Lindström's Theorem

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June 15, 2015

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We work out a proof of Lindström's Theorem, that in any proper expansion of first-order logic (meeting certain natural conditions), either the Compactness Theorem or the (downward) Löwenheim–Skolem Theorem is lost. Our main reference is Chang & Keisler [1, §2.5, pp. 127–135]; but Hodges [2, §3.3, pp. 102–111] is also useful. The sections of these notes are as follows.

§1 establishes some notation.

§2 reviews the basic definitions of model theory, so that we can be precise about what an *abstract logic* is.

§3 reviews notions that Chang & Keisler introduce *ad hoc,* but Hodges develops generally, in terms of *games* (namely Ehrenfeucht–Fraïssé games).

§4 proves Lindström's Theorem, in a way that seems simpler than Chang & Keisler's.

The last major editing of these notes was on or before August 11, 2014. Some mild adjustments were made in June, 2015.

Contents

1 Tuples

If A and Ω are sets, then a **relation from Ω to A** is a subset of the Cartesian product $\Omega \times A$. If R is such a relation, and it contains (b, a), we write

$$b\ R\ a.$$

If for every b in Ω there is exactly one a in A for which this holds, then R is a **function** from Ω to A, and we use the notation

$$R(b) = a, \qquad (*)$$

so that $R(b)$ can be used as another name for a. We denote by

$$A^{\Omega}$$

the class of all functions from Ω to A. (We could also let A, but not Ω, be a proper class.) A typical element of A^{Ω} can be written as

$$(a_i : i \in \Omega).$$

We let ω be the set of von Neumann natural numbers. If $n \in \omega$, then a typical element of A^n can be written in any of three forms:

$$(a_k : k < n), \qquad (a_0, \dots, a_{n-1}), \qquad \boldsymbol{a}.$$

Such functions are n-**tuples** of A. There is a unique 0-tuple, namely 0, so that

$$A^0 = 1.$$

There is a function, actually a bijection,

$$(\boldsymbol{a}, b) \mapsto (a_0, \dots, a_{n-1}, b) \text{ from } A^n \times A \text{ to } A^{n+1}. \tag{†}$$

Likewise, if also $m \in \omega$, then there is a bijection

$$(\boldsymbol{a}, \boldsymbol{b}) \mapsto (a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1}) \text{ from } A^n \times A^m \text{ to } A^{n+m}. \tag{‡}$$

We let

$$A^{<\omega} = \bigcup_{n \in \omega} A^n.$$

Then, by taking the corresponding unions of the functions in (†) and (‡), we have embeddings

$$\left.\begin{aligned} (\boldsymbol{a}, b) &\mapsto (a_0, \dots, b) \text{ from } A^{<\omega} \times A \text{ to } A^{<\omega}, \\ (\boldsymbol{a}, \boldsymbol{b}) &\mapsto (a_0, \dots, b_0, \dots) \text{ from } A^{<\omega} \times A^{<\omega} \text{ to } A^{<\omega} \end{aligned}\right\} \tag{§}$$

(which we shall need on page 15). We may also write these embeddings as if they were inclusions. That is, we may consider $(\boldsymbol{a}, b)$ and $(\boldsymbol{a}, \boldsymbol{b})$ as elements of $A^{<\omega}$.

2 Model theory

2.1 Structures

If $n \in \omega$, then an n**-ary relation** on a set A is a subset of A^n; an n**-ary operation** on A is a function from A^n to A. Here n is the number of **arguments** of the relation or operation. By means of the bijection in (†), we may consider n-ary operations as $(n+1)$-ary relations, though we retain the special notation for functions given in $(*)$. Normally we have no use for nullary (0-ary) relations, but an nullary operation on A is identified with an element of A.

A **structure** is a set, possibly given together with some operations and relations on it. If the set is A, the structure can be denoted by

$$\mathfrak{A},$$

and then A is the **universe** of $\mathfrak{A}$. Each of the operations and relations of $\mathfrak{A}$ is assigned a symbol, and these symbols compose the **signature** of $\mathfrak{A}$. Each symbol carries the information of whether it denotes an operation or relation, along with its number of arguments. We shall use

$$\mathscr{S}$$

to denote an arbitrary signature, and we shall denote the class of structures having this signature by

$$\mathbf{Str}_{\mathscr{S}}.$$

2.2 Sentences

We shall define a set of *first order sentences* in the signature $\mathscr{S}$. In any such sentence may occur **individual variables** from the list $(v_0, v_1, v_2, \dots)$. We use expressions from the list $(x_0, x_1, x_2, \dots)$, as well as x, y, z, and w, to stand for individual variables. An **unnested atomic formula** of $\mathscr{S}$ is an expression of one of the forms

$$x_0 = x_1, \qquad Rx_0 \cdots x_n, \qquad Fx_0 \cdots x_{n-1} = x_n,$$

where R is an $(n+1)$-ary relation symbol of $\mathscr{S}$ and F is an n-ary operation symbol of $\mathscr{S}$. We obtain arbitrary **atomic formulas** by closing under the operation of replacing a variable by an expression like $Fx_0 \cdots x_{n-1}$. Alternatively, we first define a **term** to be either a variable or an expression

$$Ft_0 \cdots t_{n-1},$$

where the t_k are now terms; then an atomic formula is an expression

$$t_0 = t_1, \qquad\qquad Rt_0 \cdots t_n,$$

where the t_k are terms. Then we obtain the set

$$\mathrm{Fm}_{\mathscr{S}}$$

of all **first order formulas** of $\mathscr{S}$ by closing the set of atomic formulas under the operations denoted by $\neg$, $\vee$, and $\exists x$: thus, if φ and ψ are formulas, then so are the expressions denoted by

$$\neg\varphi, \qquad\qquad (\varphi \vee \psi), \qquad\qquad \exists x\ \varphi;$$

these are, respectively,

1) the **negation** of φ,
2) the **disjunction** of φ and ψ, and
3) the **instantiation** at x of φ.

