Strongly Rayleigh measures and the Kadison-Singer Problem

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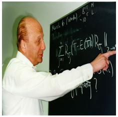
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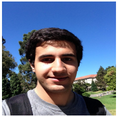
Nima Anari

Negative dependance

In this talk we only work with binary ($\{0,1\}$ valued) random variables.
$\mu \in \mathcal{P}(2^{[n]})$, i.e PM on subsets of $[n] = \{1, \cdots, n\}$.

1. **Positive dependance** well understood.

$$\begin{aligned} &PLC(\,+\text{ lattice condition}) \quad \mu(S)\mu(T) \leq \mu(S \cup T)\mu(S \cap T), \quad \forall S, T \subset [n]. \\ &PA(\text{positive association}) \quad \mathbb{E}(f)\mathbb{E}(g) \leq \mathbb{E}(fg), \quad \forall f, g : 2^{[n]} \to \mathbb{R}, \uparrow . \end{aligned}$$

   FKG(Fortuin-Kasteleyn-Ginibre) theorem, 1971 : $PLC \implies PA$, local-global.

2. **Negative dependance:** Analogous notions for repelling random variables :

$$\begin{aligned} &NLC(\text{ - lattice condition}) \quad \mu(S)\mu(T) \geq \mu(S \cup T)\mu(S \cap T), \quad \forall S, T \subset [n]. \\ &NA(\text{negative association}) \quad \mathbb{E}(f)\mathbb{E}(g) \leq \mathbb{E}(fg), \quad \forall f, g : 2^{[n]} \to \mathbb{R}, \uparrow \\ &\qquad\qquad supp(f) \cap supp(g) = \emptyset. \end{aligned}$$

   However, $NLC \not\Longrightarrow NA$

Popularized by Robin Pemantle (2000 - ...)

Various definitions

Given $\mu \in \mathcal{P}_n := \mathcal{P}(2^{[n]})$, we consider the multi-affine (generating) polynomial (of μ),

$$P_\mu = \sum_{S \subset [n]} \mu(S) z^S.$$

$X_1, \cdots, X_n$: co-ordinate random variables, $X_i(S) = 1$ if $i \in S$, else 0.

1. $\mu \in \mathcal{P}_n$ is *pairwise negative correlated* (p-NC) if

$$\mathbb{E}(X_i)\mathbb{E}(X_j) \geq \mathbb{E}(X_i X_j), \; i \neq j \in [n] \iff \partial_i P_\mu(1) \partial_j P_\mu(1) \geq \partial_{ij} P_\mu(1).$$

2. $\mu \in \mathcal{P}_n$ satisfies the *strong hereditary negative lattice condition* (h-NLC+) if,

$$\mu(S)\mu(T) \geq \mu(S \cup T)\mu(S \cap T), \qquad \forall S, T \subset [n].$$

   and the same holds for

   1. Projections : Projection onto 2^X where $X \subset [n]$, $\tilde{\mu}(S) \sim \sum_{T = S \cup X^c} \mu(T)$ for every $X \subset S$.
   2. Application of external fields : Given $(a_1, \cdots, a_n) \in \mathbb{R}^n_+$, $\tilde{\mu}(S) \sim \mu(S) \prod_{i \in S} a_i$.

3. $\mu \in \mathcal{P}_n$ is *strongly conditionally negatively associated* (CNA+) if it is conditionally negatively associated and the same holds upon applying projections and external fields.

Examples

1. *Determinantal measures:* $\mu \in \mathcal{P}_n \mid \exists$ PSD $A \in M_n(\mathbb{R})$ such that,

$$\mu(\{T \subset [n] | S \subset T\}) = \sum_{S \subset T} \mu(T) = \det[A(S)], \qquad \forall S \subset [n].$$

   (Lyons, 2003) : μ determinantal measure in $\mathcal{P}_n$. If the associated PSD matrix is a contraction, then it is CNA+.

2. *Symmetric exclusion processes* Take n points on $\{0,1\}^k$. These points jump to neighbours with fixed probabilities but jumps to occupied spots are forbidden.
   Goal : Come up with a notion of negative dependance that is preserved under these transitions.
   None of p-NC, h-NLC+ or CNA+ are.

