Exponential polynomials

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Topics

- Exponential rings, exponential fields and exponential polynomial ring
- Ritt’s Factorization Theorem
- Schanuel’s Conjecture and Shapiro’s Conjecture
**Definition:** An exponential ring, or $E$-ring, is a pair $(R, E)$ where $R$ is a ring (commutative with 1) and

$$E : (R, +) \to (U(R), \cdot)$$

a morphism of the additive group of $R$ into the multiplicative group of units of $R$ satisfying

1. $E(x + y) = E(x) \cdot E(y)$ for all $x, y \in R$
2. $E(0) = 1$.

$(K, E)$ is an $E$-field if $K$ is a field.
Examples:

1. \((\mathbb{R}, e^x); (\mathbb{C}, e^x)\);

2. \((R, E)\) where \(R\) is any ring and \(E(x) = 1\) for all \(x \in R\).

3. \((S[t], E)\) where \(S\) is \(E\)-field of characteristic 0 and \(S[t]\) the ring of formal power series in \(t\) over \(S\). Let \(f \in S[t]\), where \(f = r + f_1\) with \(r \in S\)

\[
E(f) = E(r) \cdot \sum_{n=0}^{\infty} \frac{(f_1)^n}{n!}
\]

4. \(K[X]^E\) \(E\)-ring of exponential polynomials over \((K, E)\)
Exponential polynomials

Sketch of the construction:

Let $R$ be a partial $E$-ring, $R = D \oplus \Delta$, where $D = \text{dom}(E)$.

Let $t^\Delta$ be a multiplicative copy of $\Delta$, and consider $R[t^\Delta]$.

Extend $E$ to $R$ by defining $E(\delta) = t^\delta$, $\delta \in \Delta$.

Decompose $R[t^\Delta] = R \oplus t^\Delta - \{0\}$. Iterate $\omega$ times, and get $E$ total.

1. Let $R = \mathbb{Z}[\overline{X}]$, $D = (0)$ and $\Delta = R$.
   
   The limit of previous construction is $[\overline{X}]^E$, the free $E$-ring on $\overline{X}$.

2. Let $(K, E)$ be an $E$-field and $R = K[\overline{X}]$. Decompose $K[\overline{X}] = K \oplus \Delta$, where $D = K$ and $\Delta = \{ f : f(\overline{0}) = 0 \}$.

   The limit of previous construction is the $E$-ring $K[\overline{X}]^E$ of exponential polynomials in $\overline{X}$ over $K$. 
An exponential polynomial in \([x, y]^E\) is represented as

\[
P(x, y) = -3x^2y - x^5y^7 + (2xy + 5y^2)e^{-7x^3 + 11x^5y^4}
\]

\[
+(6 - 2xy^5)e^{(5x + 2x^7y^2)}e^{5x - 10y^2}
\]

**Theorem**

Let \((R, E)\) be an E-domain. Then \(R[\overline{X}]^E\) is an integral domain whose units are \(uE(f)\), where \(u\) is invertible in \(R\) and \(f \in R[\overline{X}]^E\).

**Definition**

An element \(f \in R[\overline{X}]^E\) is irreducible if there are no non-units \(g\) and \(h\) in \(R[\overline{X}]^E\) such that \(f = gh\).
**Definition**

Let \( f = \sum_{i=1}^{N} a_i t^{\alpha_i} \) be an exponential polynomial. Then the support of \( f = \text{supp}(f) = \mathbb{Q}\)-space generated by \( \alpha_1, \ldots, \alpha_N \).

**Definition**

An exponential polynomial \( f(x) \) is simple if \( \dim \text{supp}(f) = 1 \).

\[
\sin(2\pi x) = \frac{e^{2\pi ix} - e^{-2\pi ix}}{2i}
\]
Ritt in 1927 studied factorizations of exponential polynomial

\[ 1 + \beta_1 e^{\alpha_1 z} + \ldots + \beta_k e^{\alpha_k z} \]

over \( \mathbb{C} \), using factorizations in fractional powers of classical polynomials in many variables.

Gourin (1930) and Macoll (1935) gave a refinement of Ritt’s factorization theorem for exponential polynomials of the form

\[ p_1(z)e^{\alpha_1 z} + \ldots + p_k(z)e^{\alpha_k z} \]

with \( \alpha_i \in \mathbb{C} \), and \( p_i(z) \in \mathbb{C}[z] \).

