The Markov-Zariski topology of an infinite group

Dikran Dikranjan

Mimar Sinan Güzel Sanatlar Üniversitesi
Istanbul
January 23, 2014

joint work with Daniele Toller and Dmitri Shakhmatov

1. Markov's problem 1 and 2
2. The three topologies on an infinite group
3. Problem 1 and 2 in topological terms
4. The Markov-Zariski topology of an abelian group
5. Markov's problem 3.

Markov's problem 1

Definition

A group G is *topologizable* if G admits a non-discrete Hausdorff group topology.

Problem 1. [Markov Dokl. AN SSSR 1944]

Does there exist a (countably) infinite non-topologizable group?

- Yes (under CH): Shelah, *On a problem of Kurosh, Jonsson groups, and applications*. In *Word Problems II*. (S. I. Adian, W. W. Boone, and G. Higman, Eds.) (North-Holland, Amsterdam, 1980), pp.373–394.
- Yes (in ZFC): Ol'shanskij, *A note on countable non-topologizable groups*. *Vestnik Mosk. Gos. Univ. Mat. Mekh.* (1980), no. 3, 103.

Markov's problem 2

Definition (Markov)

A subset S of a group G is called:

(a) elementary algebraic if $S = \{x \in G : a_1 x^{n_1} a_2 x^{n_2} a_3 \dots a_m x^{n_m} = 1\}$ for some natural m, integers $n_1, \dots, n_m$ and elements $a_1, a_2, \dots, a_m \in G$.

(b) algebraic, if S is an intersection of finite unions of elementary algebraic subsets.

(c) unconditionally closed, if S is closed in *every* Hausdorff group topology of G.

Every centralizer $c_G(a) = \{x \in G : axa^{-1}x^{-1} = 1\}$ is an elementary algebraic set, so $Z(G)$ is an algebraic set.

(a) $\to$ (b) $\to$ (c)

Problem 2. [Markov 1944]

Is (c) $\to$ (b) always true ?

The Zariski topology

$\mathfrak{E}_G$ the family of elementary algebraic sets of G.
$\mathfrak{A}_G^a$ the family of all finite unions of elementary algebraic sets of G.
$\mathfrak{A}_G$ the family of all algebraic sets of G.
The *Zariski* topology $\mathfrak{Z}_G$ of G has $\mathfrak{A}_G$ as family of all closed sets.
It is a T_1-topology as $\mathfrak{E}_G$ contains al singletons.

Example

(a) $\mathfrak{E}_\mathbb{Z} = \{\mathbb{Z}, \emptyset\} \cup \{\{n\} : n \in \mathbb{Z}\}$, so $\mathfrak{A}_G = \mathfrak{A}_G^a = \{\mathbb{Z}\} \cup [\mathbb{Z}]^{<\omega}$. Hence, $\mathfrak{Z}_\mathbb{Z}$ is the cofinite topology of $\mathbb{Z}$.

(b) Analogously, if G is a torsion-free abelian group and $S = \{x \in G : nx + g = 0\} \in \mathfrak{E}_G$, then either $S = G$ or $|S| \leq 1$, so again $\mathfrak{Z}_G$ is the cofinite topology of G.

(c) [Banakh, Guran, Protasov, Top. Appl. 2012] $\mathfrak{Z}_{Sym(X)}$ coincides with the point-wise convergence topology of the permutation group $Sym(X)$ of an infinite set X.

(a) and (b) show that $\mathfrak{Z}_G$ need not be a group topology.

Bryant, Roger M. *The verbal topology of a group.* J. Algebra 48 (1977), no. 2, 340–346.

Wehrfritz's MR-review to Bryant's paper:

This paper is beautiful, short, elementary and startling. It should be read by every infinite group theorist. The author defines on any group (by analogy with the Zariski topology) a topology which he calls the verbal topology. He is mainly interested in groups whose verbal topology satisfies the minimal condition on closed sets; for the purposes of this review call such a group a VZ-group.

The author proves that various groups are VZ-groups. By far the most surprising result is that every finitely generated abelian-by-nilpotent-by-finite group

is a VZ-group.

Less surprisingly, every abelian-by-finite group is a VZ-group. So is every linear group. Also, the class of VZ-groups is closed under taking subgroups and finite direct products.