Real Stable polynomials

Polynomial $p(z_1, \cdots, p_m)$ called *stable* if

$$p(z_1, \cdots, z_n) \neq 0, \quad \forall (z_1, \cdots, z_n) \mid \operatorname{Im}(z_k) > 0 \; \forall k \in [m].$$

Real Stable: Stable + real coefficients.

1. Univariate real stability = Real rootedness.
2. $p = \det[A + z_1 B_1 + \cdots + z_n B_n]$, where A is symmetric and the B_i are PSD.
3. Real stability preservers,
   1. $p \to \partial_i p$.
   2. (Julius Borcea, Petter Branden, 2006) *Lieb-Sokal lemma:* $p \to q(\partial_1, \cdots, \partial_n)p$.
   3. $p(z_1, \cdots, z_n) \to p(t, z_1, \cdots, z_n)$ for $t \in \mathbb{R}$.
4. *Convexity:* (Adam Marcus, Daniel Spielman, Nikhil Srivastava 2013, Terence Tao 2013, Alexander Scott, Alan Sokal, 2010) $a \in \mathbb{R}$ called above the roots of p or $a \in Ab_p$ if $p(a + z) > 0 \; \forall z \in \mathbb{R}^n_+$. Then,

   $$\text{Complete monotonicity} \qquad (-1)^k \partial_j^k \left(\frac{\partial_i p}{p} \right)(a) \geq 0 \quad \forall a \in Ab_p,\; i, j \in [n],\; k \geq 0.$$

5. William Helton and Victor Vinnikov 2005 : Bivariate real stable polynomials are determinantal.

Strongly Rayleigh measures

Definition (Julius Borcea, Petter Branden, Thomas Liggett 2009)

$\mu \in P_n$ is called *Strongly Rayleigh* if $P_\mu := \sum_{S \subset [n]} \mu(S) z^S$ is *real stable*.

Product measures, Determinantal measures, uniform spanning tree measures are *SR*.
Preserved under symmetric exclusion processes.
Implies p-NC, h-NLC+ and CNA+.
X random vector taking values in $\mathbb{Z}_+^n$. Define,

$$P_X = \sum \mathbb{P}(X = a) z^a.$$

M(X) : Maximum degree of the variables z_i.

Theorem (Multivariate CLT, Ghosh, Pemantle, Liggett, 2016)

X_n sequence of random vectors with sequence s_n such that there is a matrix A satisfying,

$$\frac{Cov(X_n)}{s_n^2} \to A, \qquad \frac{M(X_n)^{1/3}}{s_n} \to 0.$$

Then,

$$\frac{X - \mathbb{E}(X)}{s_n} \to N(0, A).$$

Bourgain-Tzafriri's RIP

$T : \mathbb{R}^m \to \mathbb{R}^n$ linear map.

$s_1(T) \geq \cdots \geq s_m(T)$ singular values of T, i.e eigenvalues of $(T^*T)^{1/2}$.

Question : Find large subspace on which T is well invertible.

$$\text{srank}(T) = \frac{||T||_2^2}{||T||^2} = \frac{\text{Trace}(T^*T)}{||T||^2} = \frac{\sum s_k(T)^2}{s_1(T)^2}.$$

Remark (Singular vector basis)

$\{v_1, \cdots, v_m\}$ basis of singular vectors for T, i.e eigenbasis for T^*T.
$V = \text{span}\{v_1, \cdots, v_k\}$, where $k = c\,\text{srank}(T)$ for some $c < 1$. Then,

$$\text{smin}\ T_{|V} \geq \sqrt{(1-c)}\,\sqrt{\frac{\text{srank}(T)}{m}}.$$

Similar statement holds for any basis! One version,

Theorem (The restricted invertibility principle, B-T, Spielman-Srivastava)

$\{v_1, \cdots, v_m\}$ orthonormal basis. Then, for any $c < 1$, there exists $\sigma \subset [m]$ of size $k = c\ \text{srank}(T)$,

$$\text{smin}\ T_{|P_\sigma \mathbb{R}^m} \geq \frac{1}{5}\sqrt{(1-c)}\,\sqrt{\frac{\text{srank}(T)}{m}}.$$

Theorem (R, 2016)

