$T(\sigma_u) = \alpha \sigma_u + \beta \sigma_v$

$T(\sigma_v) = \gamma \sigma_u + \delta \sigma_v$

$T = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$

Eğriliği (Yüzeyler için)

1

<u>Hatırlatma</u> $S \subset \mathbb{R}^3$

$\sigma : V \subset \mathbb{R}^2 \to S$

<u>Geometrik Yaklaşma</u>

$S \ni \sigma(u,v) \xrightarrow{G} \overline{N}(\sigma(u,v))$

$\sigma \xrightarrow[\text{Gauss Dönüşüm}]{} \overline{N}$

[$\overline{N}$: Birim normal / dik vektör]

$\boxed{G : S \to S^2}$

<u>Not</u> : Düzlem için G sabittir
küre için G Birim fonksiyon

Eğriliği anlamak için Gauss Dönüşümün Türev önemlidir.

$$\mathcal{W} := -DG$$

(Gauss dönüşümün Türev $\times (-1)$)

Not : Bir nokta $p \in S$ alalım o zaman,

$$\mathcal{W}_p : T_p(S) \to T_p(S)$$

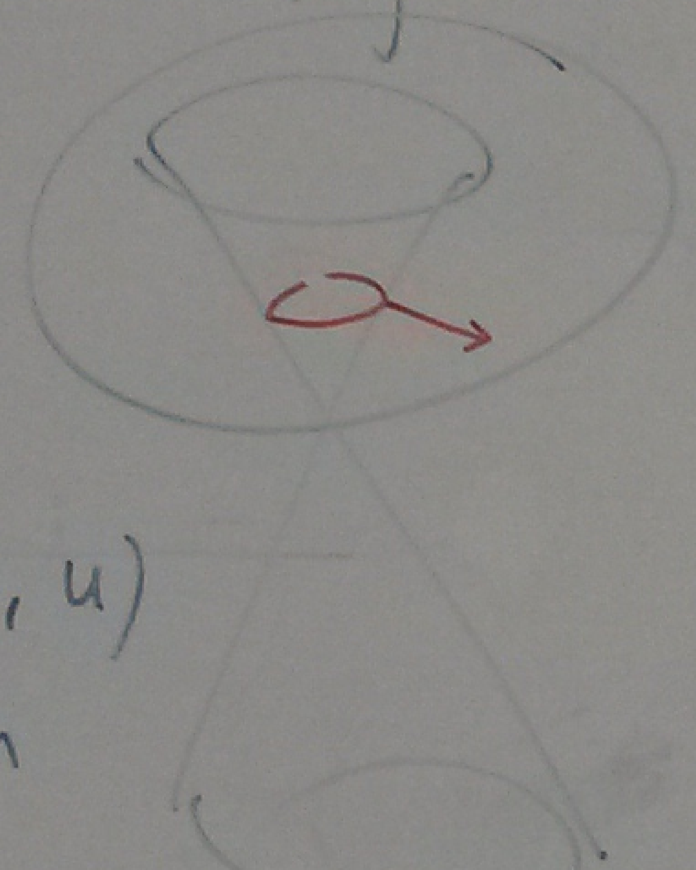
Not : $\mathcal{W}_p$ bir lineer dönüşümdür.

