Countable Borel Equivalence Relations

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A topological space \((X, \tau)\) is called a Polish space if it is separable and completely metrizable.

A measurable space \((X, \Omega)\) is called a standard Borel space if \(\Omega\) is the Borel \(\sigma\)-algebra \(B(\tau)\) of some Polish topology \(\tau\) on \(X\).

E.g. \(\mathbb{R}, [0, 1], 2^{\omega}, \omega^{\omega}, [0, 1]^{\omega}\)

Let \(X, Y\) be standard Borel spaces.

A map \(\varphi : X \to Y\) is called Borel if it is measurable, i.e. \(f^{-1}[B]\) is Borel for all Borel subsets \(B \subseteq Y\).

Equivalently, \(\varphi : X \to Y\) is Borel if \(\text{graph}(\varphi)\) is a Borel subset of \(X \times Y\).
Theorem

A subspace $S$ of a Polish space $X$ is Polish if and only if it is a $G_δ$ subset of $X$.

This means that we cannot pass to some arbitrary subspace if we want to keep the induced topology same and Polish. On the other hand:

Theorem

Let $(X, τ)$ be a Polish space and $S ⊆ X$ be any Borel subset. Then there exists a Polish topology $τ_S ⊇ τ$ on $X$ such that $B(τ_S) = B(τ)$ and $S$ is clopen in $τ_S$.

Corollary

If $(X, B)$ is a standard Borel space and $Y ∈ B$, then $(Y, B \upharpoonright Y)$ is a standard Borel space.
The Borel Isomorphism Theorem

**Definition**

Two standard Borel spaces \((X, \Omega_1)\) and \((Y, \Omega_2)\) are called isomorphic if there exists a bimeasurable bijection between \(X\) and \(Y\).

A bimeasurable version of Schroder-Bernstein theorem holds for standard Borel spaces. For any uncountable standard Borel space \(X\), by embedding \(2^\omega\) into \(X\), \(X\) into \([0, 1]^\omega\) and \([0, 1]^\omega\) into \(2^\omega\) in a bimeasurable way, we have:

**Theorem (Kuratowski)**

*Any two uncountable standard Borel spaces are isomorphic.*
Let $\mathcal{L} = \{R_i : i \in I\}$ be a countable language where $R_i$ is an $n_i$-ary relation symbol and let $X_{\mathcal{L}} = \prod_{i \in I} 2^{\omega^{n_i}}$. Then $X_{\mathcal{L}}$ is a Polish space elements of which code $\mathcal{L}$-structures with universe $\omega$ as follows. For any $x = (x_i)_{i \in I} \in X_{\mathcal{L}}$, the structure

$$M_x = (\omega, \{R_i^x\}_{i \in I})$$

represented by $x$ is defined by:

$$R_i^x(k_1, \ldots, k_{n_i}) \Leftrightarrow x_i(k_1, \ldots, k_{n_i}) = 1$$

Example

If we let $\mathcal{L}$ consist of a single binary relation $E$, then the Polish space $2^{\omega \times \omega}$ codes the space of countable graphs with underlying set $\omega$. For any such "graph" $x \in 2^{\omega \times \omega}$, there is an edge between the vertices $i$ and $j$ if and only if $x(i, j) = 1$
Remark

If we consider the infinite symmetric group \( \text{Sym}(\omega) \) as a subspace of the Baire space \( \omega^\omega \), it becomes a Polish group with a natural Borel action on \( X_L \). Then \( x, y \in X_L \) are in the same \( \text{Sym}(\omega) \)-orbit if and only if \( M_x \cong M_y \).

Given any \( \mathcal{L}_{\omega_1,\omega} \) sentence \( \psi \), the class of all structures with underlying set \( \omega \) that models \( \psi \), \( \text{Mod}(\psi) = \{ x \in X_L : M_x \models \psi \} \) is an isomorphism-invariant Borel subset of \( X_L \).

Example

Let \( \mathcal{L} \) consist of a single ternary relation. If we associate any countable group \( (\omega, \cdot) \) with the characteristic function of \( \cdot \subseteq \omega \times \omega \times \omega \), then the class of countable groups, being axiomatized by a \( \mathcal{L}_{\omega_1,\omega} \)-sentence, is a Borel subset of \( X_L \) and thus itself is a standard Borel space.
The isomorphism relation on $\text{Mod}(\psi)$ is given by

$$x \cong y \iff \exists g \in \text{Sym}(\omega) \ g \cdot x = y$$

and is an analytic equivalence relation, being the projection of graph of a Borel action, and need not be Borel in general.

**Example (Mekler)**

The isomorphism relation on the space of countable groups $\cong_G$ is not Borel.

