Quantum Groups, R-Matrices and Factorization

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Algebra

Definition

Let $A$ be a vector space over $\mathbb{k}$ and $\mu : A \otimes A \rightarrow A$ and $\eta : \mathbb{k} \rightarrow A$ be linear maps. The triple $(A, \mu, \eta)$ is said to be an algebra if the following diagrams commute:

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\
\text{id} \otimes \mu & \downarrow & \mu \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
\]

\[
\begin{array}{ccc}
k \otimes A & \xrightarrow{\eta \otimes \text{id}} & A \otimes A \\
\mu & \downarrow & \mu \\
A & \xrightarrow{\mu} & A
\end{array}
\]

\[
\begin{array}{ccc}
k \otimes A & \xleftarrow{\text{id} \otimes \eta} & A \otimes k \\
\mu & \downarrow & \mu \\
A & \xleftarrow{\mu} & A
\end{array}
\]
Coalgebra

Definition

Let $A$ be a vector space over $k$ and $\Delta : A \to A \otimes A$ and $\varepsilon : A \to k$ be linear maps. The triple $(A, \Delta, \varepsilon)$ is said to be a coalgebra if the following diagrams commute:

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\downarrow \Delta & & \downarrow \text{id} \otimes \Delta \\
A \otimes A & \xrightarrow{\Delta \otimes \text{id}} & A \otimes A \otimes A
\end{array}
\]

\[
\begin{array}{ccc}
k \otimes A & \xleftarrow{\varepsilon \otimes \text{id}} & A \otimes A \\
\downarrow \text{id} \otimes \varepsilon & & \downarrow \Delta \\
A & \xrightarrow{\Delta} & A \otimes k
\end{array}
\]
Notation

(Sweedler’s sigma notation) In order avoid the complexity of index notation we write

\[ \Delta(x) = \sum_{(x)} x' \otimes x'' \]

for any \( x \in A \).
If \((A, \mu, \eta)\) is an algebra then so is \((A \otimes A, \mu \otimes \mu, \eta \otimes \eta)\).
• If \((A, \mu, \eta)\) is an algebra then so is \((A \otimes A, \mu \otimes \mu, \eta \otimes \eta)\).

• If \((A, \Delta, \varepsilon)\) is a coalgebra then so is \\
\((A \otimes A, (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta), \varepsilon \otimes \varepsilon)\), where \(\tau(a \otimes b) = b \otimes a\).
Bialgebra

Definition

Let \((A, \mu, \eta)\) be an algebra and \((A, \Delta, \varepsilon)\) is a coalgebra. The quintuple \((A, \mu, \eta, \Delta, \varepsilon)\) is said to be a bialgebra if the maps \(\mu\) and \(\eta\) are morphisms of coalgebras or equivalently, the maps \(\Delta\) and \(\varepsilon\) are morphisms of algebras.
Hopf Algebra

Definition

Let \((H, \mu, \eta, \Delta, \varepsilon)\) be a bialgebra. An endomorphism \(S\) of \(H\) is called an antipode for the bialgebra \(H\) if

\[
\sum \frac{S(x')x''}{(x)} = \sum \frac{x'S(x'')}{(x)} = \varepsilon(x)1
\]

for all \(x \in H\).

A Hopf algebra is a bialgebra with an antipode.
R-matrix

**Definition**

Let $V$ be a vector space. An automorphism $c$ of $V \otimes V$ is called an R-matrix if it satisfies the Yang-Baxter equation

$$(c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_V \otimes c)(c \otimes id_V)(id_V \otimes c)$$

which holds in the automorphism group of $V \otimes V \otimes V$.
A bialgebra \((H, \mu, \eta, \Delta, \varepsilon)\) is called quasi-cocommutative if there exists an invertible element \(R\) of the algebra \(H \otimes H\) such that for all \(x \in H\) we have

\[
\Delta^{\text{op}}(x) = R\Delta(x)R^{-1}.
\]

Here \(\Delta^{\text{op}} = \tau_{H,H} \circ \Delta\) where \(\tau_{H,H}(h_1 \otimes h_2) = h_2 \otimes h_1\). \(R\) is called the universal R-matrix of the bialgebra \(H\). A Hopf algebra is quasi-cocommutative if its underlying bialgebra is quasi-cocommutative.
Braided Hopf Algebra

