Exam 2 solutions

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Instructions: Please, be accurate and show clearly the logic of your solutions. Only the answers are not enough: indicate your calculations and arguments.

Problem 1. (9 pts) *Of the set* $\{1,2,3,4,5,6,7,9\}$*, let* $\sigma \in S_9$ *be the permutation*

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 8 & 2 & 9 & 5 \end{bmatrix}.$$

(a) Write σ *as a product of disjoint cycles.*

Solution. $\sigma = (134)(27)(5689)$.

(b) Find σ^{100}*. (Hint: find the order of* σ*).*

Solution. The length of σ is the least common multiple of the length of the disjoint cycles. Hence $|\sigma| = \mathrm{lcm}(3,2,4) = 12$. Thus

$$\sigma^{100} = \sigma^{96}\sigma^{4} = \sigma^{4} = (134).$$

(c) Determine whether permutation σ *is even or odd.*

Solution. Since we can write $\sigma = (14)(13)(27)(59)(58)(56)$ as a product of even number of transpositions, σ is an even permutation.

Problem 2. (6 pts) *Consider transpositions* $\alpha = (12)$*,* $\beta = (23)$*,* $\gamma = (34)$ *in* S_4*.*

(a) Find the product $\alpha\beta\gamma$*.*

Solution. $\alpha\beta\gamma = (12)(23)(34) = (1234)$.

(b) Find $g \in S_4$ *such that* $\beta = g\alpha g^{-1}$*.*

Solution. Note that g is not unique. Since $g(1) = 2$ and $g(2) = 3$ we may write g as

$$g = (12)(23) = (123)$$

.

Problem 3. (8 pts)

(a) *Find the number of cosets for subgroups $3\mathbb{Z} \subset \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{Q}$. In the first case, give an explicit list of these cosets.*

Solution. The cosets of $3\mathbb{Z}$ in $\mathbb{Z}$ are $3\mathbb{Z}$, $1 + 3\mathbb{Z}$, and $2 + 3\mathbb{Z}$: there are three of them.

$\mathbb{Z}$ has infinitely many cosets in $\mathbb{Q}$ [since if a and b are distinct elements of $\mathbb{Q}$ between 0 and 1, then $a + \mathbb{Z}$ and $b + \mathbb{Z}$ are distinct cosets].

(b) *Find the group multiplication table of $\mathbb{Z}/3\mathbb{Z}$ (pay attention that the group operation is addition).*

Solution.

	$3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$3\mathbb{Z}$	$3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$1+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$	$3\mathbb{Z}$
$2+3\mathbb{Z}$	$2+3\mathbb{Z}$	$3\mathbb{Z}$	$1+3\mathbb{Z}$

(c) *Prove that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.*

Solution. Every element of $\mathbb{Q}/\mathbb{Z}$ is $k/n+\mathbb{Z}$ for some k and n in $\mathbb{Z}$ such that $n > 0$. Then $n(k/n + \mathbb{Z}) = k + \mathbb{Z} = \mathbb{Z}$, the identity in $\mathbb{Q}/Z$. Thus the order of $k/n + \mathbb{Z}$ is less than or equal to n: in particular, it is finite.

Problem 4. (7 pts) *Assume that G is a group of prime order p.*

(a) *What can be the order of an element $g \in G$? (Explain !)*

Solution. By the Lagrange Theorem, the order of an element divides the order of a group. In particular, the order of an element of G divides p; so the order is 0 or 1.

(b) *Prove that G is a cyclic group.*

Solution. The only element of G of order 1 is the identity, e. If $a \in G \setminus \{e\}$, then the order of a is p, by part (a). So the cyclic subgroup $\langle a \rangle$ of G has the same order as G itself; so that cyclic subgroup *is* G.

Problem 5. (10 pts) *Suppose that H and N are subgroups of a group G. Assume that subgroup N is normal, and let $K = H \cap N$.*

(a) Prove that K is a normal subgroup of H.

Solution. Since H, N are subgroups of G then $e \in H$ and $e \in N$, so $e \in H \cap N = K$ i.e. K is non-empty. Let $a, b \in K$ consider ab^{-1}, since $a, b \in H$ and $a, b \in N$ and they are subgroups of G then $ab^{-1} \in H$ and $ab^{-1} \in N$, so $ab^{-1} \in K$. Therefore, K is a subgroup of H.
Let $h \in H$ and $a \in K$, then $a \in H$ and $a \in N$, so $hah^{-1} \in H$ and since N is normal $hah^{-1} = n$ for some $n \in N$ which means $hah^{-1} \in N$, therefore $hah^{-1} \in H \cap N = K$. Hence, K is a normal subgroup of H.(OR: $ha = bh$ for some $b \in N$ since N is normal, but $b = hah^{-1}$ which is in H as well, so $b \in K$, hence $hK = Kh$ for all $h \in H$.)