On the other hand, for the structures that are of "finite rank" in a broad sense, the isomorphism relation is a Borel relation. E.g. Finitely generated groups, finite rank torsion-free abelian groups, connected locally finite graphs,...
Standard Borel Space of Torsion-Free Abelian Groups of rank $n$

Recall that, up to isomorphism, torsion-free abelian groups of rank $n$ are exactly additive subgroups of $\mathbb{Q}^n$ with $n$ linearly independent elements. Then, for any $n \geq 1$, we can regard the set

$$R(\mathbb{Q}^n) = \{ x \in 2^{\mathbb{Q}^n} : x \leq \mathbb{Q}^n \wedge "x \text{ contains } n \text{ linearly independent elements"} \}$$

as the space of torsion-free abelian groups of rank $n$. Observe that this set is a Borel subset of $2^{\mathbb{Q}^n}$ and is itself a standard Borel space.

**Remark**

If $A, B \in R(\mathbb{Q}^n)$, then

$$A \cong B \iff \exists \varphi \in GL_n(\mathbb{Q}) \varphi[A] = B$$

This shows that $\cong$ on $R(\mathbb{Q}^n)$ is a Borel equivalence relation.
An example from topological dynamics

Fix some $n \geq 2$.

**Definition**

- A closed infinite subset $S$ of the Cantor space $n^\mathbb{Z}$ is called a subshift if it is invariant under the shift operator $(\sigma(x))(k) = x(k + 1)$.
- Two subshifts $S$ and $T$ are called topologically conjugate if there exists a homeomorphism $\psi : S \rightarrow T$ such that $\psi \circ \sigma = \sigma \circ \psi$. 

![Diagram showing the shift operator and subshifts](image-url)
**Definition**

Let $X$ be a Polish space and $K(X)$ be the set of all non-empty compact subsets of $X$. Then the Vietoris topology on $K(X)$ generated by the sets 
\[ \{ K \in K(X) : K \subseteq U \} \text{ and } \{ K \in K(X) : K \cap U \neq \emptyset \} \] for $U$ open in $X$ is a Polish topology. If $d$ is a complete metric on $X$ inducing its Polish topology, then the Hausdorff metric

\[ \delta_H(K, L) = \max\{ \max_{x \in K} d(x, L), \max_{x \in L} d(x, K) \} \]

is a compatible metric for the Vietoris topology.

**Theorem**

The collection $S_n$ of subshifts of $n^\mathbb{Z}$ is a Borel subset of $K(n^\mathbb{Z})$, and hence is itself a standard Borel space and the topological conjugacy relation on it is a Borel equivalence relation.
Borel Equivalence Relations

Definition

- Let $X$ be a standard Borel space. An equivalence relation $E \subseteq X^2$ is called Borel if it is a Borel subset of $X \times X$. A Borel equivalence relation is called countable if every $E$-equivalence class is countable.

- Let $G$ be a Polish group. A standard Borel $G$-space is a standard Borel space $X$ equipped with a Borel $G$-action. The corresponding orbit equivalence relation is denoted by $E_X^G$.

Example

Let $G$ be a countable group endowed with discrete topology and $X$ be a standard Borel $G$-space. Then, $E_X^G$ is a countable Borel equivalence relation.
Borel Reducibility

Definition

Let $E, F$ be Borel equivalence relations on standard Borel spaces $X$ and $Y$ respectively.

- We say $E$ is Borel reducible to $F$, denoted by $E \leq_B F$, if there exists a Borel map $f : X \to Y$ such that for all $x, y \in X$

\[ x \ E \ y \iff f(x) \ F \ f(y) \]

In this case, $f$ is said to be a reduction from $E$ to $F$.

- $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$.

- $E \prec_B F$ if $E \leq_B F$ but $F \not\leq_B E$.

If $E$ is Borel reducible to $F$, then the classification with respect to $E$ is, intuitively speaking, no harder than the classification with respect to $F$. The intuition behind the requirement that $f$ is Borel is that Borel maps are thought as “explicit computations”.
Remark

If we have a Borel equivalence relation on a countable standard Borel space of cardinality \( n \) (for \( 1 \leq n \leq \omega \)), then it is trivially reducible to the identity relation \( \Delta_n \) since any function that chooses an element from each class is a reduction.

Theorem (Silver)

Let \( E \) be a Borel equivalence relation on a standard Borel space. Then either \( E \leq_B \Delta_\omega \) or \( \Delta_{2\omega} \leq_B E \).