Definition

A quasi-cocommutative bialgebra \((H, \mu, \eta, \Delta, \varepsilon, R)\) or a quasi-cocommutative Hopf algebra \((H, \mu, \eta, \Delta, \varepsilon, S, R)\) is braided if the universal \(R\)-matrix satisfies the following relations:

\[
(\Delta \otimes id_H)(R) = R_{13}R_{23}
\]

\[
(id_H \otimes \Delta)(R) = R_{13}R_{12}.
\]
R-matrix from a Braided Hopf Algebra

Let \((H, \mu, \eta, \Delta, \varepsilon, R)\) be a braided bialgebra and \(V\) be an \(H\)-module. The automorphism \(c^R_{V,V}\) of \(V \otimes V\) defined by

\[
c^R_{V,V}(v \otimes w) = \tau_{V,V}[R(v \otimes w)]
\]

is an R-matrix.
Module-coalgebra

Definition

Let \((H, \mu, \eta, \Delta_H, \varepsilon_H)\) be a bialgebra and \((C, \Delta_C, \varepsilon_C)\) be a coalgebra. \(C\) is said to be a module-coalgebra over \(H\) if there exists a morphism of coalgebras \(\phi : H \otimes C \to C\) inducing an \(H\)-module structure on \(C\), that is,

\[
(\phi \otimes \phi)\Delta_H \otimes C = \Delta_C \phi \\
\varepsilon_H \otimes C = \varepsilon_C \phi \\
\phi(\mu \otimes \text{id}_C) = \phi(\text{id}_H \otimes \phi) \\
\phi(\eta \otimes \text{id}_C) = \text{id}_C
\]
Matched pair

**Definition**

A pair \((X, A)\) of bialgebras is matched if there exist linear maps \(\alpha : A \otimes X \to X\) and \(\beta : A \otimes X \to A\) turning \(X\) into a module-coalgebra over \(A\), and turning \(A\) into a right module-coalgebra over \(X\), such that, if we set

\[
\alpha(a \otimes x) = a \cdot x \quad \text{and} \quad \beta(a \otimes x) = a^x,
\]

the following conditions are satisfied:
Definition

\[ a \cdot (xy) = \sum (a' \cdot x')(a''x'' \cdot y), \]
\[ a \cdot 1 = \varepsilon(a)1, \]
\[ (ab)x = \sum a^{b' \cdot x'} b''x'' , \]
\[ 1^x = \varepsilon(x)1, \]
\[ \sum a'^{x'} \otimes a'' \cdot x'' = \sum a''^x \otimes a' \cdot x' \]

for all \( a, b \in A \) and \( x, y \in X \).
Theorem

Let \((X, A)\) be a matched pair of bialgebras. There exists a unique bialgebra structure on the vector space \(X \otimes A\), called the bicrossed product of \(X\) and \(A\) and denoted by \(X \Join A\), such that its product, unit, coproduct and counit are given by

- \((x \otimes a)(y \otimes b) = \sum_{(a)(y)} x(a' \cdot y') \otimes a''y'' b\),
- \(\eta(1) = 1 \otimes 1\),
- \(\Delta(x \otimes a) = \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'')\),
- \(\varepsilon(x \otimes a) = \varepsilon(x)\varepsilon(a)\) for all \(x, y \in X\) and \(a, b \in A\).
Theorem

Let \((X, A)\) be a matched pair of bialgebras. There exists a unique bialgebra structure on the vector space \(X \otimes A\), called the bicrossed product of \(X\) and \(A\) and denoted by \(X \boxtimes A\), such that its product, unit, coproduct and counit are given by

1. \((x \otimes a)(y \otimes b) = \sum (a)(y)x(a' \cdot y') \otimes a''y'' b,\)
2. \(\eta(1) = 1 \otimes 1,\)
3. \(\Delta(x \otimes a) = \sum (a)(x)(x' \otimes a') \otimes (x'' \otimes a''),\)
4. \(\varepsilon(x \otimes a) = \varepsilon(x)\varepsilon(a)\)
   for all \(x, y \in X\) and \(a, b \in A.\)
5. If the bialgebras \(X\) and \(A\) have antipodes, \(X \boxtimes A\) is a Hopf algebra.
Theorem

