Locally$^\circ$ solvable$^\circ$ groups of finite Morley rank

Adrien Deloro and Eric Jaligot

May 26, 2007

Not much model theory

Let G be a group.

Call a set definable if you can express it with a first-order formula using the group language, and perhaps some extra structure.

Allow quotients and parameters.

Example : let $g \in G$.
Then $C_G(g)/Z(G)$ is definable in that sense.

Ranked groups

A group G is ranked if to every definable subset is associated an integer such that... some reasonable axioms are satisfied.

Hence $\text{rk} : \{\text{definable subsets of } G\} \to \mathbb{N}$.

Think of rk as a *dimension*.

You can do computations :
for example if $K \triangleleft H$ are definable subgroups,

$$\text{rk}(H) = \text{rk}(K) + \text{rk}(H/K).$$

The most natural examples of ranked groups are algebraic groups over alg. closed fields...

... with the Zariski dimension.

Analogies with algebraic groups.

Let G be a ranked group.

- There is a degree function on definable sets.

- There is a DCC on definable subgroups.

- There is a generation lemma for definable, indecomposable subgroups: $< X_i > = < X_{i_1}, \ldots, X_{i_n} >$ for some i_k.

- If $H < G$ is a definable subgroup, there is a connected component H°.

Unfortunately we can't use topological methods (they aren't "first order").

The Cherlin-Zilber conjecture and the Borovik program

Algebraicity Conjecture
A simple infinite group of finite Morley rank is isomorphic to some algebraic group over an algebraically closed field.

Borovik's idea is that ranked groups are a "generalization" of finite groups.

Hence one should mimic CFSG, and focus on 2-elements.

Odd type

There is a good 2-Sylow theory for ranked groups. So we'll focus on a special case.

Assumption. Here S will be toral-by finite, that is :

$$S^\circ \simeq \mathbb{Z}_{2^\infty}^n.$$

The integer $n > 0$ is called the Pruefer 2-rank of the group.

This assumption means that if a field appears in the group, its characteristic should be $\neq 2$. We say that G has odd type.

Locally$^\circ$ solvable$^\circ$ groups

Call a group G locally$^\circ$ solvable$^\circ$ if :

> whenever $1 \neq H < G$ is definable, connected, and solvable, so is $N_G^\circ(H)$.

The terminology follows from the tradition of calling $N_G^\circ(H)$ a local subgroup.

Such groups would appear in an inductive approach to the algebraicity conjecture.

What about reality ?

In the algebraic world, the only locally$^\circ$ solvable$^\circ$ (quasi-)simple groups are SL_2 and PSL_2.

Hence an important step would be

Relativised Algebraicity Conjecture
Let G be a (quasi-)simple, infinite, ranked group.
Assume : - G is locally$^\circ$ solvable$^\circ$, and
- G has odd type.

Then G is either $\mathrm{SL}_2(K)$ or $\mathrm{PSL}_2(K)$, where K is an alg. closed field of characteristic $\neq 2$.

This is an analog of Thompson's classification in the finite case.

Not easy !

Results... so far

Theorem

Let G be an infinite, connected, non-solvable, ranked group.
Assume : - G is locally$^\circ$ solvable$^\circ$, and
- G has odd type.

Also assume : $\forall i \in I(G)$, $C^\circ(i)$ is solvable.

Then :

- the involutions are conjugate
- the Pruefer 2-rank is 1 or 2
- the Weyl group is cyclic of order 1 or 2 (Pruefer rank = 1), or 3 (Pr. rank = 2)

Moreover :

- either $G \simeq \mathrm{PSL}_2(K)$ (char $K \neq 2$)
- or $C^\circ(i)$ is a Borel subgroup

Care for a proof ?

If some involution i has a sufficient action inside a Borel subgroup $B > C^\circ(i)$,
$\forall w \in i^G \setminus N(B)$ let $T[w] := \{t \in B,\ t^w = t^{-1}\}$.
Then split $B = F^\circ(B) \rtimes T[w]$ and use a theorem by Nesin to prove $G \simeq \mathrm{PSL}_2$.

If not, work harder.
Use $T[w]$ sets to prove $C^\circ(i)$ is a Borel.
Kill strongly embedded configurations and prove the Pruefer rank can't be ≥ 3.
Some crossed $T[w]$ sets will eventually prove conjugacy.

The whole relies on heavy use of 0-unipotence theory and 0-Sylow theory by Burdges.