \boldsymbol{S}(b) = \boldsymbol{S}(a + b)$,

even though the Recursion Theorem does not apply generally to all recursive structures.

{lem:+}

9 Lemma. *Suppose* $\boldsymbol{F}\colon \boldsymbol{B} \to \boldsymbol{C}$ *and* $\boldsymbol{G}\colon \boldsymbol{C} \to \boldsymbol{C}$. *Then there is a unique function* $\boldsymbol{H}$ *from* $\boldsymbol{B} \times \mathbb{N}$ *to* $\boldsymbol{C}$ *such that*

(i) $a\,\boldsymbol{H}\,0 = \boldsymbol{F}(a)$,

(ii) $a\,\boldsymbol{H}\,n^+ = \boldsymbol{G}(a\,\boldsymbol{H}\,n)$.

Proof. By the Recursion Theorem, there is a unique sub-class $\boldsymbol{R}$ of $\mathbb{N} \times (\boldsymbol{B} \times \boldsymbol{C})$ such that

(i) $0\boldsymbol{R} = \boldsymbol{F}$,

(ii) $n \in \mathbb{N} \Rightarrow n^+\boldsymbol{R} = \big((x, y) \mapsto (x, \boldsymbol{G}(y))\big)[n\boldsymbol{R}] = \{(x, \boldsymbol{G}(y))\colon n\ \boldsymbol{R}\ (x, y)\}$.

By induction, if $n \in \mathbb{N}$, then $n\boldsymbol{R}$ is a function from $\boldsymbol{B}$ to $\boldsymbol{C}$. Indeed, $0\boldsymbol{R}$ is such a function (namely $\boldsymbol{F}$), and if $n\boldsymbol{R}$ is such a function, then $n^+\boldsymbol{R}$ is its composition with $\boldsymbol{G}$. Let the function $n\boldsymbol{R}$ be denoted by $\boldsymbol{K}_n$; then $\boldsymbol{K}_{n^+} = \boldsymbol{G} \circ \boldsymbol{K}_n$. We can now define the binary function $\boldsymbol{H}$ as $(x, y) \mapsto \boldsymbol{K}_y(x)$ on $\boldsymbol{A} \times \mathbb{N}$. Then

(i) $a \boldsymbol{H}\, 0 = \boldsymbol{K}_0(a) = \boldsymbol{F}(a)$,

(ii) $a \boldsymbol{H}\, n^+ = \boldsymbol{K}_{n^+}(a) = \boldsymbol{G}(\boldsymbol{K}_n(a)) = \boldsymbol{G}(a \boldsymbol{H}\, n)$.

So $\boldsymbol{H}$ is as desired. To see that $\boldsymbol{H}$ is unique, note that $\boldsymbol{R}$ determines $\boldsymbol{H}$, and conversely. Indeed,

$$\begin{aligned}\boldsymbol{H} &= \{((x,y),z) \colon y \;\boldsymbol{R}\; (x,z)\},\\ \boldsymbol{R} &= \{(y,(x,z)) \colon x \boldsymbol{H}\, y = z\}.\end{aligned}$$

Since $\boldsymbol{R}$ uniquely satisfies the given conditions, so does $\boldsymbol{H}$. □

{thm:+}

10 Theorem and Definition. *Suppose $(\boldsymbol{A}, \boldsymbol{S}, i)$ is recursive. Then there is a unique binary operation of **addition** on $\boldsymbol{A}$ given by*

(i) $a + i = a$,

(ii) $a + \boldsymbol{S}(b) = \boldsymbol{S}(a + b)$.

Proof. By the lemma, there is a unique function $\boldsymbol{H}$ from $\boldsymbol{A} \times \mathbb{N}$ to $\boldsymbol{A}$ such that

(i) $a \boldsymbol{H}\, 0 = a$,

(ii) $a \boldsymbol{H}\, n^+ = \boldsymbol{S}(a \boldsymbol{H}\, n)$.

So $\boldsymbol{H}$ is recursively defined in its second argument. We shall show that it is also recursively definable in its first argument. First, let $\boldsymbol{F}$ be the function $x \mapsto i \boldsymbol{H}\, x$ from $\mathbb{N}$ into $\boldsymbol{A}$. Then

$$\begin{aligned}\boldsymbol{F}(0) &= i,\\ \boldsymbol{F}(n^+) &= \boldsymbol{S}(\boldsymbol{F}(n)).\end{aligned} \qquad (1)$$ {eqn:+i}