D’A. and Terzo (2011) gave a factorization theorem for general exponential polynomials \( f(\overline{X}) \in K[\overline{X}]^E \), where \( K \) is an algebraically closed field of characteristic 0 with an exponentiation.
Ritt’s basic idea

Ritt: reduce the factorization of an exponential polynomial to that of a classical polynomial in many variables in fractional powers.

If \( Q(Y_1, \ldots, Y_n) \in K[Y_1, \ldots, Y_n] \) is an irreducible polynomial over \( K \), it can happen that for some \( q_1, \ldots, q_n \in \mathbb{N}_+ \), \( Q(Y_1^{q_1}, \ldots, Y_n^{q_n}) \) becomes reducible:

**Ex:** \( X - Y \) irreducible, but \( X^3 - Y^3 = (X - Y)(X^2 + XY + Y^2) \)

**Definition**

A polynomial \( Q(Y) \) is power irreducible (over \( K \)) if for each \( \bar{q} \in \mathbb{N}_n \), \( Q(\bar{Y}^{\bar{q}}) \) is irreducible.

A factorization of \( Q(Y) \) gives a factorization of \( Q(\bar{Y}^{\bar{q}}) \)

A factorization of \( Q(\bar{Y}^{\bar{q}}) = Q(Y_1^{q_1}, \ldots, Y_n^{q_n}) \) gives a factorization of \( Q(Y_1, \ldots, Y_n) \) in fractional powers of \( Y_1, \ldots, Y_n \).
Let $f(\overline{X}) = \sum_{h=1}^{m} a_h t^{b_h}$, where $a_h \in K[\overline{X}]$ and $b_h \in \Delta$ and let $\{\beta_1, \ldots, \beta_l\}$ be a $\mathbb{Z}$-basis of $\text{supp}(f)$.

*Modulo a monomial* we consider $f$ as polynomial in $e^{\beta_1}, \ldots, e^{\beta_l}$, with coefficients in $K[\overline{X}]$. Let $Y_i = e^{\beta_i}$, for $i = 1, \ldots, l$.

\[ f(\overline{X}) \in K[\overline{X}]^E \sim Q(Y_1, \ldots, Y_l) \in K[\overline{X}][Y_1, \ldots, Y_l] \]

**monomial**: $Y_1^{m_1} \cdot \ldots \cdot Y_n^{m_n}$, where $m_1, \ldots, m_n \in \mathbb{Z}$, i.e. an invertible element in $K[\overline{X}]^E$

Simple exponential polynomials correspond to a single variable classical polynomials
If \( Q(Y) = Q_1(Y) \cdot \ldots \cdot Q_r(Y) \) then \( f(X) = f_1(X) \cdot \ldots \cdot f_r(X) \)
and for any \( \bar{q} \) positive integers

if \( Q(Y^\bar{q}) = R_1(Y) \cdot \ldots \cdot R_p(Y) \) then \( f(X) = g_1(X) \cdot \ldots \cdot g_p(X) \).

All the factorizations of \( f(X) \) are obtained in this way.

**Lemma**

Let \( f(X) \) and \( g(X) \) be in \( K[X]^E \). If \( g(X) \) divides \( f(X) \) then \( \text{supp}(ag) \) is contained in \( \text{supp}(bf) \), for some units \( a, b \).

**Remark:** If \( f \) is a simple polynomial and \( g \) divides \( f \) then \( g \) is also simple.
**Problem:** How many tuples $\overline{q}$ are there such that $Q(\overline{Y^q})$ is reducible?

Have to avoid $Y - Z$ since $Y^{\frac{1}{k}} - Z^{\frac{1}{k}}$ is a factor for all $k > 0$.

**Theorem:**

There is a uniform bound for the number of irreducible factors of

$$Q(Y_1^{q_1}, \ldots, Y_l^{q_l})$$

for $Q(Y_1, \ldots, Y_l)$ irreducible with more than two terms and arbitrary $q_1, \ldots, q_l \in \mathbb{N}_+$. The bound depends only on

$$M = \max\{d_{Y_1}, \ldots, d_{Y_l}\}.$$
Factorization Theorem

**Theorem (Ritt)**

Let \( f(z) = \lambda_1 e^{\mu_1 z} + \ldots + \lambda_N e^{\mu_N z} \), where \( \lambda_i, \mu_i \in \mathbb{C} \). Then \( f \) can be written uniquely up to order and multiplication by units as

\[
f(z) = S_1 \cdot \ldots \cdot S_k \cdot I_1 \cdot \ldots \cdot I_m
\]

where \( S_j \) are simple polynomials with \( supp(S_{j_1}) \neq supp(S_{j_2}) \) for \( j_1 \neq j_2 \), and \( I_h \) are irreducible exponential polynomials.
**Theorem (D’A-Terzo)**