The Markov topology and the $\mathfrak{P}$-Markov topology

The *Markov* topology $\mathfrak{M}_G$ of G has as closed sets all unconditionally closed subsets of G, in other words

$$\mathfrak{M}_G = \inf\{\text{all Hausdorff group topologies on } G\},$$

where inf taken in the lattice of all topologies on G.

$\mathfrak{P}_G = \inf\{\text{all precompact group topologies on } G\}$ - *precompact* Markov topology (a group is precompact if its completion is compact).
Clearly, $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$ are T_1 topologies.

Problem 2. [topological form]

Is $\mathfrak{Z}_G = \mathfrak{M}_G$ always true ?

- Perel′man (unpublished): Yes, for abelian groups
- Markov [1944]: Yes, for countable groups.
- Hesse [1979]: No in ZFC (Sipacheva [2006]: under CH Shelah's example works as well).

Markov's first problem through the looking glass of $\mathfrak{M}_G$

A group G $\mathfrak{Z}$-discrete (resp., $\mathfrak{M}$-discrete, $\mathfrak{P}$-discrete), if $\mathfrak{Z}_G$ (resp., $\mathfrak{M}_G$, resp., $\mathfrak{P}_G$) is discrete. Analogously, define $\mathfrak{Z}$-*compact*, etc.

- G is $\mathfrak{Z}$-discrete if and only if there exist $E_1, \ldots, E_n \in \mathfrak{E}_G$ such that $E_1 \cup \ldots \cup E_n = G \setminus \{e_G\}$;
- G is $\mathfrak{M}$-discrete iff G is non-topologizable. So, G is non-topologizable whenever G is $\mathfrak{Z}$-discrete.

Ol′shanskij proved that for Adian group $G = A(n, m)$ the quotient $G/Z(G)^m$ is a countable $\mathfrak{Z}$-discrete group, answering positively Porblem 1.

Example

(a) Klyachko and Trofimov [2005] constructed a finitely generated torsion-free $\mathfrak{Z}$-discrete group G.

(b) Trofimov [2005] proved that every group H admits an embedding into a $\mathfrak{Z}$-discrete group.

Example (negative answer to Problem 2)

(Hesse [1979]) There exists a $\mathfrak{M}$-discrete group G that is not $\mathfrak{Z}$-discrete.

Criterion [Shelah]

An uncountable group G is $\mathfrak{M}_G$-discrete whenever the following two conditions hold:

(a) there exists $m \in \mathbb{N}$ such that $A^m = G$ for every subset A of G with $|A| = |G|$;

(b) for every subgroup H of G with $|H| < |G|$ there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G$ such that the intersection $\bigcap_{i=1}^{n} x_i^{-1} H x_i$ is finite.

(i) The number n in (b) may depend of H, while in (a) the number m is *the same* for all $A \in [G]^{|G|}$.

(ii) Even the weaker form of (a) (with m depending on A), yields that every proper subgroup of G has size $< |G|$ (if $|G| = \omega_1$, groups with this property are known as *Kurosh groups*).

(iii) Using the above criterion, Shelah produced an example of an $\mathfrak{M}$-discrete group under the assumption of CH. Namely, a torsion-free group G of size ω_1 satisfying (a) with $m = 10000$ and (b) with $n = 2$. So every proper subgroup H of G is malnormal (i.e., $H \cap x^{-1}Hx = \{1\}$), so G is also simple.

Proof.

Let $\mathcal{T}$ be a Hausdorff group topology on G. There exists a $\mathcal{T}$-neighbourhood V of e_G with $V \neq G$. Choose a $\mathcal{T}$-neighbourhood W of e_G with $W^m \subseteq V$. Now $V \neq G$ and (a) yield $|W| < |G|$. Let $H = \langle W \rangle$. Then $|H| = |W| \cdot \omega < |G|$. By (b) the intersection $O = \bigcap_{i=1}^{n} x_i^{-1}Hx_i$ is finite for some $n \in \mathbb{N}$ and elements $x_1, \ldots, x_n \in G$. Since each $x_i^{-1}Hx_i$ is a $\mathcal{T}$-neighbourhood of 1, this proves that $1 \in O \in \mathcal{T}$. Since $\mathcal{T}$ is Hausdorff, it follows that $\{1\}$ is $\mathcal{T}$-open, and therefore $\mathcal{T}$ is discrete. □

$\mathfrak{Z}$-Noetherian groups

A topological space X is Noetherian, if X satisfies the ascending chain condition on open sets (or, equivalently, the minimal condition on closed sets). Obviously, a Noetherian space is compact, and a subspace of a Noetherian space is Noetherian itself. Actually, a space is Noetherian iff all its subspaces are compact (so an infinite Noetherian spaces are never Hausdorff).

