On groups with all subgroups subnormal or soluble of bounded derived length

Maria Tota
(joint work with K. Ersoy and A. Tortora)

Università degli Studi di Salerno
Dipartimento di Matematica

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A subgroup H of a group G is said to be **subnormal** if H is a term of a finite series of G, i.e. if there exist distinct subgroups $H_0, H_1, \dots, H_{n-1}, H_n$ such that

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If G is a nilpotent group then every subgroup of G is subnormal.

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Is a group with all subgroups subnormal nilpotent?

Heineken and Mohamed, 1968

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A group G is of **Heineken-Mohamed type** if G is not nilpotent and all of its proper subgroups are subnormal and nilpotent.

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Menegazzo (1995) gave examples of soluble Heineken-Mohamed p-groups of arbitrary derived length.

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Let G be a locally (soluble-by-finite) group with all subgroups subnormal or nilpotent. Then G is soluble.
Moreover, if G is torsion-free, then G is nilpotent.

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Let G be a locally graded group and suppose that, for some $n \geq 1$, every non-nilpotent subgroup of G is subnormal of defect at most n in G. Then G is soluble.

A group is **locally graded** if every non-trivial finitely generated subgroup has a non-trivial finite quotient, e.g. locally (soluble-by-finite) groups and residually finite groups.

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This restriction is made in order to avoid **Tarski groups**, i.e. infinite 2-generator simple groups with all proper non-trivial subgroups of prime order.

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(*v*) $Sz(2^p)$, where p is any odd prime.

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$PSL(2, p)$, $p > 3$, $p^2 + 1 \equiv 0$ $(mod\ 5)$, $p^2 - 1 \equiv 0$ $(mod\ 16)$ has a sbgp. of der. length 3 and every proper sbgp. of d.l. ≤ 3.

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Hence: Every proper subgroup of a finite minimal simple group has derived length at most 5.

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In (ii) one cannot expect that G is an extension of a soluble group by a finite minimal simple group : it suffices to consider the direct product of any abelian group by the symmetric group of degree 5.

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Moreover, $G^{(s)}$ is not soluble and every proper subgroup of $G^{(s)}$ is soluble of length at most d. Thus $G^{(s)}$ is finite by Zaicev's result, a contradiction.

Proposition B

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- (i) G is soluble of derived length not exceeding a function depending on n and d, or
- (ii) $G^{(r)}$ is finite for some integer $r = r(n)$ and G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.