Let $f(\overline{X}) \in K[\overline{X}]^E$, where $(K, E)$ is an algebraically closed $E$-field of char 0 and $f \neq 0$. Then $f$ factors, uniquely up to units and associates, as finite product of irreducibles of $K[\overline{X}]$, a finite product of irreducible polynomials $Q_i$ in $K[\overline{X}]^E$ with support of dimension bigger than 1, and a finite product of polynomials $P_j$ where $\text{supp}(P_{j_1}) \neq \text{supp}(P_{j_2})$, for $j_1 \neq j_2$ and whose supports are of dimension 1.

**Corollary**

If $f$ is irreducible and the dimension of $\text{supp}(f) > 1$ then $f$ is prime.
Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \),

\[
\text{l.d.}(\alpha_1, \ldots, \alpha_n) = \text{linear dimension of } <\alpha_1, \ldots, \alpha_n >_{\mathbb{Q}}
\]

\[
\text{tr.d.}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n) = \text{transcendence degree of } \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \text{ over } \mathbb{Q}.
\]

\[
(\text{SC}) \quad \text{tr.d.}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq \text{l.d.}(\alpha_1, \ldots, \alpha_n)
\]

**Generalized Schanuel Conjecture** Assume \((R, E)\) is an \(E\)-ring and \(\text{char}(R) = 0\). Let \(\lambda_1, \ldots, \lambda_n \in R\) then

\[
\text{tr.d.}_{\mathbb{Q}}(\lambda_1, \ldots, \lambda_n, e^{\lambda_1}, \ldots, e^{\lambda_n}) - \text{l.d.}(\lambda_1, \ldots, \lambda_n) \geq 0
\]
Generalization of Lindemann-Weierstrass Theorem:
Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers which are linearly independent over $\mathbb{Q}$. Then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraic independent over $\mathbb{Q}$.

1. $\lambda = 1$ transcendence of $e$ (Hermite 1873)
2. $\lambda = 2\pi i$ transcendence of $\pi$ (Lindemann 1882)
3. $\lambda = (\pi, i\pi)$ then $\text{tr.d.}(\pi, i\pi, e, e^{i\pi}) = 2$, i.e. $\pi, e^\pi$ are algebraically independent over $\mathbb{Q}$ (Nesterenko 1996)
4. (SC) is true for power series $\mathbb{C}[[t]]$ (Ax 1971)
The Schanuel machine

\[ \overline{\lambda} = (1, \pi i), \]

\[ \text{SC} \Rightarrow t.d(1, i\pi, e, e^{i\pi}) \geq l.d.(1, i\pi). \]

Then \( e, \pi \), are algebraically independent over \( \mathbb{Q} \);

\[ \overline{\lambda} = (1, i\pi, e) \]

\[ \text{SC} \Rightarrow t.d(1, i\pi, e, e^{i\pi}, e^{e}) \geq l.d.(1, i\pi, e). \]

Then \( \pi, e, e^e \) are algebraically independent over \( \mathbb{Q} \);

\[ \overline{\lambda} = (1, i\pi, i\pi^2, e, e^{e}, e^{i\pi^2}), \]

\[ \text{SC} \Rightarrow t.d(1, i\pi, i\pi^2, e, e^{e}, e^{i\pi^2}, e, e^{e}, e^{ee}, e^{ei\pi^2}) \]

\[ \geq l.d.(1, i\pi, i\pi^2, e, e^{e}, e^{i\pi^2}). \]

Then \( \pi, e, e^{ee}, e^{i\pi^2}, e^{ei\pi^2} \) are algebraically independent / \( \mathbb{Q} \).
Theorem (Macintyre)
Suppose $S$ is an $E$-ring satisfying (SC), and $S_0$ is the $E$-subring of $S$ generated by $1$. Then the natural $E$-morphism $\varphi : [\emptyset]^E \rightarrow S_0$ is an $E$-isomorphism, i.e. $S_0$ is isomorphic to $E$-free ring on the empty set.