(So $\boldsymbol{F}$ is the unique homomorphism from $(\mathbb{N}, {}^+, 0)$ into $(\boldsymbol{A}, \boldsymbol{S}, i)$ guaranteed by Corollary 7.) By induction, $\operatorname{rng}(\boldsymbol{F}) = \boldsymbol{A}$; indeed, $i \in \operatorname{rng}(\boldsymbol{F})$, and $a \in \operatorname{rng}(\boldsymbol{F}) \Rightarrow \boldsymbol{S}(a) \in \operatorname{rng}(\boldsymbol{F})$. The equation

$$\boldsymbol{S}(a) \boldsymbol{H}\, n = \boldsymbol{S}(a \boldsymbol{H}\, n) \qquad (2)$$ {eqn:+n}

holds when $n = 0$, since $\boldsymbol{S}(a) \boldsymbol{H}\, 0 = \boldsymbol{S}(a) = \boldsymbol{S}(a \boldsymbol{H}\, 0)$. Suppose (2) holds for some n in $\mathbb{N}$. Then

$$\begin{aligned}\boldsymbol{S}(a) \boldsymbol{H}\, n^+ &= \boldsymbol{S}(\boldsymbol{S}(a) \boldsymbol{H}\, n) && \text{[by definition of } \boldsymbol{H}\text{]}\\ &= \boldsymbol{S}(\boldsymbol{S}(a \boldsymbol{H}\, n)) && \text{[by inductive hypothesis]}\\ &= \boldsymbol{S}(a \boldsymbol{H}\, n^+). && \text{[by definition of } \boldsymbol{H}\text{]}\end{aligned}$$

So (2) holds for all n in $\mathbb{N}$. Therefore each of the operations $x \mapsto x \boldsymbol{H}\, n$ is the operation $\boldsymbol{G}_n$ recursively defined by

(i) $\boldsymbol{G}_n(i) = \boldsymbol{F}(n)$,

(ii) $\boldsymbol{G}_n(\boldsymbol{S}(a)) = \boldsymbol{S}(\boldsymbol{G}_n(a))$.

In particular,
$$\boldsymbol{G}_m = \boldsymbol{G}_n \Leftrightarrow \boldsymbol{F}(m) = \boldsymbol{F}(n).$$
Now we can define addition on $\boldsymbol{A}$ by
$$a + b = c \Leftrightarrow \exists x\ (\boldsymbol{F}(x) = b\ \&\ \boldsymbol{G}_x(a) = c).$$
Then $a + i = \boldsymbol{G}_0(a) = a\,\boldsymbol{H}\,0 = a$. Also, if $b = \boldsymbol{F}(n)$, so that $\boldsymbol{S}(b) = \boldsymbol{F}(n^+)$, then
$$a + \boldsymbol{S}(b) = \boldsymbol{G}_{n^+}(a) = a\,\boldsymbol{H}\,n^+ = \boldsymbol{S}(a\,\boldsymbol{H}\,n) = \boldsymbol{S}(\boldsymbol{G}_n(a)) = \boldsymbol{S}(a + b).$$
Thus $+$ is as desired; it is unique by Theorem 4. □

• ¶ 5.2.3 should have a reference to Landau. ¶ 5.2.4 can be slightly rewritten:

11. Suppose (a, s, i) is a recursive *set,* so that all operations on a are sets. Then we can establish addition on a as follows. By Corollary 5, for each b in a, there is at most one singulary operation f_b on a such that

(i) $f_b(i) = b$,

(ii) $f_b \circ s = s \circ f_b$.

Let a_0 be the subset of a comprising those b such that f_b does exist; note well how this definition of a_0 requires f_b to be a set. Then f_i exists and is id_a, so $i \in a_0$. If $b \in a_0$, then $f_{s(b)}$ exists and is $s \circ f_b$. By induction, $a_0 = a$. Now we can define $b + c = f_b(c)$.

• At the beginning of § 5.2 should be inserted the following:

{lem:.}

12 Lemma. *Suppose* $\boldsymbol{F}\colon \boldsymbol{B} \to \boldsymbol{C}$ *and* $\boldsymbol{G}\colon \boldsymbol{C} \times \boldsymbol{B} \to \boldsymbol{C}$*. Then there is a unique function* $\boldsymbol{H}$ *from* $\boldsymbol{B} \times \mathbb{N}$ *to* $\boldsymbol{C}$ *such that*

(i) $a\,\boldsymbol{H}\,0 = \boldsymbol{F}(a)$,

(ii) $a\,\boldsymbol{H}\,n^+ = (a\,\boldsymbol{H}\,n)\,\boldsymbol{G}\,a$.

Proof. By the Recursion Theorem, there is a unique sub-class $\boldsymbol{R}$ of $\mathbb{N} \times (\boldsymbol{B} \times \boldsymbol{C})$ such that

(i) $0\boldsymbol{R} = \boldsymbol{F}$,

(ii) $n \in \mathbb{N} \Rightarrow n^+\boldsymbol{R} = \big((x, y) \mapsto (x, y\,\boldsymbol{G}\,x)\big)[n\boldsymbol{R}] = \{(x, y\,\boldsymbol{G}\,x))\colon n\ \boldsymbol{R}\ (x, y)\}$.