küre için G Birim fonksiyon

Hesaplama

$S = \{(x,y,z) \mid z^2 = x^2 + y^2\}$

Konik

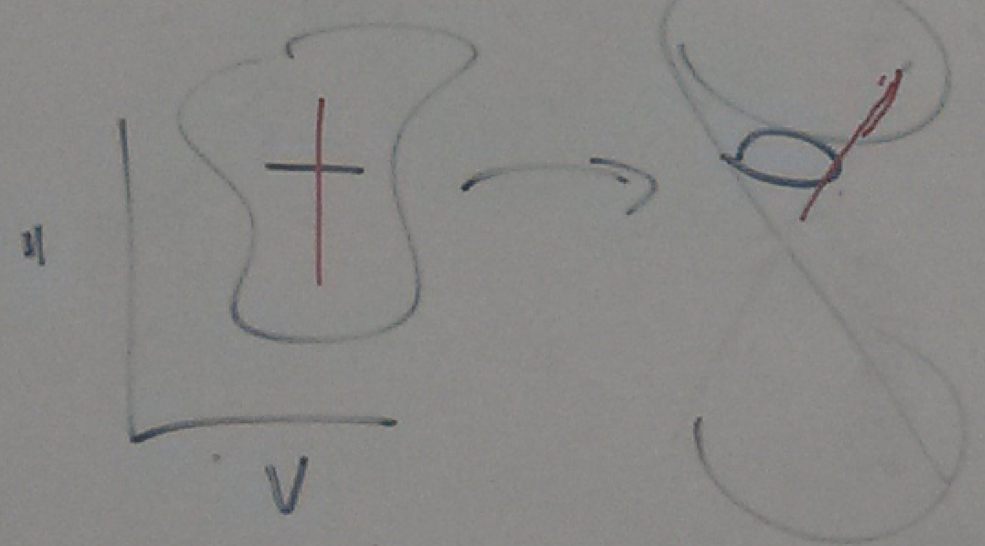


Parametrizasyon:

$\sigma(u,v) = (u\cos v, u\sin v, u)$

üst yarı için bir chart.

$u \in \mathbb{R}_+ \quad v \in (0, 2\pi)$



Gauss Dönüşüm

$(u\cos v, u\sin v, u) \longrightarrow \left(\frac{1}{\sqrt{2}}\cos v, \frac{1}{\sqrt{2}}\sin v, \frac{-1}{\sqrt{2}}\right)$

Weingarten Dönüşüm

$\sigma_u = (\cos v, \sin v, 1) \xrightarrow{\mathcal{W}} -\frac{d}{du}(\cdot) = 0$

$\sigma_v = (-u\sin v, u\cos v, 0) \xrightarrow{\mathcal{W}} -\frac{d}{dv}(\cdot) = \left(\frac{1}{\sqrt{2}}\sin v, \frac{-1}{\sqrt{2}}\cos v, 0\right) = \frac{-1}{\sqrt{2}u}\sigma_v$

$\mathcal{W} : \{\sigma_u, \sigma_v\}$ baz ile yazsak,

$$\mathcal{W} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{u\sqrt{2}} \end{bmatrix}$$

$T(\sigma_u) = \alpha\sigma_u + \beta\sigma_v$

$T(\sigma_v) = \gamma\sigma_u + \delta\sigma_v$

$$T = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

3

Örnek $\sigma(u,v)$

Helikoid $= (u\cos v,\ u\sin v,\ v)$

$u \in \mathbb{R} \quad v \in [0, 2\pi)$

Gauss Dönüşüm

$\sigma_u = (\cos v,\ \sin v,\ 0)$

$\sigma_v = (-u\sin v,\ u\cos v,\ 1)$

Birim normal

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(\sin v,\ -\cos v,\ u)}{\sqrt{u^2+1}}$$

$$\begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

Gauss Dönüşüm

$$(u\cos v,\ u\sin v,\ v) \xrightarrow{\mathcal{G}} \frac{(\sin v,\ -\cos v,\ u)}{\sqrt{u^2+1}}$$

Weingarten

$$\sigma_u \longrightarrow \left(\sin v \cdot u(u^2+1)^{-3/2},\ -\cos v \cdot u \cdot (u^2+1)^{-3/2},\ -(u^2+1-u)(u^2+1)^{-3/2}\right)$$

$$\mathcal{W} = -D\mathcal{G}$$

$$\sigma_v \longrightarrow \left(\frac{-\cos v}{\sqrt{u^2+1}},\ \frac{-\sin v}{\sqrt{u^2+1}},\ 0\right)$$

$$\mathcal{W}(\sigma_v) = \frac{-1}{\sqrt{u^2+1}}\sigma_u$$

$$\mathcal{W}(\sigma_u) = \alpha\sigma_u - (u^2+1-u)(u^2+1)^{3/2}$$

$$\mathcal{W} = \begin{pmatrix} \alpha & \frac{-1}{\sqrt{u^2+1}} \\ \frac{-(u^2+1-u)}{(u^2+1)^{3/2}} & 0 \end{pmatrix}$$