Corollary (SC)
There is an algorithm which decides if two exponential constants coincide.
Theorem (Terzo)

(SC) Let \([x, y]_E\) be the free \(E\)-ring generated by \(\{x, y\}\) and let \(\psi\) be the \(E\)-morphism:

\[
\psi : [x, y]_E \to (\mathbb{C}, \exp)
\]

defined by \(\psi(x) = \pi\) and \(\psi(y) = i\). Then there exists a unique isomorphism

\[
f : [x, y]_E / \text{Ker}\psi \to \langle i, \pi \rangle_\mathbb{E}
\]

and

\[
\text{Ker}\psi = \langle e^{xy} + 1, y^2 + 1 \rangle_\mathbb{E}.
\]

Corollary

(SC) The only algebraic relations among \(\pi\), \(e\) and \(i\) over \(\mathbb{C}\) are

\[
e^{i\pi} = -1 \text{ and } i^2 = -1
\]
**Theorem (Terzo)**

(SC) Let $[x]^E$ be the free $E$-ring generated by $\{x\}$ and let $R$ be the $E$-subring of $(\mathbb{R}, \exp)$ generated by $\pi$. Then the $E$-morphism

$$
\varphi: [x]^E \to (R, \exp)
$$

$$
\begin{align*}
\varphi(x) &= \pi
\end{align*}
$$

is an $E$-isomorphism.

**Corollary**

(SC)

1. There is an algorithm for deciding if two exponential polynomials in $\pi$ and $i$ are equal in $\mathbb{C}$.
2. There is an algorithm for deciding if two exponential polynomials in $\pi$ are equal in $\mathbb{R}$. 
\( K[\overline{X}]^E \) is sharply Schanuel

\( [\overline{X}]^E \) satisfies Schanuel Conjecture

**Theorem (D., Macintyre and Terzo)**

Let \((K, E)\) be an exponential field satisfying Schanuel Conjecture. Suppose that

\[ \gamma_1, \ldots, \gamma_n \in K[\overline{X}]^E - K \text{ are } \mathbb{Q}\text{-linearly independent over } K. \]

Then

\[ t.d. K(\gamma_1, \ldots, \gamma_n, E(\gamma_1), \ldots, E(\gamma_n)) \geq n + 1. \]
Shapiro’s Conjecture (1958): If two exponential polynomials \( f, g \) of the form
\[
    f = c_1 e^{\lambda_1 z} + \ldots + c_n e^{\lambda_n z}
\]
\[
    g = b_1 e^{\mu_1 z} + \ldots + b_m e^{\mu_m z},
\]
where \( c_i, b_j, \lambda_i, \mu_j \in \mathbb{C} \) have infinitely many zeros in common they are both multiples of some exponential polynomial.

This conjecture comes out of complex analysis (and early work of Polya, Ritt and many other). It was formulated by H.S. Shapiro in a paper entitled:

*The expansion of mean-periodic functions in series of exponentials.*
Shapiro’s Conjecture

Remark

The factorization theorem implies that we need to consider only two cases of the Shapiro problem.

Case 1. At least one of the exponential polynomial is a simple polynomial.

Case 2. At least one of the exponential polynomials is irreducible.
Case 1.

Over \( \mathbb{C} \) answer is positive unconditionally

**Theorem** (van der Poorten and Tijdeman, 1975)

Let \( f(z) = \sum \alpha_j e^{\beta_j z} \), with \( \alpha_j, \beta_j \in \mathbb{C} \), be a simple exponential polynomial and let \( g(z) \) be an arbitrary exponential polynomial such that \( f(z) \) and \( g(z) \) have infinitely many common zeros. Then there exists an exponential polynomial \( h(z) \), with infinitely many zeros, such that \( h \) is a common factor of \( f \) and \( g \) in the ring of exponential polynomial.
Ingredients of the proof

**Theorem (Ritt)**

If every zero of an exponential polynomial $f(z)$ is a zero of $g(z)$ then $f(z)$ divides $g(z)$.

**Theorem (Skolem, Mahler, Lech)**

Let $f(z) = \sum \alpha_j e^{\beta_j z}$, be an exponential polynomial, where $\alpha, \beta \in K$ where $K$ is a field of characteristic 0. If $f(z)$ vanishes for infinitely many integers $z = z_i$, then there exists an integer $d$ and certain set of least residues modulo $d, d_1, \ldots, d_l$ such that $f(z)$ vanishes for all integers $z \equiv d_i (mod d)$, for $i = 1, \ldots, l$, and $f(z)$ vanishes only finitely often on other integers.
Case 1.