Theorem

- *(Bryant) A subgroup of a $\mathfrak{Z}$-Noetherian group is $\mathfrak{Z}$-Noetherian,*
- *(D.D. - D. Toller) A group G is $\mathfrak{Z}$-Noetherian iff every countable subgroup of G is $\mathfrak{Z}$-Noetherian.*

Using the fact that linear groups are $\mathfrak{Z}$-Noetherian, and the fact that countable free groups are isomorphic to subgroups of linear groups, one gets

Theorem (Guba Mat. Zam.1986, indep., D. Toller - DD, 2012)

Every free group is $\mathfrak{Z}$-Noetherian.

The Zariski topology of a direct product

The Zariski topology $\mathfrak{Z}_G$ of the direct product $G = \prod_{i \in I} G_i$ is coarser than the product topology $\prod_{i \in I} \mathfrak{Z}_{G_i}$.

These two topologies need not coincide (for example $\mathfrak{Z}_{\mathbb{Z} \times \mathbb{Z}}$ is the co-finite topology of $\mathbb{Z} \times \mathbb{Z}$, so neither $\mathbb{Z} \times \{0\}$ nor $\{0\} \times \mathbb{Z}$ are Zariski closed in $\mathbb{Z} \times \mathbb{Z}$, whereas they are closed in $\mathfrak{Z}_{\mathbb{Z}} \times \mathfrak{Z}_{\mathbb{Z}}$).

Item (B) of the next theorem generalizes Bryant's result.

Theorem (DD - D. Toller, Proc. Ischia 2010)

(A) Direct products of $\mathfrak{Z}$-compact groups are $\mathfrak{Z}$-compact.

(B) $G = \prod_{i \in I} G_i$ is $\mathfrak{Z}$-Noetherian iff every G_i is $\mathfrak{Z}$-Noetherian and all but finitely many of the groups G_i are abelian.

According to Bryant's theorem, abelian groups are $\mathfrak{Z}$-Noetherian.

Corollary

A nilpotent group of nilpotency class 2 need not be $\mathfrak{Z}$-Noetherian.

Take an infinite power of finite nilpotent group, e.g., Q_8.

$\mathfrak{Z}$-Hausdorff groups and $\mathfrak{M}$-Hausdroff groups

If $\{F_i \mid i \in I\}$ is a family of finite groups, and $G = \prod_{i \in I} F_i$, then the product $\prod_{i \in I} \mathfrak{Z}_{F_i}$ is a compact Hausdorff group topology, so

$$\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G \subseteq \prod_{i \in I} \mathfrak{Z}_{F_i}.$$

(1) G is $\mathfrak{Z}$-Hausdorff if and only if $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \prod_{i \in I} \mathfrak{Z}_{F_i}$.

(2) G is $\mathfrak{M}$-Hausdorff if and only if $\mathfrak{M}_G = \mathfrak{P}_G = \prod_{i \in I} \mathfrak{Z}_{F_i}$.

Theorem (DD - D. Toller, Proc. Ischia 2010)

If $\{F_i \mid i \in I\}$ is a non-empty family of finite center-free groups, and $G = \prod_{i \in I} F_i$, then $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \prod_{i \in I} \mathfrak{Z}_{F_i}$ is a Hausdorff group topology on G.

Theorem (Gaughan Proc. Nat. Acad. USA 1966)

The permutation group $Sym(X)$ of an infinite set X is $\mathfrak{M}$-Hausdorff.

Since $\mathfrak{Z}$-Hausdorff $\Rightarrow$ $\mathfrak{M}$-Hausdorff, this follows also from Banakh-Guran-Protasov theorem. In particular, $\mathfrak{M}_{Sym(X)} = \mathfrak{Z}_{Sym(X)}$ coincides with the point-wise convergence topology of $Sym(X)$.