Gaussian Eğriliği

$$K = \det(\mathcal{W})$$

$$\xrightarrow{K} \frac{u^2+1-u}{(u^2+1)^2}$$

$\cos v \quad \sin v \quad 0$

$-u\sin v \quad u\cos v \quad 1$

$$W = \begin{pmatrix} \alpha & \frac{-1}{\sqrt{u^2+1}} \\ -\frac{(u^2+1-u)}{(u^2+1)^{3/2}} & 0 \end{pmatrix} \xrightarrow{K} \frac{u^2+1-u}{(u^2+1)^2}$$

4

Cebirsel Yaklaşım

$$\left[\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)\right] \cdot \bar{N}$$

Düzlem için 0.

Bunun değer eğrilik için bir ifadesi verir.

Taylor formül kullanarak

$$\approx \sigma_{uu} \cdot \bar{N} (\Delta u)^2 + 2\sigma_{uv} \cdot \bar{N} \Delta u \Delta v + \sigma_{vv} \cdot \bar{N} (\Delta v)^2$$

$L := \sigma_{uu} \cdot \bar{N}$

$M := \sigma_{uv} \cdot \bar{N}$

$N := \sigma_{vv} \cdot \bar{N}$

İkinci Temel Formu

$$L\,du^2 + 2M\,du\,dv + N\,dv^2$$

Bu bir iç çarpım verir.

$\langle\langle \sigma_u, \sigma_u \rangle\rangle = L$

$\langle\langle \sigma_u, \sigma_v \rangle\rangle = M$

$\langle\langle \sigma_v, \sigma_v \rangle\rangle = N$

ve lineer şeklinde uzatırız.

$$\langle\langle \alpha\sigma_u + \beta\sigma_v, \gamma\sigma_u + \delta\sigma_v \rangle\rangle = \alpha\gamma \langle\langle \sigma_u, \sigma_u \rangle\rangle + (\alpha\delta + \beta\gamma)\langle\langle \sigma_u, \sigma_v \rangle\rangle + \beta\delta \langle\langle \sigma_v, \sigma_v \rangle\rangle$$

Eğriliği (Yüzeyler için)

5

Üçüncü Yaklaşım

Eğriler kullanarak.

$p \in S$ γ p'den geçen

S üzerinde olan birim

hız bir eğri olsun.

$\dot{\gamma} \perp \ddot{\gamma}$ $\{\bar{N}, \dot{\gamma}, \bar{N} \times \dot{\gamma}\}$

orthonormal baz

$$\ddot{\gamma} = k_n \bar{N} + k_g \bar{N} \times \dot{\gamma}$$

şeklinde yazılabilir.

$k_n :=$ Normal eğrilik

$k_g :=$ Geodezik eğrilik

Not $k_n = \ddot{\gamma} \cdot \bar{N}$

Teorem IV.7

$$k_n = \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle$$

İspat Gauss dönüşümün tanımından dolayı

$$\mathcal{G}(\gamma) = \bar{N}$$

O zaman

$$W(\dot{\gamma}) = -\frac{d\bar{N}}{dt} \qquad \left(W(\dot{\gamma}(s)) = -\frac{d}{dt} \bar{N}(\gamma(t)) \Big|_s \right)$$

$\bar{N}$ ve $\dot{\gamma}$ birbirine dik

$\Rightarrow \bar{N} \cdot \dot{\gamma} \equiv 0$

$\Rightarrow \frac{d\bar{N}}{dt} \cdot \dot{\gamma} + \bar{N} \cdot \ddot{\gamma} \equiv 0 \Rightarrow k_n = -\frac{d\bar{N}}{dt} \cdot \dot{\gamma}$

$$= \langle W(\dot{\gamma}), \dot{\gamma} \rangle$$

κ_g := Geodezik eğrilik

$\frac{}{dt}$

$= \langle W(\dot\gamma), \dot\gamma \rangle$

$L = \langle \sigma_{uu}, \bar N \rangle$

$M = \langle \sigma_{uv}, \bar N \rangle$

$N = \langle \sigma_{vv}, \bar N \rangle$

$\kappa_n = \langle W(\dot\gamma), \dot\gamma \rangle$

Ne göstermek lazım?