**Setting:** Over \((K, E)\) algebraically closed field with an exponentiation, \(\text{char}(K) = 0,\ ker(E) = \omega\mathbb{Z},\ E\ \text{surjective}\)

answer is positive unconditionally

<table>
<thead>
<tr>
<th><strong>Lemma (DMT)</strong></th>
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<tbody>
<tr>
<td>Let (h(z) = \lambda_1 e^{\mu_1 z} + \ldots + \lambda_N e^{\mu_N z}), where (\lambda_j, \mu_j \in K). If (h) vanishes over all integers then (\sin(\pi z)) divides (h).</td>
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We use Vandermonde determinant.

<table>
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<th><strong>Theorem (DMT)</strong></th>
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<td>Let (f) be a simple exponential polynomial, and let (g) be an arbitrary exponential polynomial such that (f) and (g) have infinitely many common roots. Then there exists an exponential polynomial which divides both (f) and (g).</td>
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Theorem (A. Skhop, 2010)

(SC) Let $f$ and $g$ be exponential polynomials as above with $c_i, b_j, \lambda_i, \mu_j \in \mathbb{Q}^{alg}$. If $f$ and $g$ have no common factors except monomials then $f$ and $g$ have only finitely many common zeros.

Theorem (D’A, Macintyre, Terzo, 2011)

Schanuel’s conjecture implies Shapiro’s conjecture.

The proof uses no logic, but substantial work by Bombieri, Masser and Zannier, and work of Evertse, Schlickewei and Schmidt on linear functions of elements of finite rank groups.
Consider the following system:

\[
\begin{align*}
    f(z) &= \lambda_1 e^{\mu_1 z} + \ldots + \lambda_N e^{\mu_N z} = 0 \\
    g(z) &= l_1 e^{m_1 z} + \ldots + l_M e^{m_M z} = 0
\end{align*}
\]  

(1)

where \( \lambda_i, \mu_i, l_j, m_j \in K \).

Let \( D = l.d.(\text{supp}(f) \cup \text{supp}(g)) \), \( b_1, \ldots, b_D \) a \( \mathbb{Z} \)-basis, and \( Y_i = e^{b_i z} \) for \( i = 1, \ldots, D \).

To system (1) associate:

\[
\begin{align*}
    F(Y_1, \ldots, Y_D) &= 0 \\
    G(Y_1, \ldots, Y_D) &= 0
\end{align*}
\]  

(2)

where \( F(Y_1, \ldots, Y_D), G(Y_1, \ldots, Y_D) \in \mathbb{Q}(\bar{\lambda}, \bar{l})[Y_1, \ldots, Y_D] \).
Let $L = \mathbb{Q}(\lambda, \bar{l})^{alg}, \ t.d.\mathbb{Q}(L) < \infty$. Let $S$ be the infinite set of non-zero common solutions of system (1).

**Remark**

If $s \in S$ then $(e^{b_1 s}, \ldots, e^{b_D s})$ is a solution of system (2).

**Theorem (D’A, Macintyre and Terzo)**

**(SC)** The $\mathbb{Q}$-vector space generated by $S$ is finite dimensional.

**(SC)** gives bounds on linear dimensions and transcendence degrees of finite subsets of $S$ and their exponentials.

Let $V$ be an irreducible component of the subvariety of the algebraic group $G_m^D$ defined by (2) over $L$ containing $(e^{b_1 s}, \ldots, e^{b_D s})$ for infinitely many $s \in S$. 

**Definition**

An irreducible subvariety $W$ of $V$ is anomalous in $V$ if $W$ is contained in an algebraic subgroup $\Gamma$ of $G_m^D$ with

$$\dim W > \max\{0, \dim V - \text{codim}\Gamma\}$$

---

**Theorem (Bombieri, Masser, Zannier (2007))**

Let $V$ be an irreducible variety in $G_m^D$ of positive dimension defined over $\mathbb{C}$. There is a finite collection $\Phi_V$ of proper tori $H$ such that $1 \leq D - \dim H \leq \dim V$ and every maximal anomalous subvariety $W$ of $V$ is a component of the intersection of $V$ with a coset $H\theta$ for some $H \in \Phi_V$ and $\theta \in G_m^D$.
Second case of Shapiro’s Conjecture

**Remark**

BMZ result holds for every algebraically closed field $K$ with $\text{char}(K) = 0$

For a finite sequence $\overline{s} = s_1, \ldots, s_k \in S$ consider the variety $W_{\overline{s}} \subseteq V^k$ generated by $(e^{\overline{b}s_1}, \ldots, e^{\overline{b}s_k})$, where $\overline{b} = b_1, \ldots, b_D$.

