A Method for Companionability, Applied to Group Actions and Valuations

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Ἢν στρατεύηται ἐπὶ Πέρσας, μεγάλην ἀρχὴν μιν καταλύσειν

—Oracle to Croesus, as reported by Herodotus (I.53)

Three theories of education:

1. Learn the word of God.
2. Learn skills.
3. Learn freedom.

Mathematics teaches freedom by being
(a) *personal* (none can command you to accept a theorem),
(b) *universal* (disagreement must be settled peacefully).

> [Formal proof] was one of the great discoveries of the early 20th century, largely due to Frege, Russell, and Whitehead . . . This discovery has had a profound impact on mathematics, because it means that *any dispute about the validity of a mathematical proof can always be resolved.*
>
> —Timothy Gowers, *Mathematics: A Very Short Introduction*

Nesin Mathematics Village, Şirince
Selçuk, İzmir (Ephesus, Ionia), July 9, 2016

A PROBLEM OF MODEL THEORY: identify complete theories and their properties, such as being axiomatizable or not.

Presburger 1930. $\mathrm{Th}(\mathbb{N}, +)$ is axiomatizable.

Gödel 1931. $\mathrm{Th}(\mathbb{N}, +, \times)$ is not.

USEFUL DEFINITIONS:

$$\mathrm{diag}(\mathfrak{A}) = \mathrm{Th}(\text{structures in which } \mathfrak{A} \text{ embeds}),$$
$$T_\forall = \mathrm{Th}(\text{structures embedding in models of } T).$$

A. Robinson 1956. A theory T is **model-complete** if, whenever $\mathfrak{A} \models T$, then $T \cup \mathrm{diag}(\mathfrak{A})$ is complete.

"Eli Bers" 1969. A model-complete theory T^* is the **model-companion** of any theory T for which ${T^*}_\forall = T_\forall$.

Fields may have

1) a valuation ring $\mathcal{O}$ (with max. ideal $\mathcal{M}$),
2) a derivation δ,
3) an automorphism σ.

the theory of fields	has model companion	source
—	ACF	Tarski 1948
with $\mathcal{O}$	ACVF	Robinson 1956
of char. 0 with δ	DCF_0	Robinson 1959
of char. p with δ	DCF_p	Wood 1973
with σ	ACFA	Macintyre 1997 Chatzidakis–Hrushovski 1999
of char. 0 with δ, σ	yes	P. 2004
of char. p with δ, σ	no	P. 2004
with $\mathcal{O}$, σ	yes	Beyarslan–Hoffmann–Onay–P. 201?

THEOREM (generalizing **Robinson 1957**). If it exists, the model-companion T^* of a theory T is axiomatized by $T_\forall$ and sentences

$$\forall \boldsymbol{x}\ \forall \boldsymbol{y}\ \exists \boldsymbol{z}\ \big(\vartheta(\boldsymbol{x},\boldsymbol{y}) \Rightarrow \varphi(\boldsymbol{x},\boldsymbol{z})\big),$$

where

- φ is a system of atomic and negated atomic formulas,
- ϑ is from a set Θ_φ of formulas,
- for all models $\mathfrak{M}$ of $T_\forall$ with parameters $\boldsymbol{a}$,

$$\vartheta(\boldsymbol{a},\boldsymbol{y}) \text{ is soluble in } \mathfrak{M} \text{ for some } \vartheta \text{ in } \Theta_\varphi \iff$$
$$\varphi(\boldsymbol{a},\boldsymbol{z}) \text{ is soluble in a model of } T_\forall \cup \operatorname{diag}(\mathfrak{M}).$$

Not every system φ need be considered, but "enough" of them.

To axiomatize DCF_0, Robinson considered all systems.

Blum 1968: one-variable systems are enough.

For the model-companion of any theory T, it is enough to consider *unnested* systems.

EXAMPLE. Over a field K with σ, $\mathcal{O}$, and $\mathcal{M}$, one need only understand systems

$$\bigwedge_{f\in I_0} f = 0 \wedge \bigwedge_{i<m} X_i{}^{\sigma} = X_{\tau(i)} \wedge \bigwedge_{\ell\in\lambda} X_\ell \in \mathcal{O} \wedge \bigwedge_{k\in\kappa} X_k \in \mathcal{M},$$

where, for some n in ω,

- I_0 is a finite subset of $K[X_j : j < n]$,
- $m \leqslant n$ and $\tau : m \rightarrowtail n$,
- $\kappa \subseteq \lambda \subseteq n$.