$$\boxed{\langle\langle \dot\gamma, \dot\gamma \rangle\rangle = \langle W(\dot\gamma), \dot\gamma \rangle}$$

Bunu göstermek yeter

L $\langle\langle \sigma_u, \sigma_u \rangle\rangle = \langle W(\sigma_u), \sigma_u \rangle$

M $\langle\langle \sigma_u, \sigma_v \rangle\rangle = \langle W(\sigma_u), \sigma_v \rangle$

M $\langle\langle \sigma_v, \sigma_u \rangle\rangle = \langle W(\sigma_v), \sigma_u \rangle$

N $\langle\langle \sigma_v, \sigma_v \rangle\rangle = \langle W(\sigma_v), \sigma_v \rangle$

$L = \sigma_{uu} \cdot \bar N$

$\sigma_u \perp \bar N$

$\Rightarrow \sigma_u \cdot \bar N = 0$

Türev alsak

$\sigma_{uu} \cdot \bar N = - \sigma_u \cdot \bar N_u$

$\sigma(u,v) \xrightarrow{G} \bar N(\sigma(u,v))$

Türev alsak ($u \to \sigma(u,v)$ eğri)

$\sigma_u \xrightarrow{W} - \bar N_u$

$$\Rightarrow L = \sigma_{uu} \cdot \bar N = - \sigma_u \cdot \bar N_u$$
$$= \langle -\bar N_u, \sigma_u \rangle$$
$$= \langle W(\sigma_u), \sigma_u \rangle$$

$\Rightarrow \langle\langle \sigma_u, \sigma_u \rangle\rangle = \langle W(\sigma_u), \sigma_u \rangle$ (1)

$\langle\langle \sigma_u, \sigma_v \rangle\rangle = M = \sigma_{uv} \cdot \bar N$

$\sigma_u \cdot \bar N \equiv 0$

Türev alsak (v'e göre)

$\sigma_{uv} \cdot \bar N = - \sigma_u \cdot \bar N_v$

$W(\sigma_v) = - \bar N_v$

$$\Rightarrow \langle\langle \sigma_u, \sigma_v \rangle\rangle = - \sigma_u \cdot \bar N_v$$
$$= \langle W(\sigma_v), \sigma_u \rangle$$

Benzer şekilde

$\langle\langle \sigma_v, \sigma_v \rangle\rangle = \langle W(\sigma_v), \sigma_v \rangle$

Sonuç: Her $\xi, \eta \in T_p(S)$ için

$$\boxed{\langle\langle \xi, \eta \rangle\rangle = \langle W(\xi), \eta \rangle}$$

$\mathcal{W}(\sigma_u)\cdot\sigma_u = \alpha\langle\sigma_u,\sigma_u\rangle + \beta\langle\sigma_v,\sigma_u\rangle$

$= \alpha E + \beta F$

$\Rightarrow \mathcal{W} = \mathcal{F}_I \; \mathcal{F}_{II}$

$= \bar{N}\cdot\sigma_{vv}$

$= N$

$\mathcal{W}(\sigma_v)\cdot\sigma_v = \gamma F + \delta G$

Sonuç:

$\kappa_n = \langle \mathcal{W}(\dot{\gamma}), \dot{\gamma}\rangle$

$= \langle\langle \dot{\gamma}, \dot{\gamma}\rangle\rangle.$

Corollary

$\gamma \subset S$ birim hız olsun.

Normal eğrilik şu şekillerde ifade edilebilir

a) $\kappa_n = \ddot{\gamma}\cdot\bar{N}$

b) $\kappa_n = \langle \mathcal{W}(\dot{\gamma}), \dot{\gamma}\rangle$

c) $\kappa_n = \langle\langle \dot{\gamma}, \dot{\gamma}\rangle\rangle$

7

Hesaplama IV.8

$\gamma \subset S^2$ birim hız.

O zaman onun normal eğriliği nedir?