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear operator. Then, for any $0 \leq \delta \leq 1$, there is a subset σ of size $|\sigma| = \delta \frac{||T||_2^4}{||T||_4^4}$ and such that, letting $c = \frac{|\sigma|}{m}$, we have,

$$s_{min}(T\,|_{P_\sigma \mathbb{R}^m}) \geq \sqrt{\frac{\text{srank}(T)}{m}}\left[\sqrt{1-c} - \sqrt{\delta - c}\right].$$

Theorem (Joel Anderson's Paving problem, Adam Marcus, Daniel Spielman, Nikhil Srivastava 13)

There are universal constants $\epsilon < 1$ and $r \in \mathbb{N}$ so that for any zero diagonal contraction $A \in M_n(\mathbb{R})^{sa}$, there are diagonal projections $Q_1, \cdots, Q_r$ with $Q_1 + \cdots + Q_r = I$,

$$\lambda_1(Q_i A Q_i) < \epsilon, \quad 1 \le i \le r.$$

MSS(2014) : $r = 12$. R(2016) : $r = 4$. Expected: $r = 2 + \epsilon$. Known: $r > 2$.

Restricted Invertibility in analogous form,

Theorem (Restricted Invertibility, R 2016)

For any trace zero contraction $A \in M_n(\mathbb{R})^{sa}$ and any $c \le \frac{1}{2}$ there is a principal submatrix $A(S)$ of size cn such that

$$\lambda_1[A(S)] \le 2\sqrt{c - c^2}.$$

An equivalence

Casazza, Speegle, Tremain, Weber 2006: Equivalent to fundamental problems in Geometric Functional Analysis, Convex geometry, Signal processing, Harmonic analysis, Frame theory (Feichtinger conjecture), Coding theory, ...

$$I \subset \mathbb{Z}. \qquad S(I) := \overline{\text{span}(\{e^{int} : n \in S\})}^{||}.$$

Theorem (Weyl)

Given any $[a, b] \subset [0, 1], \epsilon > 0$ *there is a partition* $X_1 \cup \cdots \cup X_n = \mathbb{Z}$ *such that* $\forall f \in S(X_j)$, $1 \leq j \leq n$,

$$(1 - \epsilon)||f||_2^2 \leq \frac{||f\chi[a, b]||_2^2}{b - a} \leq (1 + \epsilon)||f||_2^2.$$

Does the same hold for any measurable set E? Equivalent to Kadison-Singer.

$\mu \in \mathcal{P}_n$ *Strongly Rayleigh*, $A \in M_n(\mathbb{R})^{sa}$ self adjoint.

Sample principal submatrices of A, picking A_S with probability $\mu(S)$.
A_S : Principal submatrix of A with rows and columns from S removed.
Sublime idea of MSS : Take expectation not of largest eigenvalue, but of the characteristic polynomial!

Theorem (MSS, Nima Anari and Oveis-Gharan 2014, R 2016)

$$\mathbb{E}\chi[A_S] = \sum_{S \subset [n]} \mu(S)\chi[A_S],$$

is real rooted and further,

$$\mathbb{P}\left[\lambda_1 \chi[A_S] \leq \lambda_1 \mathbb{E}\chi[A_S]\right] > 0.$$

Further,

$$\mathbb{E}\chi[A_S] = P_\mu(\partial_1, \cdots, \partial_n) \det[Z - A] \mid_{Z = xI} .$$

SR and RI + KSP

Restricted invertibility : Uniform measure on $n-k$ element subsets of $[n]$,

$$P_\mu = \binom{n}{k}^{-1} \sum_{|S|=n-k} z^S = \binom{n}{k}^{-1} (\partial_1 + \cdots + \partial_n)^k z_1 \cdots z_n.$$

Kadison-Singer : Pick subsets of $[n] \times [n]$ of the form $T \times T^c$.

$$P_{\mu_2} = 2^{-n} \left(\prod_{i=1}^{n} (\partial_{z_i} + \partial_{y_i}) \right) (z_1 \cdots z_n)(y_1 \cdots y_n).$$

Cauchy-Poincare, R.C.Thompson and MSS' Markov principle

Theorem (Cauchy-Poincare)

$A \in M_n(\mathbb{R})^{sa}$. Then, the eigenvalues of $\chi[A]$ and $\chi[A_i]$ interlace.