Let $H = (H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ be a finite-dimensional Hopf algebra and $X = (H^{op})^* = (H^*, \Delta^*, \varepsilon^*, (\mu^{op})^*, \eta^*, (S^{-1})^*, S^*)$ be the dual of the opposite Hopf algebra. Let $\alpha : H \otimes X \to X$ and $\beta : H \otimes X \to H$ be the linear maps given by

$$\alpha(a \otimes f) = a \cdot f = \sum_{(a)} f(S^{-1}(a'')a'),$$

and

$$\beta(a \otimes f) = a^f = \sum_{(a)} f(S^{-1}(a'''a')a'')$$

for $a \in H$ and $f \in X$, where $f(S^{-1}(a'')a')$ is the map defined by $f(S^{-1}(a'')a')(x) = f(S^{-1}(a'')xa')$, for all $x \in H$. Then the pair $(H, X)$ is matched.
Quantum double

**Definition**

The quantum double of $H$ is defined by

$$D(H) = X \bowtie H$$

where $H$ is a finite-dimensional Hopf algebra with invertible antipode and $X = (H^\text{op})^*$. 
Theorem

Let \( \{ e_i \}_{i \in I} \) be a basis of \( H \) and \( \{ e^i \}_{i \in I} \) be its dual basis. \( D(H) \) is a braided Hopf algebra with the universal R-matrix

\[
R = \sum_{i \in I} (1 \otimes e_i) \otimes (e^i \otimes 1).
\]
Definition

Let $p$ and $q$ be nonzero elements of a field $K$ and $M_{p,q}(n) = K\{a_{ij} | i, j \in \{1, 2, ..., n\}\}/I$ be the quotient of the free algebra generated by the generators $\{a_{ij} | i, j \in \{1, 2, ..., n\}\}$ over $K$ by the two-sided ideal $I$ generated by the relations

\[
\begin{align*}
    a_{il}a_{ik} &= pa_{ik}a_{il}, \\
    a_{jk}a_{ik} &= qa_{ik}a_{jk}, \\
    a_{jk}a_{il} &= p^{-1}qa_{il}a_{jk}, \\
    a_{jl}a_{ik} &= a_{ik}a_{jl} + (p - q^{-1})a_{jk}a_{il}
\end{align*}
\]

whenever $j > i$ and $l > k$. 
Bialgebra Structure of $M_{p,q}(n)$

Define coproduct and counit on the generators as follows:

$$
\Delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj}
$$

$$
\varepsilon(a_{ij}) = \delta_{ij}
$$

where $\delta_{ij}$ is the Kronecker delta and extend these maps to $M_{p,q}(n)$ as algebra maps.
Special Cases

- $p = q \implies M_q(n)$
Special Cases

- $p = q \implies M_q(n)$
- $\det_q = 1 \implies SL_q(n)$
Special Cases

- $p = q \implies M_q(n)$
- $det_q = 1 \implies SL_q(n)$
- $det_q \neq 0 \implies GL_q(n)$
Definition

Let $U_q gl(n)$ be the algebra generated by $e_i, f_i, k_j, k_j^{-1}, i = 1, 2, ..., n-1, j = 1, 2, ..., n$ with the following relations:

\[
\begin{align*}
k_i k_j &= k_j k_i, \\
k_i e_j k_i^{-1} &= q^{\delta_{i,j} - \delta_{i,j+1}} e_j, \\
k_i f_j k_i^{-1} &= q^{-\delta_{i,j} + \delta_{i,j+1}} f_j, \\
e_i f_j - f_j e_i &= \delta_{i,j} \frac{k_i k_{i+1}^{-1} - k_i^{-1} k_{i+1}}{q - q^{-1}}, \\
e_i e_j &= e_j e_i, f_i f_j = f_j f_i, \text{ if } |i - j| \geq 2, \\
e_i^2 e_i \pm 1 + e_i \pm 1 e_i^2 &= (q + q^{-1}) e_i e_i \pm 1 e_i, \\
f_i^2 f_i \pm 1 + f_i \pm 1 f_i^2 &= (q + q^{-1}) f_i f_i \pm 1 f_i.
\end{align*}
\]
Hopf Algebra Structure of $U_q gl(n)$