S^2 için $\mathcal{W} = -I = -\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \kappa_n = \langle -\dot{\gamma}, \dot{\gamma}\rangle = -1$ (sabit).

Soru

$\mathcal{W}$ için matris yazalım (σ_u, σ_v baz ile).

$\mathcal{W}(\sigma_u) = -\bar{N}_u$

$\mathcal{W}(\sigma_v) = -\bar{N}_v$

$\sigma(u,v) \xrightarrow{\mathcal{G}} \bar{N}$

$\sigma_u \xrightarrow{\mathcal{W}} -\bar{N}_u$

$$w(\sigma_v) = -\bar{N}_v \qquad \sigma_u \xrightarrow{w} -\bar{N}_u$$

$E\,du^2 + 2F\,du\,dv + G\,dv^2 \rightarrow$ Birinci Temel formu (8)

$L\,du^2 + 2M\,du\,dv + N\,dv^2 \rightarrow$ İkinci Temel formu

$w(\sigma_u) = \alpha\sigma_u + \beta\sigma_v$ şeklinde

$w(\sigma_v) = \gamma\sigma_u + \delta\sigma_v$ yazalım.

(1) σ_u ile iç çarpım alalım

$$w(\sigma_u)\cdot\sigma_u = -\bar{N}_u\cdot\sigma_u$$

$$w(\sigma_u)\cdot\sigma_u = \alpha\langle\sigma_u,\sigma_u\rangle + \beta\langle\sigma_v,\sigma_u\rangle$$

Ama, $\bar{N}\cdot\sigma_u = 0$

Türev alsak,

$$-\bar{N}_u\cdot\sigma_u = \bar{N}\cdot\sigma_{uu} = L$$

$\Rightarrow$ $w(\sigma_u)\cdot\sigma_u = -\bar{N}_u\cdot\sigma_u = L$

$$w(\sigma_u)\cdot\sigma_u = \alpha\langle\sigma_u,\sigma_u\rangle + \beta\langle\sigma_v,\sigma_u\rangle = \alpha E + \beta F$$

(1) $L = \alpha E + \beta F$

(2) $M = \alpha F + \beta G$

(3) $M = \gamma E + \delta F$

(4) $N = \gamma F + \delta G$

$$\underbrace{\begin{bmatrix} L & M \\ M & N \end{bmatrix}}_{\mathcal{F}_{II}} = \underbrace{\begin{bmatrix} E & F \\ F & G \end{bmatrix}}_{\mathcal{F}_{I}} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

$$\boxed{w = \mathcal{F}_I^{-1}\,\mathcal{F}_{II}}$$

(2) $w(\sigma_u)\cdot\sigma_v = -\bar{N}_u\cdot\sigma_v = \bar{N}\cdot\sigma_{uv} = M$

$$w(\sigma_u)\cdot\sigma_v = \alpha F + \beta G$$

(3) $w(\sigma_v)\cdot\sigma_u = -\bar{N}_v\cdot\sigma_u = \bar{N}\cdot\sigma_{uv} = M$

$$w(\sigma_v)\cdot\sigma_u = \gamma E + \delta F$$

(4) $w(\sigma_v)\cdot\sigma_v = -\bar{N}_v\cdot\sigma_v = \bar{N}\cdot\sigma_{vv} = N$

$$w(\sigma_v)\cdot\sigma_v = \gamma F + \delta G$$

$$\sigma_{uv} = (b \sin u \sin v, -b \sin u \cos v, 0)$$

$$\sigma_{vv} = (-(a+b\cos u)\cos v, -(a+b\cos u)\sin v, 0)$$

$$= \begin{bmatrix} \frac{1}{b^2} & 0 \\ \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & (a+b\cos u)\cos u \end{bmatrix}$$