$\mathfrak{P}$-discrete groups

A group G is $\mathfrak{P}$-discrete iff G admits no precompact group topologies (i.e., G is not maximally almost periodic, in terms of von Neumann).
In parfticular, examples of $\mathfrak{P}$-discrete groups are provided by all minimally almost periodic (again in terms of von Neumann, these are the groups G such that every homomorphism to a compact group K is trivial).

Example

(a) (von Neumann and Wiener) $SL_2(\mathbb{R})$;
(b) The permutation group $Sym(X)$ of an infinite set X (as $\mathfrak{M}_{Sym(X)}$ is not precompact).

Theorem (DD - D. Toller, Topology Appl. 2012)

Every divisible solvable non-abelian group is $\mathfrak{P}$-discrete.

Proof.

Let G be a divisible solvable non-abelian group. It suffices to see that G admits no precompact group topology. To this end we show that every divisible precompact solvable group must be abelian.
Let G be a divisible precompact solvable group. Then its completion K is a connected group. On the other hand, K is also solvable. It is enough to prove that K is abelian.
Arguing for a contradiction, assume that $K \neq Z(K)$, is not abelian. By a theorem of Varopoulos, $K/Z(K)$ is isomorphic to a direct product of simple connected compact Lie groups, in particular, $K/Z(K)$ cannot be solvable. On the other hand, $K/Z(K)$ has to be solvable as a quotient of a solvable group, a contradiction. □

Corollary

For every field K with $\mathrm{char} K = 0$ the Heisenberg group

$$H_K = \begin{pmatrix} 1 & K & K \\ & 1 & K \\ & & 1 \end{pmatrix} \text{ is } \mathfrak{P}\text{-discrete.}$$

The Zariski topology of an abelian group: Markov's problem 3

Definition (Markov, Izv. AN SSSR 1945)

A subset A of a group G is potentially dense in G if there exists a Hausdorff group topology $\mathcal{T}$ on G such that A is $\mathcal{T}$-dense in G.

Example (Markov)

Every infinite subset of $\mathbb{Z}$ is potentially dense in $\mathbb{Z}$.

By Weyl's uniform disitribution theorem for every infinite $A = (a_n)$ in $\mathbb{Z}$ there exists $\alpha \in \mathbb{R}$ such that $(a_n\alpha)$ is uniformly distributed modulo 1, so the subset $(a_n\overline{\alpha})$ of $\mathbb{R}/\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$ (so in $\langle\overline{\alpha}\rangle$ as well). Now the topology $\mathcal{T}$ on $\mathbb{Z}$ induced by $\mathbb{Z} \cong \overline{\alpha} \hookrightarrow \mathbb{R}/\mathbb{Z}$ works.

Problem 3 [Markov]

Characterize the potentially dense subsets of an abelian group.

A hint. [two necessary conditions]

- a potentially dense set is Zarisky-dense;
- if G has a countable potentially dense set, then $|G| \leq 2^{\mathfrak{c}}$.

Theorem (Tkachenko-Yaschenko, Topology Appl. 2002)

If an Abelian group with $|G| \leq \mathfrak{c}$ is either torsion-free or has exponent p, then every infinite set of G is potentially dense.

Question [Tkachenko-Yaschenko]

Can this be extended to groups with $|G| \leq 2^{\mathfrak{c}}$?

The answer is (more than) positive:

Theorem (DD - D. Shakhmatov, Adv. Math. 2011)

For a countably infinite subset A of an Abelian group G TFAE:
(i) A is potentially dense in G,
(ii) there exists a precompact Hausdorff group topology on G such that A becomes $\mathcal{T}$-dense in G,
(iii) $|G| \leq 2^{\mathfrak{c}}$ and A is Zarisky dense in G.

The proof if based on a realization theorem for the Zariski closure by means of (metrizable) precompact group topologies.

For $n \in \omega$ and $E \subseteq G$ let
$G[n] = \{x \in G : nx = 0\}$ and $nE = \{nx : x \in E\}$.
$\forall E \in \mathfrak{E}_G, \exists a \in G, n \in \omega$ such that

$$E = a + G[n] = \{x \in G : nx = na\}.$$

So $\mathfrak{E}_G$ is stable under finite intersections:
$(a+G[n])\cap(b+G[m])=c+G[d]$, with $d=GCD(m,n)$ (if $\neq \emptyset$)

Lemma

If G is abelian, then $\mathfrak{A}_G$ consists of finite unions of elementary algebraic sets $\mathfrak{E}_G$, i.e., $\mathfrak{A}_G = \mathfrak{A}_G^a$. Moreover:
(a) $(G, \mathfrak{Z}_G)$ is Noetherian (hence, compact).
(b) $\mathfrak{Z}_G|_H = \mathfrak{Z}_H$ and $\mathfrak{M}_G|_H = \mathfrak{M}_H$ or every subgroup H of G.