Define coproduct, counit and antipode on the generators as follows:

$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$
$$\Delta(e_i) = e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i,$$
$$\Delta(f_i) = f_i \otimes 1 + k_{i+1}^{-1} k_i \otimes f_i,$$
$$\varepsilon(k_i^{\pm 1}) = 1,$$
$$\varepsilon(e_i) = \varepsilon(f_i) = 0,$$
$$S(k_i) = k_i^{-1},$$
$$S(e_i) = -e_i k_i^{-1} k_{i+1},$$
$$S(f_i) = -k_i k_{i+1}^{-1} f_i.$$

and extend $\Delta$ and $\varepsilon$ on $U_q gl(n)$ as algebra homomorphisms and $S$ as an algebra antihomomorphism.
Definition

Let

\[ R_{p,q}(n) = K\{x_i^{(k)}, y_i^{(k)} | k \in \{1, 2, ..., n-1\}, i \in \{1, 2, ..., 2n-1\}\}/J \]

be the quotient of the free algebra over \( K \) generated by the generators \( \{x_i^{(k)}, y_i^{(k)} | k \in \{1, 2, ..., n-1\}, i \in \{1, 2, ..., 2n-1\}\} \) by the two-sided ideal \( J \) generated by the relations...
Definition

\[
x^{(k)}_{2i} x^{(k)}_{2i-1} = px^{(k)}_{2i-1} x^{(k)}_{2i},
\]
\[
x^{(k)}_i x^{(k)}_j = x^{(k)}_j x^{(k)}_i,
\]
\[
y^{(k)}_{2i+1} y^{(k)}_{2i} = py^{(k)}_{2i} y^{(k)}_{2i+1},
\]
\[
y^{(k)}_i y^{(k)}_j = y^{(k)}_j y^{(k)}_i,
\]
\[
x^{(k_3)}_i y^{(k_4)}_l = y^{(k_4)}_l x^{(k_3)}_i
\]

for every \(i, j, k, l, k_1, k_2, k_3, k_4\) where \(k_1 \neq k_2, |j - i| \geq 2\).
$X^{(k)} = \begin{pmatrix}
  x_1^{(k)} & x_2^{(k)} & x_3^{(k)} & x_4^{(k)} & \cdots & \cdots \\
  x_2^{(k)} & x_3^{(k)} & \cdots & \cdots & \cdots & \cdots \\
  x_3^{(k)} & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & x_4^{(k)} & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & x_{2n-2}^{(k)} & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & x_{2n-1}^{(k)}
\end{pmatrix}$
Quantum Groups
Factorization
Duality Between $U_{q \mathfrak{gl}(n)}$ and $M_q(n)$

Factorization of $M_{p,q}(n)$
P.B.W. Basis of $U_{q \mathfrak{gl}(n)}$

$Y(k) = \begin{pmatrix}
y_1^{(k)} & y_2^{(k)} & y_3^{(k)} & y_4^{(k)} & y_5^{(k)} \\
y_2^{(k)} & y_3^{(k)} & y_4^{(k)} & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
& & & y_{2n-2}^{(k)} & y_{2n-1}^{(k)}
\end{pmatrix}$
Theorem

The map $\phi : M_{p,q}(n) \rightarrow R_{p,q}(n)$ mapping $a_{ij}$ to $\hat{a}_{ij}$, where $\hat{a}_{ij}$ is the $ij$th entry of the matrix $\hat{A} = X^{(1)} X^{(2)} ... X^{(n-1)} Y^{(1)} Y^{(2)} ... Y^{(n-1)}$, is well-defined, i.e. the entries of $\hat{A} = (\hat{a}_{ij})$ satisfy relations

$$
\hat{a}_{il} \hat{a}_{ik} = p \hat{a}_{ik} \hat{a}_{il},
\hat{a}_{jk} \hat{a}_{ik} = q \hat{a}_{ik} \hat{a}_{jk},
\hat{a}_{jk} \hat{a}_{il} = p^{-1} q \hat{a}_{il} \hat{a}_{jk},
\hat{a}_{jl} \hat{a}_{ik} = \hat{a}_{ik} \hat{a}_{jl} + (p - q^{-1}) \hat{a}_{jk} \hat{a}_{il}
$$

whenever $j > i$ and $l > k$. 
Sketch of the Proof

Use infinite matrices $\tilde{X}^{(1)}, \tilde{X}^{(2)}, \ldots, \tilde{X}^{(n)}$.