EXAMPLE. A **group action** is (P, A), where

1) $P = \{\text{functions}\}$,

2) $A = \{\text{points}\}$,

3) there is $(\xi, y) \mapsto \xi\, y$ from $P \times A$ to A whereby

   (a) functions have inverses: $\forall \xi\ \exists \eta\ \forall z\ (\xi\, \eta\, z = z \wedge \eta\, \xi\, z = z)$;

   (b) two functions have a composite: $\forall \xi\ \forall \eta\ \exists \zeta\ \forall v\ \xi\, \eta\, v = \zeta\, v$;

   (c) there is an identity: $\exists \xi\ \forall y\ \xi\, y = y$.

Let $\mathsf{GA} = \mathrm{Th}(\text{group actions})$.

Then $\mathsf{GA}_\forall$ is that functions are injective:

$$\forall \xi\ \forall y\ \forall z\ (y \neq z \Rightarrow \xi\, y \neq \xi\, z).$$

KEY OBSERVATION: Compositions not preserved in extensions.

To find GA^*, one need only consider systems

$$\bigwedge \alpha\, x = y \wedge \bigwedge \xi\, t = u,$$

where t and u are point variables or constants.

GA^* is complete and says,

1) any $n!$ distinct functions act like $\mathrm{Sym}(n)$ on some n points;

2) on any n distinct points, some $n!$ functions act like $\mathrm{Sym}(n)$;

3) there are at least two points.

GA includes Th(**parametrized permutations**), axiomatized by

$$\forall \xi\ \forall y\ \forall z\ (y \neq z \Rightarrow \xi\, y \neq \xi\, z), \qquad \forall \xi\ \forall y\ \exists z\ \xi\, z = y.$$

Shelah 1993: $\mathsf{T}_{\mathrm{feq}}$, namely

Th(**parametrized equivalence relations**).

$\mathsf{T}_{\text{feq}}{}^* = \text{Th}(\text{Fraïssé limit of the class of finite models of } \mathsf{T}_{\text{feq}})$.

It was shown that $\mathsf{T}_{\text{feq}}{}^*$ has TP_2 and, ultimately, NSOP_1.

Thus $\mathsf{T}_{\text{feq}}{}^*$ occupies an undivided region of the *Map of the Universe* (**Conant 2013–,** `forkinganddividing.com`).

As for GA^*, so for for $\mathsf{T}_{\text{feq}}{}^*$, one can obtain axioms:

1) a partition of n points is effected by some relation,

2) the intersection of classes of n distinct relations is nonempty,

3) there are n relations and n classes of each.

Like $\mathsf{T}_{\text{feq}}{}^*$, GA^* has TP_2 and NSOP_1.

Chernikov–Ramsey 2016: In a finite relational signature,

if the theory of the Fraïssé limit of a Fraïssé class with Strong Amalgamation is simple,

then the theory of *parametrized* models has NSOP_1,

because it has a certain independence relation $\bigcirc\!\!\!\!\!\!\perp$ with independent amalgamation of types.

THEOREM. GA^* also has NSOP_1, because of $\bigcirc\!\!\!\!\!\!\perp$ given by

$$(\mathrm{P}, A) \underset{(\mathrm{T},C)}{\bigcirc\!\!\!\!\!\!\perp} (\Sigma, B) \iff$$
$$\mathrm{P} \cap \Sigma \subseteq \mathrm{T} \ \ \& \ \ \langle A \cup C \rangle_{\mathrm{P}\cup\mathrm{T}} \cap \langle B \cup C \rangle_{\Sigma\cup\mathrm{T}} \subseteq \langle C \rangle_{\mathrm{T}},$$

where $\langle X \rangle_{\Xi} = \{\xi^n x \colon \xi \in \Xi \ \ \& \ \ n \in \mathbb{Z} \ \ \& \ \ x \in X\}$.

Τέττιξ on tree, Marmara Island (Proconnesus), July 26, 2012

τῇ δὲ πρεσβυτάτῃ Καλλιόπῃ *καὶ τῇ μετ' αὐτὴν* Οὐρανίᾳ *τοὺς ἐν φιλοσοφίᾳ διάγοντάς τε καὶ τιμῶντας τὴν ἐκείνων μουσικὴν ἀγγέλλουσιν, αἳ δὴ μάλιστα τῶν* Μουσῶν *περί τε οὐρανὸν καὶ λόγους οὖσαι θείους τε καὶ ἀνθρωπίνους ἱᾶσι καλλίστην φωνήν* —Plato, *Phaedrus* 259D