A functor approach to modular representations of GL_n

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Çeşme, May 2012

$\mathcal{P}_d$ — the category of homogeneous polynomial functors of degree d over $\mathbf{k}$.

$$F_{V,W} : \mathrm{Hom}(V,W) \longrightarrow \mathrm{Hom}(F(V),F(W))$$

is homogeneous polynomial of degree d.

Examples of polynomial functors:

$$\begin{array}{ll}
V \rightsquigarrow V^{\otimes d} & (I^d), \\
V \rightsquigarrow (V^{\otimes d})_{\Sigma_d} & (S^d), \\
V \rightsquigarrow (V^{\otimes d})^{\Sigma_d} & (D^d), \\
V \rightsquigarrow ((V^{\otimes d})^{alt})^{\Sigma_d} \simeq ((V^{\otimes d})^{alt})_{\Sigma_d} & (\Lambda^d),
\end{array}$$

If char($\mathbf{k}$)=p,

$$\begin{array}{ll}
V \rightsquigarrow V^{(1)} & (I^{(1)}), \\
F^{(i)} := F \circ I^{(1)} \circ \ldots \circ I^{(1)}. &
\end{array}$$

$$\begin{array}{ccc}
\mathcal{P}_d & \longrightarrow & GL_n^{rat}(\mathbf{k}) - \mathrm{mod} \\
F & \mapsto & F(\mathbf{k}^n)
\end{array}$$

$\mathrm{char}(\mathbf{k}) = p > 0$,

Theorem (Friedlander–Suslin) For $n \geq d$,

$$\mathrm{Ext}^*_{\mathcal{P}_d}(F,G) \simeq \mathrm{Ext}^*_{GL_n^{rat}}(F(\mathbf{k}^n), G(\mathbf{k}^n)).$$

Young diagram of weight d: $\lambda = (\lambda_1, \ldots, \lambda_k)$,
$\Sigma\, \lambda_j = d$.

$\{F_\lambda\}$ – complete set of simple objects in $\mathcal{P}_d$

Problem Compute $\mathrm{Ext}^*_{\mathcal{P}_d}(F_\mu, F_\lambda)$.

$S_\lambda := \mathrm{im}(\Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_k} \longrightarrow I^d \longrightarrow S^{\widetilde{\lambda}_1} \otimes \ldots \otimes S^{\widetilde{\lambda}_s})$,
$W_\lambda := \mathrm{im}(D^{\widetilde{\lambda}_1} \otimes \ldots \otimes D^{\widetilde{\lambda}_s} \longrightarrow I^d \longrightarrow \Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_k})$,
$F_\lambda \hookrightarrow S_\lambda$, $W_\lambda \twoheadrightarrow F_\lambda$.

The Lusztig Conjecture
$\mathrm{Ext}^*_{\mathcal{P}_d}(F_\mu, S_\lambda)$ has parity.

$$\dim(\mathrm{Ext}^n_{\mathcal{P}_d}(W_\mu, S_\lambda) = \begin{cases} 1 & \text{if } \ \mu = \lambda, n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Program Compute $\mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(W^{(i)}_\mu, S_\lambda)$ for $i > 0$.

Computing $\mathrm{Ext}^*_{\mathcal{P}_{dp}}(W_\mu^{(1)}, S_\lambda)$.

$\lambda \mapsto c(\lambda), q(\lambda) = \{\lambda^0; \ldots; \lambda^{p-1}\}$,
$|\lambda| = |c(\lambda)| + p\,\Sigma_{j=0}^{p-1}|\lambda^j|$.

Fact If $c(\lambda) \neq \emptyset$, then $\mathrm{Ext}^*_{\mathcal{P}_{dp}}(W_\mu^{(1)}, S_\lambda) = 0$.

$$\begin{array}{rcl} F & \mapsto & F^{(1)} \\ \mathbf{C} : \mathcal{DP}_d & \longrightarrow & \mathcal{DP}_{dp} \end{array}$$

$\mathbf{K}^r(F)(V) := \mathrm{RHom}_{\mathcal{DP}_{dp}}(D^d(V^* \otimes I^{(1)}), F)$,
$\mathbf{K}^l(F) := \mathbf{K}^r(F^\#)^\#$.

Theorem 1 There is a two-sided adjunction

$$\mathbf{C} : \mathcal{DP}_d \leftrightarrows \mathcal{DP}_{dp} : \mathbf{K}^r, \mathbf{K}^l.$$

$$\mathrm{Ext}^*_{\mathcal{P}_{dp}}(W_\mu^{(1)}, S_\lambda) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(W_\mu, \mathbf{K}^r(S_\lambda))$$

$$\mathrm{H}^*(\mathbf{K}^r(F))(V) := \mathrm{Ext}^*_{\mathcal{P}_{dp}}(D^d(V^* \otimes I^{(1)}), F)),$$
$$A := \mathrm{Ext}^*_{\mathcal{P}_p}(I^{(1)}, I^{(1)}) \simeq \mathbf{k}[x]/x^p, \text{ for } \deg(x) = 2.$$

$\mathcal{P}_d^{af} := \{F : \text{A–mod}^{fr} \longrightarrow \mathbf{k}\text{–mod}^{gr},$
homogeneous polynomial of degree d over $\mathbf{k}\}$.

Examples of affine strict polynomial functors:
For $0 \leq j \leq p-1$, $\chi_j(V) := x^j V / x^{j+1} V[-j]$,
$S^{af}_{(\emptyset;\ldots;\lambda;\ldots;\emptyset)}(V) := S_\lambda(\chi_j(V))$,
$S^{af}_{((1);\ldots;(1))}(V) := \Lambda^p(V) \otimes_{D^p(A)} \mathbf{k}$.

Theorem 2

$$\mathbf{K}^r(S_\lambda)(V) = S^{af}_{q(\lambda)}(V \otimes A)[-h(\lambda)].$$

$A = \mathbf{k}[x]/x^p, \deg(x) = 2.$

$\mathcal{P}_d^{af} := \{F : \text{A–mod}^{fr} \longrightarrow \mathbf{k}\text{–mod}^{gr},$

homogeneous polynomial of degree d over $\mathbf{k}\}.$

$$\begin{array}{ccc} \mathcal{P}_d^{af} & \longrightarrow & GL_n^{rat}(A) - \mathrm{mod}^{gr} \\ F & \mapsto & F(A^n) \end{array}$$

$$A \simeq \mathbf{k}[\mathbf{Z}/p] \quad \longleftrightarrow \quad \mathbf{k}[\mathbf{Z}] \simeq \mathbf{k}[x, x^{-1}]$$

$$GL_n^{rat}(\mathbf{k}[x]/x^p) - \mathrm{mod}^{gr} \quad \longleftrightarrow \quad GL_n^{rat}(\mathbf{k}[x, x^{-1}]) - \mathrm{mod}$$