**Lemma**

$$
(\tilde{X}^{(1)} \tilde{X}^{(2)} \ldots \tilde{X}^{(n)})_{ij} = \begin{cases} 
  x_{2i-1}^{(1)} x_{2i-1}^{(2)} \ldots x_{2i-1}^{(n)} & \text{if } i = j \\
  \sum_{k_1=1}^{n} \sum_{k_2=2}^{k_3-1} \omega & \text{if } 0 < j - i \leq n \\
  0 & \text{otherwise}
\end{cases}
$$

where $\omega = x_{2i-1}^{(1)} x_{2i-1}^{(2)} \ldots x_{2i-1}^{(k_1-1)} x_{2i}^{(k_1)} x_{2i+1}^{(k_1+1)} \ldots x_{2j-3}^{(k_j-i-1)} x_{2j-1}^{(k_j-i)} x_{2j-1}^{(k_j-i+1)} \ldots x_{2j-1}^{(n-1)} x_{2j-1}^{(n)}$. 
\[ \tilde{A}^{(n)}_{ij} = \tilde{A}^{(n-1)}_{ij-1} x_{2j-2} + \tilde{A}^{(n-1)}_{ij} x_{2j-1}, \]

If \( d > c \)

\[
\tilde{A}^{(n)}_{ad} \tilde{A}^{(n)}_{ac} = (\tilde{A}^{(n-1)}_{ad-1} x_{2d-2} + \tilde{A}^{(n-1)}_{ad} x_{2d-1})(\tilde{A}^{(n-1)}_{ac-1} x_{2c-2} + \tilde{A}^{(n-1)}_{ac} x_{2c-1})
\]

\[
= \tilde{A}^{(n-1)}_{ad-1} x_{2d-2} \tilde{A}^{(n-1)}_{ac-1} x_{2c-2} + \tilde{A}^{(n-1)}_{ad} x_{2d-1} \tilde{A}^{(n-1)}_{ac-1} x_{2c-2}
\]

\[
+ \tilde{A}^{(n-1)}_{ad-1} x_{2d-2} \tilde{A}^{(n-1)}_{ac} x_{2c-1} + \tilde{A}^{(n-1)}_{ad} x_{2d-1} \tilde{A}^{(n-1)}_{ac} x_{2c-1}
\]

\[
= p \tilde{A}^{(n-1)}_{ac-1} x_{2c-2} \tilde{A}^{(n-1)}_{ad-1} x_{2d-2} + p \tilde{A}^{(n-1)}_{ac-1} x_{2c-2} \tilde{A}^{(n-1)}_{ad} x_{2d-1}
\]

\[
+ p \tilde{A}^{(n-1)}_{ac} x_{2c-1} \tilde{A}^{(n-1)}_{ad-1} x_{2d-2} + p \tilde{A}^{(n-1)}_{ac} x_{2c-1} \tilde{A}^{(n-1)}_{ad} x_{2d-1}
\]

\[
= p(\tilde{A}^{(n-1)}_{ac-1} x_{2c-2} + \tilde{A}^{(n-1)}_{ac} x_{2c-1})(\tilde{A}^{(n-1)}_{ad-1} x_{2d-2} + \tilde{A}^{(n-1)}_{ad} x_{2d-1})
\]

\[
= p \tilde{A}^{(n)}_{ac} \tilde{A}^{(n)}_{ad}
\]
We will follow the method of Marc Rosso in An Analogue of P.B.W. Theorem and the Universal R-Matrix for $U_h\mathfrak{sl}(N + 1)$
Let $\alpha = \alpha(i, j + 1) = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$, $\gamma = \alpha - \alpha_j$ and $\beta = \alpha - \alpha_i$ where $i \neq j$. Then define by induction

$$e_\alpha = \begin{cases} e_\gamma e_j - q e_j e_\gamma & \text{if } i \neq j \\ e_i & \text{if } i = j \end{cases}$$

$$f_\alpha = \begin{cases} f_i f_\beta - q^{-1} f_\beta f_i & \text{if } i \neq j \\ f_i & \text{if } i = j \end{cases}$$
We order the elements as follows:

\[ e_\alpha(i,j) < e_\alpha(k,l) \text{ if } i > k \text{ or } (i = k \text{ and } j > l) \]

\[ f_\alpha(i,j) < f_\alpha(k,l) \text{ if } i < k \text{ or } (i = k \text{ and } j < l) \]
### Proposition