Teorem IV.4

$S \subset \mathbb{R}^3$ yüzey

$\sigma : U \longrightarrow S$ chart

$E\,du^2 + 2F\,du\,dv + G\,dv^2$ I form

$L\,du^2 + 2M\,du\,dv + N\,dv^2$ II form

O zaman,

$$\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \qquad \mathcal{F}_{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

ise,

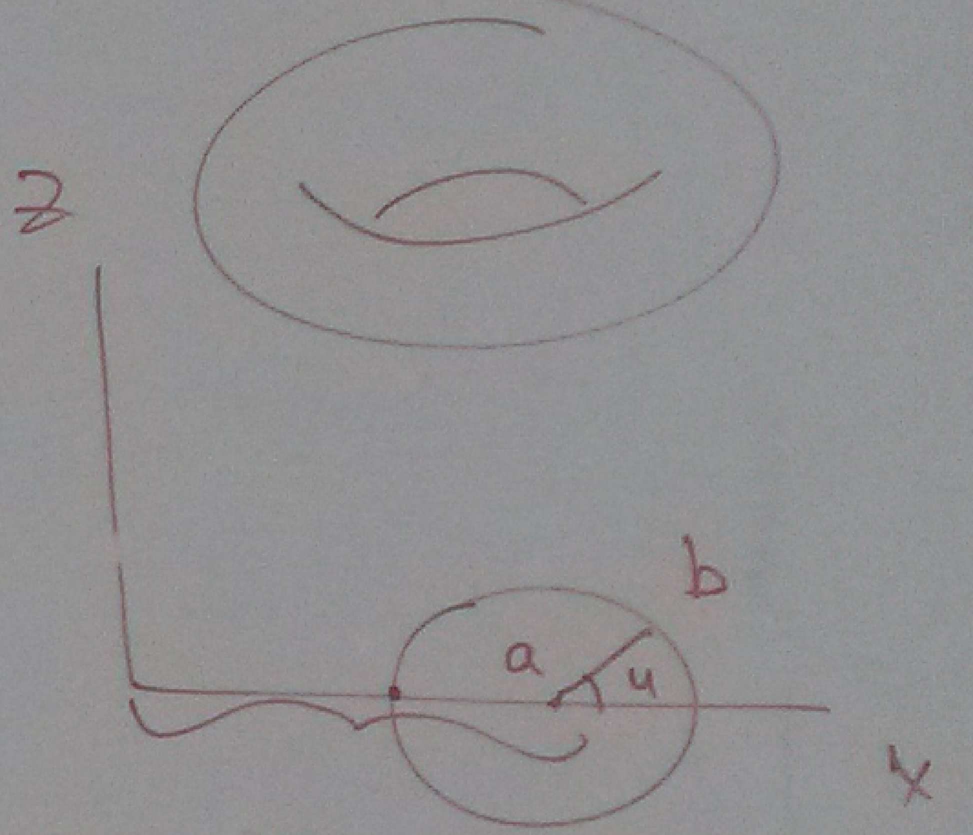
$$W = \mathcal{F}_I^{-1} \mathcal{F}_{II}$$

$\{\sigma_u, \sigma_v\}$ baz'e göre

Örnek IV.10

Torus

$$\sigma(u,v) = ((a+b\cos u)\cos v, (a+b\cos u)\sin v, b \sin u)$$



$(a+b\cos u, 0, b\sin u)$

Birinci Temel formu

$$\sigma_u = (-b\sin u \cos v, -b \sin u \sin v, b \cos u)$$

$$\sigma_v = (-(a+b\cos u)\sin v, (a+b\cos u)\cos v, 0)$$

$$E = \langle \sigma_u, \sigma_u \rangle = b^2$$

$$F = \langle \sigma_u, \sigma_v \rangle = 0$$

$$G = \langle \sigma_v, \sigma_v \rangle = (a+b\cos u)^2$$

$$\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} b^2 & 0 \\ 0 & (a+b\cos u)^2 \end{bmatrix}$$

10

$$\bar{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \left[\frac{-(a+b\cos u)\, b\cos u\cos v,\ -(a+b\cos u)\, b\cos u\sin v,\ -(a+b\cos u)\, b\sin u}{b(a+b\cos u)}\right]$$

$$= \left(-\cos u\cos v,\ -\cos u\sin v,\ -\sin u\right)$$

$$\mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & (a+b\cos u)\cos u \end{pmatrix}$$