Definition

A Borel equivalence relation \( E \) is called smooth if \( E \leq_B \Delta_X \) for some (equivalently every) uncountable standard Borel space \( X \).
Examples of smooth Borel equivalence relations

Example

Let $E$ be a finite Borel equivalence relation on $X$, that is, a Borel equivalence relation with finite classes. Fix a Borel linear ordering $\leq$ on $X$. Then, $E$ is smooth via the Borel map $f(x) = \text{the } \leq-\text{least element of } [x]_E$.

A more interesting example:

Example

The class of countable divisible abelian groups in the space of countable groups can be axiomatized by a $\mathcal{L}_{\omega_1,\omega}$-sentence $\psi$ and hence, forms a standard Borel space on its own. Let $\cong_\psi$ denote the isomorphism relation on it. Any countable divisible abelian group $G$ can be written as $\left( \bigoplus_{i \in r_0(G)} \mathbb{Q} \right) \oplus \left( \bigoplus_{p \in \mathbb{P}} \bigoplus_{j \in r_p(G)} \mathbb{Z}[p^\omega] \right)$ where $0 \leq r_0(G), r_p(G) \leq \omega$ and these ranks determine $G$ up to isomorphism. Then, the Borel map $f(G) = (r_0(G), r_2(G), r_3(G), \ldots)$ witnesses the fact that $\cong_\psi$ is smooth.
Examples of non-smooth Borel equivalence relation

Example

Let $\mathbb{Z}$ act on $S^1$ by $n \cdot e^{i\theta} \mapsto e^{i(\theta + n)}$. The orbit equivalence relation $E_{\mathbb{Z}}^{S^1}$ is non-smooth for if it were smooth, then there would be a Borel set of $S^1$ intersecting each equivalence class at exactly one point. But all such sets are necessarily non-measurable since the action is measure preserving.

Example

Let $E_0$ be the countable Borel equivalence relation on $2^\omega$ defined by:

$$x \ E_0 \ y \iff \exists n \ \forall m \geq n \ x(m) = y(m)$$

Assume that there is a Borel reduction $f : 2^\omega \to [0, 1]$ from $E_0$ to $\Delta_{[0,1]}$. If we endow $2^\omega$ with its usual product probability measure, then both $f^{-1}[0, 1/2]$ and $f^{-1}[1/2, 1]$ are Borel tail events, and one of them has to have measure 1 by Kolmogorov 0-1 law. Continuing in this manner, we see that $f$ is constant almost everywhere, which is a contradiction.
Back to climbing up in the hierarchy

It turns out that $E_0$ is the immediate successor of $\Delta_{2^\omega}$ with respect to $\leq_B$

**Theorem (Harrington-Kechris-Louvea)**

Let $E$ be a Borel equivalence relation on a standard Borel space. Then either $E \leq_B \Delta_{2^\omega}$ or $E_0 \leq_B E$.

**Example ($\cong_1 \cong_B E_0$)**

Let $\cong_1$ be the isomorphism relation for torsion-free abelian groups of rank 1. For any $G \in R(\mathbb{Q})$, $0 \neq x \in G$ and prime $p \in \mathbb{P}$, set the $p$-height of $x$ to be $h_p(x) = \sup\{n \in \omega : \exists y \in G \ p^n y = x\} \in \omega \cup \{\infty\}$ and let the characteristic of $x$ be the sequence $\chi(x) = (h_p(x))_{p \in \mathbb{P}}$. In 1937, Baer proved that $G, H \in R(\mathbb{Q})$ are isomorphic if and only if for any non-zero $x \in G$, $y \in H$, $\chi(x)$ and $\chi(y)$ take the same values on almost all primes and they take the value $\infty$ on exactly the same indices. This condition defines an equivalence relation on $(\mathbb{N} \cup \{\infty\})^\mathbb{P}$ which is bireducible with $E_0$. 
The Feldman-Moore Theorem

Recall that any countable discrete group $G$ acting a standard Borel $G$-space induces a countable Borel equivalence relation as its orbit equivalence relation. Remarkably, the converse of this is also true:

**Theorem (Feldman-Moore)**

Let $E$ be a countable Borel equivalence relation on a standard Borel space $X$. Then, there exists a countable discrete group $G$ and a Borel $G$-action on $X$ such that $E = E^X_G$. Moreover, $G$ can be chosen such that

$$x \ E \ y \iff \exists g \in G \ g^2 = 1 \land g \cdot x = y$$

In order to prove this, we will need

**Theorem (Lusin-Novikov Uniformization Theorem)**

Let $X, Y$ be standard Borel spaces and $E \subseteq X \times Y$ be a Borel relation such that each section $E_x$ is countable. Then $\text{proj}_X(E)$ is Borel and $E = \bigcup_n f_n$ where $f_n$ are partial Borel functions of $X$. 
Sketch Proof of the Feldman-Moore Theorem

Let $E \subseteq X^2$ be a countable Borel equivalence relation with countable sections $E_x$ for all $x$.