For big $k$, either $\dim W_{\overline{s}} = 0$ or $W_{\overline{s}}$ is anomalous.

If for infinitely many $k$’s $\dim W_{\overline{s}} = 0$ then we are done.

Otherwise, we are forced into anomalous case, and using BMZ we get finite dimensionality of the set of solutions.
**Corollary (DMT)**

If \( \hat{G} \) is the divisible hull of \( G \) the group generated by all \( e^{\mu_j s} \)'s where \( s \in S \) then \( \hat{G} \) has finite rank.

**Theorem (DMT)**

(SC) Let \( f(z) \) be an irreducible polynomial and suppose the following system

\[
\begin{align*}
    f(z) &= \lambda_1 e^{{\mu_1}z} + \ldots + \lambda_N e^{{\mu_N}z} = 0 \\
    g(z) &= l_1 e^{{m_1}z} + \ldots + l_M e^{{m_M}z} = 0
\end{align*}
\]

has infinitely common zeros. Then \( f \) divides \( g \).
A solution \((\alpha_1, \ldots, \alpha_n)\) of a linear equation

\[ a_1x_1 + \ldots + a_nx_n = 1 \]

over a field \(K\) is non degenerate if for every proper non empty subset \(I\) of \(\{1, \ldots, n\}\) we have \(\sum_{i \in I} a_i \alpha_i \neq 0\).

**Theorem (Evertse, Schlickewei, Schmidt)**

Let \(K\) be a field, \(\text{char}(K) = 0\), \(n\) a positive integer, and \(\Gamma\) a finitely generated subgroup of rank \(r\) of \((K^\times)^n\). There exists a positive integer \(R = R(n, r)\) such that for any non zero \(a_1, \ldots, a_n\) elements in \(K\), the equation \(a_1x_1 + \ldots + a_nx_n = 1\) does not have more than \(R\) non degenerate solutions \((\alpha_1, \ldots, \alpha_n)\) in \(\Gamma\).
Associated linear equation

By finite dimensionality of $S$, $l.d.(S) = p$, where $p \in \mathbb{N}$. Denote by \{s_1, \ldots, s_p\} a $\mathbb{Q}$-basis of $S$. For any $s \in S$ we have:

$$s = \sum_{l=1}^{p} c_l s_l$$

where $c_l \in \mathbb{Q}$.

$$0 = f(s) = \lambda_1 e^{\mu_1(\sum_{l=1}^{p} c_l s_l)} + \ldots + \lambda_N e^{\mu_N(\sum_{l=1}^{p} c_l s_l)} = \sum_{j=1}^{N} \lambda_j \prod_{l=1}^{p} (e^{\mu_j s_l})^{c_l}$$

Any solution $s \in S$ produces a solution $\bar{\omega}$ of the linear equation associated to $f$,

$$\lambda_1 X_1 + \ldots + \lambda_N X_N = 0$$

where $\omega_i = e^{\mu_i(\sum_{l=1}^{p} c_l s_l)}$, $i = 1, \ldots, N$ and $\bar{\omega} \in \widehat{G}$.
Proof of main result

Induction of length of $g(z)$.

- $M = 2$ ($g$ simple);
- $N, M > 2$, we associate to

$$g(z) \sim l_1 X_1 + \ldots + l_M X_M.$$  

By (ESS) result we have that there are infinitely many degenerate solutions. By PHP there exist a subset $I = \{i_1, \ldots, i_r\}$ of $\{1, \ldots, M\}$ such that $i_r > 2$ and

$$l_{i_1} X_{i_1} + \ldots + l_{i_r} X_{i_r} = 0$$

has infinitely many zeros.

$g(z) = g_1(z) + g_2(z)$, where $g_1(z) = l_{i_1} e^{m_{i_1} z} + \ldots + l_{i_r} e^{m_{i_r} z}$, and $g_2(z) = g(z) - g_1(z)$. By inductive hypothesis and by the irreducibility of $f$, we have that $f$ divides $g_1$ and $f$ divides $g_2$, and hence $f$ divides $g$. 