Lemma (Markov principle)

$p_1, \cdots, p_n$ be same degree monic real rooted with common interlacer. Then, $\forall k\ \exists i$,

$$\lambda_k(p_i) \leq \lambda_k(p_1 + \cdots + p_n).$$

Lemma (Obreshkoff)

$\{p_i\}_{i=1}^n$ degree k monic real rooted. Common interlacer iff every convex combination real rooted.

Theorem (MSS, 2014 + R.C.Thompson, 1963, R, 2016)

Let $A \in M_n(\mathbb{R})$ be hermitian. Then, $\exists i \in [n]$ such that,

$$\lambda_1(\chi[A_i]) \leq \lambda_1\left(\sum \chi[A_i]\right) = \lambda_1(\chi'[A]).$$

For any $k \in [n]$, there is a size k subset $S \subset [n]$ such that,

$$\lambda_1(\chi[A_S]) \leq \lambda_1\left(\sum_{|S|=k} \chi[A_S]\right) = \lambda_1\left(\chi^{(k)}[A]\right).$$

Set $Z = \operatorname{diag}(z_1, \cdots, z_n)$ diagonal matrix of variables.

Lemma

$A \in M_n(\mathbb{R})$ and $S \subset [n]$. Then,

$$\det[A_S] = \frac{\partial^S}{\partial z^S} \det[Z + A] \mid_{Z=0}, \qquad \chi[A_S] = \frac{\partial^S}{\partial z^S} \det[Z - A] \mid_{Z=xI} .$$

$\mu \in \mathcal{P}_n$ Strongly Rayleigh. $A \in M_n(\mathbb{R})^{sa}$ real symmetric.

Create tree. $n + 1$ levels.

Nodes at level k indexed by subsets of $[k-1]$. Mark node at level k by $\sum_{S \supset T} \mu(S)\chi[A_S]$.

Children of node indexed by $S \subset [k]$: $n - k$ nodes indexed by $S \cup i$, for $i \notin S$.

Leaf nodes : $\chi[A_S]$ for $S \subset [n]$.

Top node : $\sum_{S \subset [n]} \mu(S)\chi[A_S]$.

Theorem (R, 2016)

Let $A \in M_n(\mathbb{R})$ be real symmetric. Then, the sum of the characteristic polynomials of all the 2 pavings of A is real rooted and satisfies,

$$\sum_{S \amalg T = [n]} \chi[A_S \oplus A_T] = \left[\prod_{m=1}^{n} (\partial_{z_m} + \partial_{y_m})\right] \det[Z - A] \det[Y - A] \,|_{Z=Y=xI} \;.$$

Further, there is a paving $(S, T) \in \mathcal{P}_2$ such that

$$\lambda_1 \chi[A_S \oplus A_T] \leq \lambda_1 \left[\sum_{S \amalg T = [n]} \chi[A_S \oplus A_T]\right].$$

Lemma (R, 2016)

$$\left[\prod_{m=1}^{n} (\partial_{z_m} + \partial_{y_m})\right] \det[Z - A] \det[Y - A] \,|_{Z=Y=xI} = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det[Z - A]^2 \,|_{Z=xI} \;.$$

Definition (Mixed determinant)

$A, B \in M_n(\mathbb{R})$,

$$D(A,B) := \sum_{S \amalg T = [n]} \det[A(S)] \det[B(T)].$$

Definition

Given a matrix $A \in M_n(\mathbb{R})$, define

$$\det_r(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} (-1)^{\mathrm{sgn}(\sigma)} r^{c(\sigma)}, \qquad \chi_r[A] := \det_r(xI - A).$$

where $c(\sigma)$ denotes the number of cycles in σ.

Lemma (R, 2016)

$$\mathbb{E}_{\mathcal{P}_2([n])} \chi[A_{\mathcal{X}}] = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det[Z - A]^2 \mid_{Z = xI} = \chi_2[A] = D(xI - A, xI - A).$$

Conjecture

$A \in M_n(\mathbb{R})^+$, *positive contraction, diagonal entries of* A *all be at most* $\alpha \leq \frac{1}{2}$. *Then,*

$$\operatorname{maxroot} \chi_2[A] \leq \frac{1}{2} + \sqrt{\alpha(1-\alpha)} = \frac{1}{4}\left(\sqrt{2\alpha} + \sqrt{2(1-\alpha)}\right)^2.$$

$$MSS : \frac{1}{2} + \sqrt{2\alpha} + \alpha, \quad BCMS : \frac{1}{2} + \sqrt{2\alpha(1-2\alpha)}.$$