By induction, if $n \in \mathbb{N}$, then $n\boldsymbol{R}$ is a function $\boldsymbol{K}_n$ from $\boldsymbol{B}$ to $\boldsymbol{C}$. Indeed, $0\boldsymbol{R}$ is such a function, namely $\boldsymbol{F}$; this then is $\boldsymbol{K}_0$. If $n\boldsymbol{R}$ is such a function, as $\boldsymbol{K}_n$, then $n^+\boldsymbol{R}$ is $x \mapsto \boldsymbol{H}_n(x)\,\boldsymbol{G}\,x$; this then is $\boldsymbol{K}_{n^+}$. We can now define the binary function $\boldsymbol{H}$ as $(x, y) \mapsto \boldsymbol{K}_y(x)$ on $\boldsymbol{A} \times \mathbb{N}$. Then

(i) $a\,\boldsymbol{H}\,0 = \boldsymbol{K}_0(a) = \boldsymbol{F}(a)$,

(ii) $a\,\boldsymbol{H}\,n^+ = \boldsymbol{K}_{n^+}(a) = \boldsymbol{K}_n(a)\,\boldsymbol{G}\,a = (a\,\boldsymbol{H}\,n)\,\boldsymbol{G}\,a$.

So $\boldsymbol{H}$ is as desired; its uniqueness is as in Lemma 9. □

• The proof of Theorem 5.3.2 can be supplied as follows:

Proof. We follow the pattern of the proof of Theorem 10. By the lemma, there is a unique function $\boldsymbol{H}$ from $\boldsymbol{A} \times \mathbb{N}$ into $\boldsymbol{A}$ such that

(i) $a \,\boldsymbol{H}\, 0 = i$,

(ii) $a \,\boldsymbol{H}\, n^{+} = a \,\boldsymbol{H}\, n + a$.

By induction, $i \,\boldsymbol{H}\, n = i$ for all n in $\mathbb{N}$; indeed, this is given when $n = 0$, and if it holds when $n = m$, then $i \,\boldsymbol{H}\, m^{+} = i \,\boldsymbol{H}\, m + i = i \,\boldsymbol{H}\, m = i$. Let $\boldsymbol{F}$ be the unique homomorphism from $(\mathbb{N}, {}^{+}, 0)$ into $(\boldsymbol{A}, \boldsymbol{S}, i)$. The equation

$$\boldsymbol{S}(a) \,\boldsymbol{H}\, n = a \,\boldsymbol{H}\, n + \boldsymbol{F}(n) \tag{3}$$ {eqn:.n}

holds when $n = 0$, since $\boldsymbol{S}(a) \,\boldsymbol{H}\, 0 = i = i + i = a \,\boldsymbol{H}\, 0 + \boldsymbol{F}(0)$. Suppose (3) holds for some n in $\mathbb{N}$. Then

$$\begin{aligned}
\boldsymbol{S}(a) \,\boldsymbol{H}\, n^{+} &= \boldsymbol{S}(a) \,\boldsymbol{H}\, n + \boldsymbol{S}(a) && \text{[by definition of } \boldsymbol{H}\text{]} \\
&= (a \,\boldsymbol{H}\, n + \boldsymbol{F}(n)) + \boldsymbol{S}(a) && \text{[by inductive hypothesis]} \\
&= a \,\boldsymbol{H}\, n + (\boldsymbol{F}(n) + \boldsymbol{S}(a)) && \text{[by associativity of } +\text{]} \\
&= a \,\boldsymbol{H}\, n + \boldsymbol{S}(\boldsymbol{F}(n) + a) && \text{[by definition of } +\text{]} \\
&= a \,\boldsymbol{H}\, n + (\boldsymbol{S}(\boldsymbol{F}(n)) + a) && \text{[by Lemma 5.2.5]} \\
&= a \,\boldsymbol{H}\, n + (\boldsymbol{F}(n^{+}) + a) && \text{[because } \boldsymbol{F} \text{ is a homomorphism]} \\
&= a \,\boldsymbol{H}\, n + (a + \boldsymbol{F}(n^{+})) && \text{[by commutativity of } +\text{]} \\
&= (a \,\boldsymbol{H}\, n + a) + \boldsymbol{F}(n^{+}) && \text{[by associativity of } +\text{]} \\
&= a \,\boldsymbol{H}\, n^{+} + \boldsymbol{F}(n^{+}). && \text{[by definition of } \boldsymbol{H}\text{]}
\end{aligned}$$

So (3) holds for all n in $\mathbb{N}$. Therefore each of the operations $x \mapsto x \,\boldsymbol{H}\, n$ is the operation $\boldsymbol{G}_n$ recursively defined by

(i) $\boldsymbol{G}_n(i) = i$,

(ii) $\boldsymbol{G}_n(\boldsymbol{S}(a)) = \boldsymbol{G}_n(a) + \boldsymbol{F}(n)$.