Let \( \alpha = \alpha_i + \ldots + \alpha_j \) and \( \beta = \alpha_p + \ldots + \alpha_r \). Up to exchanging the roles of \( \alpha \) and \( \beta \) we may assume \( i \leq p \). Then,

\[
e_\alpha e_\beta = \begin{cases} 
e_\beta e_\alpha & \text{if } p \geq j + 2 \\
qe_\beta e_\alpha + e_\alpha + \beta & \text{if } p = j + 1 \\
q^{-1} e_\beta e_\alpha & \text{if } p = i \text{ and } r \geq j + 1 \\
e_\beta e_\alpha & \text{if } i < p < j \text{ and } r < j \\
q^{-1} e_\beta e_\alpha & \text{if } i < p \leq j \text{ and } r = j \\
e_\beta e_\alpha - (q - q^{-1}) e_{\alpha'} e_{\alpha''} & \text{if } i < p \leq j \text{ and } r \geq j + 1 \
\end{cases}
\]

where \( \alpha' = \alpha_p + \ldots + \alpha_j \) and \( \alpha'' = \alpha_i + \ldots + \alpha_r \).
Proposition

Let $\alpha = \alpha_i + \ldots + \alpha_j$ and $\beta = \alpha_p + \ldots + \alpha_r$. Upto exchanging the roles of $\alpha$ and $\beta$ we may assume $i \leq p$. Then,

\[
f_\beta f_\alpha = \begin{cases} 
  f_\alpha f_\beta & \text{if } p \geq j + 2 \\
  q(f_\alpha f_\beta - f_{\alpha+\beta}) & \text{if } p = j + 1 \\
  q^{-1}f_\alpha f_\beta & \text{if } p = i \text{ and } r \geq j + 1 \\
  f_\alpha f_\beta & \text{if } i < p < j \text{ and } r < j \\
  q^{-1}f_\alpha f_\beta & \text{if } i < p \leq j \text{ and } r = j \\
  f_\alpha f_\beta - (q - q^{-1})f_{\alpha'} f_{\alpha''} & \text{if } i < p \leq j \text{ and } r \geq j + 1 
\end{cases}
\]

where $\alpha' = \alpha_i + \ldots + \alpha_r$ and $\alpha'' = \alpha_p + \ldots + \alpha_j$. 
Theorem

1. The set \( B^0 = \{ \prod_{i=1}^{n} k_i^{c_i} : c_i \in \mathbb{Z} \} \) is a basis for \( U_q^{0} \text{gl}(n) \).
Theorem

1. The set $B^0 = \{ \prod^n_i k_i^{c_i} : c_i \in \mathbb{Z} \}$ is a basis for $U^0_q gl(n)$.

2. The set $B^+ = \{ \prod_{\alpha \in \Phi^+} e_{\alpha}^{c_{\alpha}} : c_{\alpha} \in \mathbb{N} \}$, where the product is in the order corresponding to that of the elements $e_{\alpha}$, is a basis for $U^+_q gl(n)$. 
Theorem

1. The set \( B^0 = \{ \prod_{i=1}^n k_i^{c_i} : c_i \in \mathbb{Z} \} \) is a basis for \( U_q^0 \mathfrak{gl}(n) \).

2. The set
\[
B^+ = \left\{ \prod_{\alpha \in \Phi^+} e_{\alpha}^{c_{\alpha}} : c_{\alpha} \in \mathbb{N} \right\},
\]
where the product is in the order corresponding to that of the elements \( e_{\alpha} \), is a basis for \( U_q^+ \mathfrak{gl}(n) \).