Theorem (R 2016, 2paving)

$A \in M_n(\mathbb{R})^+$, *positive contraction, diagonal entries of* A *all be at most* $\alpha \leq \frac{1}{4}$. *Then,*

$$\operatorname{maxroot} \chi_2[A] \leq \frac{1}{4}\left(\sqrt{\alpha} + \sqrt{3(1-\alpha)}\right)^2.$$

Theorem (R 2016, paving diagonal 1/2 projections)

$A \in M_n(\mathbb{R})^+$, *positive contraction, diagonal entries of* A *all be at most* $\alpha \leq \frac{1}{2}$. *Then,*

$$\operatorname{maxroot} \chi_4[A] \leq \frac{(3+\sqrt{7})^2}{32} \approx 0.996.$$

p : Real rooted degree n polynomial. For $b \geq \lambda_1(p)$ and $\varphi > 0$, define

$$\Phi_p(b) := \frac{p'}{p} = \sum \frac{1}{b - \lambda_i}, \quad \mathrm{smax}_\varphi(p) := \Phi^{-1}(\varphi) = \lambda_1(p' - \varphi p).$$

Note : For any $\varphi > 0$, we have : $\lambda_1(p) < \mathrm{smax}_\varphi(p)$.

Proposition (Marcus, 2014)

Let p be real rooted and $\varphi > 0$. Then,

$$\mathrm{smax}_\varphi(p') \leq \mathrm{smax}_\varphi(p) - \frac{1}{\varphi}, \quad \rightsquigarrow \quad \mathrm{smax}_\varphi(p^{(k)}) \leq \mathrm{smax}_\varphi(p) - \frac{k}{\varphi}.$$

Follows from concavity of $\dfrac{1}{\Phi_p}$ above $\lambda_1(p)$.

$A \in M_n(\mathbb{R})^{sa}$, set

$$p_0 = \det[Z - A]^2, \quad p_1 = \frac{\partial}{\partial z_1}\det[Z-A]^2, \cdots, \quad p_n = \frac{\partial^n}{\partial z_1 \cdots \partial z_n}\det[Z-A]^2.$$

Real stable polynomial $p(z_1, \cdots, z_n)$, say $z \in \mathbb{R}^n$ is in Ab_p if $p(z+t) \neq 0$ for any $t \in \mathbb{R}^n_+$.
(Upper) potential of p in direction j,

$$\Phi_p^j(z) = \frac{\partial_j p}{p}(z).$$

Basic fact, for any $z \in Ab_p$ and $i, j \in [n]$,

$$\Phi_p^j > 0, \quad \partial_i \Phi_p^j < 0\,(Monotonicity), \quad \partial_i^2 \Phi_p^j > 0\,(Convexity).$$

Lemma (MSS, R)

$$\Phi^j_{(1-\partial_i)p}(z + \delta e_i) \leq \Phi_p^j(z), \qquad \delta = \frac{1}{1 - \Phi_p^i}, \qquad i, i \in [n].$$

Suppose p is of degree at most 2 in z_i,

$$\Phi^j_{\partial_i p}(z - \delta e_i) \leq \Phi_p^j(z), \qquad \delta = \frac{1}{2\Phi_p^i}, \qquad j \in [n].$$

Theorem (R, 2016)

$p(z_1, \cdots, z_n)$ real stable and of degree at most 2 in each of the variables. (For instance, $p = \det[Z - A]^2$). Let $q = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} p$. Then, for any $z \in Ab_p$,

$$\Phi_q^j(z - \delta) \leq \Phi_p^j(z), \quad j \in [n] \quad \textit{where} \quad \delta = \min_{j \in [n]} \frac{1}{2\Phi_p^j(z)}.$$

Lemma (R, 2016)

Suppose $p = \det[Z - A]^2$ where A is a positive contraction and $z = zI$ where $z > \lambda_1(A)$, then,

$$\Phi_p^j(zI) \leq \frac{\delta}{z-1} + \frac{1-\delta}{z}, \quad \delta = max(A_{ii}).$$

Theorem (R, 2016)