$L = \sigma_{uu} \cdot \bar{N}$

$M = \sigma_{uv} \cdot \bar{N}$

$N = \sigma_{vv} \cdot \bar{N}$

$\sigma_{uu} = (-b\cos u\cos v,\ -b\cos u\sin v,\ -b\sin u)$

$\sigma_{uv} = (b\sin u\sin v,\ -b\sin u\cos v,\ 0)$

$\sigma_{vv} = (-(a+b\cos u)\cos v,\ -(a+b\cos u)\sin v,\ 0)$

$L = b$

$M = 0$

$N = (a+b\cos u)\cos u$

$$\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II}$$

$$= \begin{pmatrix} \frac{1}{b^2} & 0 \\ 0 & \frac{1}{(a+b\cos u)^2} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & (a+b\cos u)\cos u \end{pmatrix}$$

$w(\sigma_u)\cdot\sigma_u = \alpha\langle\sigma_u,\sigma_u\rangle + \beta\langle\sigma_v,\sigma_u\rangle$

Ama, $-\bar{N}\cdot\sigma_u = 0$

Türev alsak,

$-\bar{N}_u\cdot\sigma_u - \bar{N}\cdot\sigma_{uu} = L$

(4) $N = \gamma F + \delta G$

$$\underbrace{\begin{bmatrix} L & M \\ M & N \end{bmatrix}}_{F_{II}} = \underbrace{\begin{bmatrix} E & F \\ F & G \end{bmatrix}}_{F_I}\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

$w(\sigma_u)\cdot\sigma_v = \alpha F + \beta G$

(3) $w(\sigma_v)\cdot\sigma_u = -\bar{N}_v\cdot\sigma_u$

$= \bar{N}\cdot\sigma_{uv}$

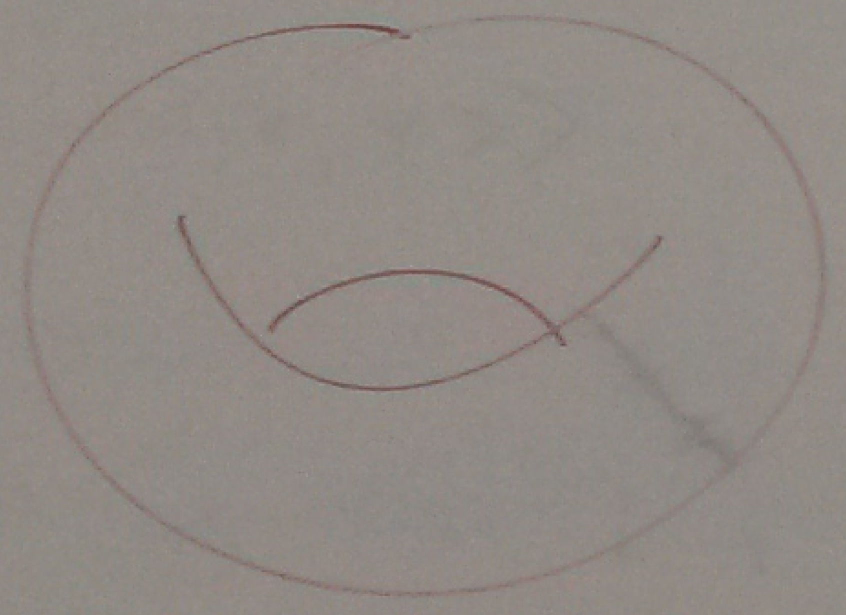
$= M$

$w(\sigma_v)\cdot\sigma_u = \gamma E + \delta F$

$$\Rightarrow w = \begin{bmatrix} \frac{1}{b} & 0 \\ 0 & \frac{\cos u}{a+b\cos u} \end{bmatrix}$$

$$k = \det(w) = \frac{\cos u}{b(a+b\cos u)}$$

Note: $|a| > |b|$



Teorem IV.11

$p\in S$. O zaman,

$$\lim_{\substack{A\ni p\\ |A|\to 0}} \frac{|\mathcal{G}(A)|}{|A|} = K(p)$$

Gaussian curvature'ı p etrafındaki bir bölge $\mathcal{G}$ altında ne kadar şişiyor.