All these propertirs are false in the non-abelian case (e.g., when G is a countable $\mathfrak{Z}$-discrete group).

Example

$\mathfrak{Z}_G$ coincides with the cofinite topology of an abelian group G iff either $r_p(G) < \infty$ for all primes p or G has a prime exponent p.

An algebraic description of the $\mathfrak{Z}$-irredducible sets

Definition

A topological space X is irreducible, if $X = F_1 \cup F_2$ with closed F_1, F_2 yields $X = F_1$ or X_2.

Lemma

For a countably infinite subset A of G TFAE:

(a) A is irreducible;

(b) A carries the cofinite tiopology;

(c) there exists $n \in \mathbb{N}$ such that for every $a \in A$

(†) *$E = A - a$ satisfies $nE = 0$ and $\{x \in E : dx = h\}$ is finite for each $h \in G$ and every divisor d of n with $d \neq n$.*

Let $\mathfrak{T}(G) = \{E \in \mathcal{P}(G) : E$ is irreducible and $0 \in cl_{\mathfrak{Z}_G}(E)\}$. For every $E \in \mathfrak{T}(G)$ the set $E_0 = E \cup \{0\}$ is still irreducible. Let $o(E) = o(E_0)$ be the number n determined by (†) and let $\mathfrak{T}_n(G) = \{E \in \mathfrak{T}(G) : o(E) = n\}$. Then $\mathfrak{T}(G) = \bigcup_n \mathfrak{T}_n(G)$, $\mathfrak{T}_1(G) = \emptyset$ and $\mathfrak{T}_m(G) \cap \mathfrak{T}_n(G) = \emptyset$ whenever $n \neq m$.

$E \in \mathfrak{T}_n(G)$ iff every infinite subset of E is $\mathfrak{Z}_G$-dense in $G[n]$.

Example

Let G be an infinite abelian group.
(a) Every countably infinite subset of G is irreducible if G is torsion-free.
(b) $\mathfrak{T}_0(G) = \emptyset$ iff G is bounded.
(c) $\mathfrak{T}_n(G) \neq \emptyset$ for some $n > 1$ iff there exists a monomorphism $\bigoplus_\omega \mathbb{Z}(n) \hookrightarrow G$.

Theorem

Let S be an infinite subset of an abelian group G. Then there exist a finite $F \subseteq S$, infinite subsets $\{S_i : i = 1, 2, \ldots, k\}$ of S and a finite set $\{a_1, a_2, \ldots, a_k\}$ of G such that
(a) $S_i - a_i \in \mathfrak{T}_{n_i}(G)$ for some $n_i \in \omega \setminus \{1\}$;
(b) $S = F \cup \bigcup_{i=1}^k S_i$;
(c) $cl_{\mathfrak{Z}_G}(S) = F \cup \bigcup_i cl_{\mathfrak{Z}_G}(S_i)$ and each S_i is $\mathfrak{Z}_G$-dense in $G[n_i]$.

The realzation theorem

Theorem (DD - D. Shakhmatov, J. Algebra 2010)

Let G be an Abelian group with $|G| \leq \mathfrak{c}$ and $\mathcal{E}$ be a countable family in $\mathfrak{T}(G)$. Then there exists a metrizable precompact group topology $\mathcal{T}$ on G such that $cl_{\mathfrak{Z}_G}(S) = cl_{\mathcal{T}}(S)$ for all $S \in \mathcal{E}$.

The realization of the Zariski closure of uncountably many sets is impossible in general.

Corollary

For an abelian group G with $|G| \leq 2^{\mathfrak{c}}$ the following are equivalent:
(a) every infinite subset of G is potentially dense in G;
(b) G is either almost torsion-free or has exponent p for some prime p;
(c) every Zariski-closed subset of G is finite.

This corollary resolves Tkachenko-Yaschenko's problem.

Corollary

$\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ for every abelian group G.