In particular,

$$\boldsymbol{G}_m = \boldsymbol{G}_n \Leftrightarrow \boldsymbol{F}(m) = \boldsymbol{F}(n).$$

Now we can define multiplication on $\boldsymbol{A}$ by

$$a \cdot b = c \Leftrightarrow \exists x \; (\boldsymbol{F}(x) = b \;\&\; \boldsymbol{G}_x(a) = c).$$

Then $a \cdot i = \boldsymbol{G}_0(a) = a \,\boldsymbol{H}\, 0 = i$. Also, if $b = \boldsymbol{F}(n)$, so that $\boldsymbol{S}(b) = \boldsymbol{F}(n^{+})$, then

$$a \cdot \boldsymbol{S}(b) = \boldsymbol{G}_{n^{+}}(a) = a \,\boldsymbol{H}\, n^{+} = a \,\boldsymbol{H}\, n + a = \boldsymbol{G}_n(a) + a = a \cdot b + a.$$

Thus $\cdot$ is as desired; it is unique by Theorem 4. □

• The proof of Theorem 5.3.5 can be replaced with a reference to Lemma 12.

Trivial changes

• ¶ 4.4.3: it can be noted that the last part of the proof is by contradiction.

• P. 4, item (i), after '**V** in these notes': insert reference to ¶ 2.2.7.

• P. 8: include Table 2.1 on p. 18 (best done by changing the Table to a Figure).

• P. 9: capitalize the letters after the hyphens in 'Replacement-scheme' and 'Power-set'.

• P. 10: transpose ¶ 1.1.1 to read:

> A **set** is a thing that **contains** other things. Those other things are called **members** or **elements** of the set. The set cannot be separated from its elements the way a box can be emptied of its contents: the set **comprises** its members, and the members **compose** the set. A set *is* its elements, considered as one thing. It is a multitude that is also a unity.

• P. 15, caption to Table 1.1: Replace sentence 'A terminal ω. . . ' with

> The vowels α, η, and ω may have an iota subscript (ᾳ, ῃ, ῳ).

• P. 16, after the first list of 3 items: Delete repeated 'recursively' (and add to index).

• After the second list of 3 items: change 'is' to 'of'; don't capitalize 'Parts'.

• P. 20, n. 4: 'The latter sequence that gives. . . ': delete 'that'.

• P. 21, ¶ 2.2.4, item (iii): 'Then $\exists x\ \varphi$ *is* true. . . '

• P. 23, ¶ 2.3.5, item (iii): '(where a is allowed to appear in σ)': change σ to φ.

• P. 23, ¶ 2.3.5: '*this* rule allows *us* to obtain the sentence τ. . . '

• P. 24: allow Fig. 2.2 to float to the top of a page?

• P. 30, ¶ 3.2.3: replace 'However, $\bigcup a$ is a set' with 'However, the union of a *set* is a set'.

• ¶ 3.5.9: the meaning of *greater than* should perhaps have been given explicitly in ¶ 3.5.5.

• P. 36, ¶ 3.6.3: In the formula displayed over two lines, the terminal & on the first line should be repeated on the second (as this is the convention I use elsewhere).

- In the following line, replace **to** with **(in)to**.

- Pp. 38 f., ¶ 3.7.1: change $\boldsymbol{E}$ to $\boldsymbol{C}$.

- P. 40: Exercise (3) should follow (5).

- P. 45, ¶ 4.1.8: slant *chain* as a technical term.

- Last line of text, but two: '. . . will *be (in ¶ 5.1.3)* another example. . . '

- ¶ 4.2.2 can be broken into 3 paragraphs.

- ¶ 4.3.1: 'This means *by ¶ 4.2.1*. . . '; in item (ii): delete *from* $\boldsymbol{C}$; afterwards: 'The five numbered conditions here *for being an arithmetic structure* are sometimes. . . '

- ¶ 4.4.1, last line but one: $\boldsymbol{C}$ should be $\boldsymbol{D}$.

- p. 55, ¶ 5.1.1, just before (5.1): 'Meanwhile *we* have'. In (5.2) and (5.3), the functions $\boldsymbol{F}$ are really *sets* and should be written that way. (Actually they are variables. . .)

- ¶ 5.1.3 (iv): change $\boldsymbol{C}$ to a (both times).

- ¶ 5.1.5: Give the numerical reference (4.3.5) for the Recursion Theorem.

- ¶ 5.3.7: Add: 'For all a and b in $\boldsymbol{A}$, and all m and n in $\mathbb{N}$'; in (ii), replace $x \mapsto x^a$ with $x \mapsto x^m$.