- By Lusin-Novikov uniformization theorem, there exists partial Borel functions on $X$ such that $E = \bigcup_n f_n$. Without loss of generality, assume that $f_m \cap f_n = \emptyset$.

- Using the isomorphism between $X$ and $[0, 1]$, find disjoint Borel subsets $A_p, B_p$ for each $p \in \omega$ such that $X^2 - \Delta_X = \bigcup_{p \in \omega} A_p \times B_p$.

- If we set $f_{nmp} = f_n \cap f_m^{-1} \cap (A_p \times B_p)$, then each $f_{nmp}$ is a partial Borel bijection whose domain and range are disjoint.

- Extend each $f_{nmp}$ to some Borel automorphism $g_{nmp}$ of $X$ so that $E = \bigcup g_{nmp}$ (this can be done in such a way that each $g_{nmp}$ is an involution).

- Then $E = E_X^G$ for $G = \langle g_{nmp} \rangle$. 
An important consequence of Feldman-Moore

**Theorem (Dougherty-Jackson-Kechris)**

There exists a universal countable Borel equivalence relation $E_\omega$, i.e. for all countable Borel equivalence relations $E$ we have $E \leq_B E_\omega$.

**Definition**

- Let $F_\omega$ be the free group on $\omega$-many generators.
- Define the Borel action of $F_\omega$ on

$$\left(2^\omega\right)^{F_\omega} = \{ f \mid f : F_\omega \to 2^\omega \}$$

by setting

$$(g \cdot p)(h) = p(g^{-1}h)$$

for all $p : F_\omega \to 2^\omega$ and let $E_\omega$ be the orbit equivalence relation of this action.
Proof that $E_\omega$ is universal

- Let $E$ be a countable Borel equivalence relation on $X$. Then, by Feldman-Moore, there exists $G$ such that $E = E^X_G$.
- $G$ is a homomorphic image of $F_\omega$, so we can find some Borel action of $F_\omega$ inducing $E$ as its orbit equivalence relation.
- Let $\{U_i\}_{i \in \omega}$ be a sequence of Borel subsets of $X$ separating points and define the Borel map $f : X \to (2^\omega)^{F_\omega}$ by $x \mapsto f_x$ where
  \[(f_x(h))(i) = 1 \iff x \in h(U_i)\]
- Since $U_i$ separates points, $f$ is injective and
  \[(g \cdot f_x(h))(i) = 1 \iff x \in (f_x(g^{-1}h))(i) = 1 \iff x \in g^{-1}h(U_i) \iff g \cdot x \in h(U_i) \iff (f_{g \cdot x}(h))(i) = 1\]
Other examples of universal countable Borel equivalence relations

Remark

More generally, for any Borel action $G \curvearrowright X$ of some countable $G$, the corresponding orbit equivalence relation $E^X_G$ is Borel reducible to the orbit equivalence relation of the shift action $G \curvearrowright (2^\omega)^G$ by the same proof.

Theorem (Clemens, 2009)

Topological conjugacy on the space of subshifts $S_n$ is a universal countable Borel equivalence relation.

Theorem (Thomas-Velickovic, 1998)

The isomorphism relation on the space of finitely generated groups $G_{fg}$ is a universal countable Borel equivalence relation.
Do we have anything between $E_0$ and $E_\omega$?

**Theorem (Hjorth, 1998 (for $n = 1$), Thomas, 2001 (for $n \geq 2$))**

Let $\cong_n$ denote the isomorphism relation of torsion-free abelian groups of rank $n$. Then, $\cong_n <_B \cong_{n+1}$ for all $n \geq 1$.

**Theorem (Adams-Kechris, 2000)**

There exists $2^{\omega}$-many incomparable countable Borel equivalence relations.
A subshift $S \subseteq n^\mathbb{Z}$ is called minimal if $S$ has no proper $\sigma$-invariant closed subsets.

The subshifts constructed by Clemens to show universality of topological conjugacy on $S_n$ are not minimal. If we restrict topological conjugacy to the standard Borel space of minimal subshifts $\mathcal{M}_n$, where does it fit in the picture?

**Theorem (Gao-Jackson-Seward, 2011)**

*The topological conjugacy relation for minimal subshifts is not smooth.*

**Conjecture (Thomas)**

*The topological conjugacy relation for minimal subshifts is a universal countable Borel equivalence relation.*