(b) Let $A = \{x \in G \colon xK = Kx\}$. Prove that $H \subseteq A$.

Solution. By part (a)K is normal in H, so we know that $hK = Kh$ for all $h \in H$, but then $h \in A$ for all $h \in H$. Hence $H \subseteq A$.

(c) Prove that A is a subgroup of G.

Solution. (i) Since $eK = K = Ke$, we have $e \in A$.

(ii) If $x \in A$, so that $xK = Kx$, then $K = x^{-1}xK = x^{-1}Kx$, so $Kx^{-1} = x^{-1}Kxx^{-1} = x^{-1}K$, and so $x^{-1} \in A$.

(iii) If $x, y \in A$, then $xK = Kx$ and $yK = Ky$, so $xyK = xKy = Kxy$, hence $xy \in A$.

Problem 6. (5 pts) *Let H and K be normal subgroups of G such that G/H has order 5, and G/K has order 3. Prove that for any $g \in G$, $g^{15} \in K \cap H$.*

Solution. Let $g \in G$, consider the element $gH \in G/H$. Since G/H has order 5 then by Lagrange Thm. $o(gH)|5$ by problem 4(a) $o(gH) = 5$ or $o(gH) = 1$.
If $o(gH) = 1$ then $gH = eH = H \Rightarrow g \in H \Rightarrow g^5 \in H$. If $o(gH) = 5$ then since H is a normal subgroup of G

$$eH = (gH)^5 = gHgHgHgHgH = gHgHgHggH = \cdots = g^5H$$

which means $g^5 \in H$.

Similarly, $o(gK)|3$ then $o(gK) = 3$ or $o(gK) = 1$. If $o(gK) = 1$ then $g^3 \in K$. If $o(gK) = 3$ then since K is a normal subgroup of G $eK = (gK)^3 = g^3K$ which means $g^3 \in K$.

Now, since $g^15 = (g^5)^3 \in H$ and $g^15 = (g^3)^5 \in K$ then $g^15 \in K \cap H$.

Problem 7. (6 pts) *Consider subsets* $4\mathbb{Z} \subset \mathbb{Z}$, $\{0\} \subset \mathbb{Z}$, $\varnothing \subset \mathbb{Z}$, $\mathbb{Z} \subset \mathbb{Q}$, $(0, \infty) \subset \mathbb{R}$, $\{[0], [1], [2], [4]\} \subset \mathbb{Z}_6$, $M_2(\mathbb{Z}) \subset M_2(\mathbb{R})$.

(a) *Which of the above examples are NOT subrings? Why they are not subrings ?*

Solution. $\varnothing \subset \mathbb{Z}$, because a subring should be non-empty,

$(0, \infty) \subset \mathbb{R}$, because for number $x \in (0, \infty)$ its opposite, $-x$, does not belong to $(0, \infty)$,

$\{[0], [1], [2], [4]\} \subset \mathbb{Z}_6$, because $[1] + [2] = [3] \notin \{[0], [1], [2], [4]\}$.

(b) *Which ones are NOT ideals? Why not ?*

Solution. An ideal is a subring, thus, the three examples in part (a) which are not rings are also not ideals. There are two more examples:

$\mathbb{Z} \subset \mathbb{Q}$ (the product of a rational and an integers may be NOT an integer), and

$M_2(\mathbb{Z}) \subset M_2(\mathbb{R})$ (the product of an integer matrix with a real matrix may be NOT an integer matrix).

(So, the only examples of ideals here are $4\mathbb{Z} \subset \mathbb{Z}$, and $\{0\} \subset \mathbb{Z}$.)

Problem 8. (9 pts) *In each of the following two subrings of* $\mathbb{Z}_{12}$*, find a unity or show that there is no unity (multiplicative identity).*

(a) $2\mathbb{Z}_{12} = \{[0], [2], [4], [6], [8], [10]\}$

Solution. If $[2n]$ is the unity, then $[2][2n] = [2]$, that is $4n = 2 \mod 12$. But $4n - 2$ is not divisible by 4, and thus, by 12. So, there is no unity in $2\mathbb{Z}_{12}$.

(b) $3\mathbb{Z}_{12} = \{[0], [3], [6], [9]\}$

Solution. $[9]$ is the unity in $2\mathbb{Z}_{12}$, because $[9][0] = [0], [9][3] = [27] = [3], [9][6] = [54] = [6], [9][9] = [81] = [9]$.

(c) *Find all the zero divisors in the ring* $\mathbb{Z}_{12}$.

Solution. The zero divisors are $[2], [3], [4], [6], [8], [9], [10]$, because in $\mathbb{Z}_{12}$, we have $[2][6] = [3][4] = [8][9] = [10][6] = [0]$.