3. The set
\[
B^- = \left\{ \prod_{\alpha \in \Phi^+} f_{\alpha}^{c_{\alpha}} : c_{\alpha} \in \mathbb{N} \right\},
\]
where the product is in the order corresponding to that of the elements \( f_{\alpha} \), is a basis for \( U_q^- \mathfrak{gl}(n) \).
Theorem

1. The set \( B^0 = \{ \prod_i^n k_i^{c_i} : c_i \in \mathbb{Z} \} \) is a basis for \( U_q^{0} gl(n) \).

2. The set
   \[
   B^+ = \{ \prod_{\alpha \in \Phi^+} e_{\alpha}^{c_{\alpha}} : c_{\alpha} \in \mathbb{N} \},
   \]
   where the product is in the order corresponding to that of the elements \( e_{\alpha} \), is a basis for \( U_q^{+} gl(n) \).

3. The set
   \[
   B^- = \{ \prod_{\alpha \in \Phi^+} f_{\alpha}^{c_{\alpha}} : c_{\alpha} \in \mathbb{N} \},
   \]
   where the product is in the order corresponding to that of the elements \( f_{\alpha} \), is a basis for \( U_q^{-} gl(n) \).

4. Hence the set \( B = B^- \otimes B^0 \otimes B^+ \) is a basis for \( U_{q} gl(n) \).
Definition

Let \((U, \mu, \eta, \Delta, \varepsilon)\) and \((H, \mu, \eta, \Delta, \varepsilon)\) be bialgebras and \(<, >\) be a bilinear form on \(U \times H\). We say that the bilinear form realizes a duality between \(U\) and \(H\), or that the bialgebras \(U\) and \(H\) are in duality if we have

\[< uv, x > = \sum_{(x)} < u, x' > < v, x'' >, \]  \hfill (1)

\[< u, xy > = \sum_{(u)} < u', x > < u'', y >, \]  \hfill (2)

\[< 1, x > = \varepsilon(x), \]  \hfill (3)

\[< u, 1 > = \varepsilon(u) \]  \hfill (4)

for all \(u, v \in U\) and \(x, y \in H\).
Definition

Moreover, if $U$ and $H$ are Hopf algebras with antipode $S$, then they are said to be in duality if the underlying bialgebras are in duality and we have

\[ \langle S(u), x \rangle = \langle u, S(x) \rangle \]

for all $u \in U$ and $x \in H$. 
Proposition

Let $\phi$ be the linear map from $U$ to the dual vector space $H^*$ and $\psi$ be the linear map from $H$ to the dual vector space $U^*$ defined by

$$\phi(u)(x) = \langle u, x \rangle \quad \quad \psi(x)(u) = \langle u, x \rangle$$

With the above notation, the relations (1) and (3) of the previous definition are equivalent to $\phi$ being an algebra morphism and the relations (2) and (4) are equivalent to $\psi$ being an algebra morphism.
Construct an algebra map $\psi$ from $M_q(n)$ to the dual algebra $U_q^*gl(n)$.

Consider the representation $\rho$ defined on the generators by

\[
\begin{align*}
\rho(e_i) &= E_{i,i+1}, \\
\rho(f_i) &= E_{i+1,i}, \\
\rho(k_i) &= D_i,
\end{align*}
\]

where $E_{ij}$ denotes the elementary matrix and $D_i$ denotes the diagonal matrix

\[
D_i = E_{11} + E_{22} + \ldots + qE_{i,i} + E_{i+1,i+1} + E_{i+2,i+2} + \ldots + E_{nn}.
\]
If $u$ is an element of $U_q gl(n)$ using the P.B.W. basis, we have

$$\rho(u) = \begin{pmatrix}
A_{11}(u) & A_{12}(u) & \ldots & A_{1n}(u) \\
A_{21}(u) & A_{22}(u) & \ldots & A_{2n}(u) \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1}(u) & A_{n2}(u) & \ldots & A_{nn}(u)
\end{pmatrix}$$

Let $\psi : H = M_q(n) \rightarrow U^* = U_q gl(n)^*$ be the algebra morphism defined on the generators by $\psi(a_{ij}) = A_{ij}$.
Theorem

The bilinear form \( \langle u, x \rangle = \psi(x)(u) \) realizes a duality between the bialgebras \( U_q \text{gl}(n) \) and \( M_q(n) \).
ψ is well-defined.
• $\psi$ is well-defined.

• $\langle, \rangle$ satisfies (1) and (3)
Thank You 😊