$A \in M_n(\mathbb{R})^+$, positive contraction, diagonal entries of A all be at most $\alpha \leq \frac{1}{4}$. Then,

$$\operatorname{maxroot} \chi_2[A] \leq \inf_{z \geq 1} z - \frac{1}{2}\left(\frac{\alpha}{z-1} + \frac{1-\alpha}{z}\right)^{-1} = \frac{1}{4}\left(\sqrt{\alpha} + \sqrt{3(1-\alpha)}\right)^2.$$

Remark

Suppose we could shift the barrier to the left by $\frac{1}{\Phi_p^j(z)}$ instead of $\frac{1}{2\Phi_p^j(z)}$, we would have the conjectured optimal estimate of maxroot $\chi_2[A] \leq \frac{1}{2} + \sqrt{\alpha(1-\alpha)}$.

Alas, not true in general. Also fails for polynomials of the form $\det[Z - A]^2$.
Similar estimates can be gotten for $\chi_3[A]$ and $\chi_4[A]$ through brute force means.

Theorem (R, 2016)

$A \in M_n(\mathbb{R})^+$, positive contraction, diagonal entries of A all be at most α. Then,

$$\operatorname{maxroot} \chi_3[A] \leq \frac{1}{9}\left(\sqrt{5(1-\alpha)} + 2\sqrt{\alpha}\right)^2, \qquad \alpha \leq \frac{4}{9}.$$

$$\operatorname{maxroot} \chi_4[A] \leq \frac{1}{16}\left(\sqrt{7(1-\alpha)} + 3\sqrt{\alpha}\right)^2, \quad \alpha \leq \frac{9}{16}.$$

Question

p and q real stable polynomials in n variables. Estimates for zero free regions of,

$$q(\partial_1, \cdots, \partial_n)p(z_1, \cdots, z_n).$$

Special case of great interest,

$$e_k(\partial_1, \cdots, \partial_n) \det[Z - A].$$

One variable case : $\varphi > 0$. Define $\text{smax}_\varphi(p) = \phi_p^{-1}(\varphi)$.

Theorem

p real rooted. Then,

$$\text{smax}_\varphi(\partial p) \leq \text{smax}_\varphi(p) - \frac{1}{\varphi}, \qquad \text{smax}_\varphi[(\partial - \alpha)p] \leq \text{smax}_\varphi(p) - \frac{1}{\varphi - \alpha}.$$

Theorem (One variable Analytic Lieb-Sokal)

p, q real rooted. Then,

$$\text{smax}_\varphi(q(\partial)p) \leq \text{smax}_\varphi(p) - \Phi_q(\varphi).$$

Conjectural Analytic Lieb-Sokal

Multivariable case : $\varphi \in \mathbb{R}^n_+$.

Let $\text{smax}_\varphi(p) = \{b \in \mathbb{R}^n : \Phi_p(b) = \varphi\}$.

Given two sets $A, B \in \mathbb{R}^n$, say $A \prec B$ if for all $b \in B$ and $h \in \mathbb{R}^n_+$, $b + h \notin A$.

Conjecture

p, q real stable in $\mathbb{R}[z_1, \cdots, z_n]$ and let $a \in \text{Ab}_p$. Then,

$$\text{smax}_\varphi(q(\partial)p) \prec \text{smax}_\varphi(p) - \Phi_q(\varphi).$$

Given a class function ϕ on S_n and a matrix A, the expression

$$\det_\phi(A) := \sum_{\sigma \in S_n} \left(\prod_{i \in [n]} a_{i\sigma(i)} \right) \phi(\sigma),$$

is called an immanant. One many define the expression,

$$\chi_\phi[A] := \det_\phi[xI - A].$$

$c(\sigma)$: number of cycles in σ.
When $\phi(\sigma) = (-1)^{sgn(\sigma)}$, we get $\chi[A]$.
When $\phi(\sigma) = (-1)^{sgn(\sigma)} r^{c(\sigma)}$, we get $\chi_r[A]$.
$r \in \mathbb{N}$: We have that $\chi_r[A]$ is real rooted for hermitian A.

Question

Which immanantal polynomials are real rooted for all hermitian arguments?

Conjecture

Those immanants such that

$$\det_\phi(A) = p(\partial_1, \cdots, \partial_n) \det[Z + A]^k \mid_{Z=0}, \quad deg(p) = (k-1)n, \quad p \text{ real stable } + \text{ symmetric.}$$