The set of **unnested formulas** is obtained by closing likewise the set of unnested atomic formulas.

Thus the set of formulas of $\mathscr{S}$ (as well as of unnested formulas) is defined recursively, and so it is possible to prove by induction that particular subsets of $\mathrm{Fm}_{\mathscr{S}}$ are actually the whole set. An important theorem, often overlooked, is that functions *on* $\mathrm{Fm}_{\mathscr{S}}$ can be defined by recursion, because of the *unique readability* of formulas. Each formula is the root of a tree whose nodes are all of the subformulas that occur in the construction of the formula; **unique readability** is that this tree is unique.

For example, by unique readability, there is a function

$$\varphi \mapsto \mathrm{fv}(\varphi)$$

assigning to each formula in $\mathrm{Fm}_{\mathscr{S}}$ its set of **free variables,** and this function can be defined recursively:

0. The free variables of an atomic formula are the variables that occur in it.
1. $\mathrm{fv}(\neg\varphi) = \mathrm{fv}(\varphi)$.
2. $\mathrm{fv}((\varphi \vee \psi)) = \mathrm{fv}(\varphi) \cup \mathrm{fv}(\psi)$.
3. $\mathrm{fv}(\exists x\ \varphi) = \mathrm{fv}(\varphi) \smallsetminus \{x\}$.

A **first order sentence** is a first order formula having no free variables. We shall denote the set of first order sentences of $\mathscr{S}$ by

$$\mathrm{Sn}_{\mathscr{S}}\ .$$

Thus $\mathrm{Sn}_{\mathscr{S}} = \{\varphi \in \mathrm{Fm}_{\mathscr{S}} \colon\ \mathrm{fv}(\varphi) = \varnothing\}$.

2.3 Truth

There is a relation of *truth,* denoted by

$$\vDash$$

or more precisely $\vDash_{\mathscr{S}}$, between $\mathbf{Str}_{\mathscr{S}}$ and $\mathrm{Sn}_{\mathscr{S}}$. There are two equivalent ways to define it, and each one has its inconveniences.

2.3.1 Definable sets

One approach to truth is to define more generally a certain function

$$(\mathfrak{A}, \varphi) \mapsto \varphi^{\mathfrak{A}}$$

on $\mathbf{Str}_{\mathscr{S}} \times \mathrm{Fm}_{\mathscr{S}}$, where $\varphi^{\mathfrak{A}} \subseteq A^{\mathrm{fv}(\varphi)}$; here $\varphi^{\mathfrak{A}}$ is the relation on A that is **defined** by φ in $\mathfrak{A}$. Once we have defined this for arbitrary φ, we can make the definition

$$\mathfrak{A} \vDash \sigma \iff \sigma^{\mathfrak{A}} = 1,$$

where $\sigma \in \mathrm{Sn}_{\mathscr{S}}$.

First we have to note that the same symbol S in $\mathscr{S}$ denotes different things in different structures; what it denotes in $\mathfrak{A}$ can be denoted more precisely by

$$S^{\mathfrak{A}}.$$

Then $S^{\mathfrak{A}}$ is the **interpretation** in $\mathfrak{A}$ of S. We proceed.

The interpretation of an operation symbol is an operation. A term t also has an interpretion $t^{\mathfrak{A}}$, which is an operation. More precisely, if $\mathrm{v}(t)$ is the set of variables occurring in t, then $t^{\mathfrak{A}}$ is a function from $A^{\mathrm{v}(t)}$ to A. In particular, if t is just v_k, then an element of $A^{\mathrm{v}(t)}$ is just a set $\{(v_k, a)\}$, whose image under ${v_k}^{\mathfrak{A}}$ is a. Also,

$$\begin{aligned}&Ft_0 \cdots {t_{n-1}}^{\mathfrak{A}}(a_x \colon x \in \mathrm{v}(Ft_0 \cdots t_{n-1}))\\ &\qquad = F^{\mathfrak{A}}({t_0}^{\mathfrak{A}}(a_x \colon x \in \mathrm{v}(t_0)), \dots, {t_{n-1}}^{\mathfrak{A}}(a_x \colon x \in \mathrm{v}(t_{n-1})))).\end{aligned}$$

Now we can define interpretations of formulas.

0. For unnested atomic formulas, we define

$$(x_0 = x_1)^{\mathfrak{A}} = \{f \in A^{\{x_0, x_1\}} \colon f(x_0) = f(x_1)\},$$
$$(Rx_0 \cdots x_n)^{\mathfrak{A}} = \{f \in A^{\{x_0, \dots, x_n\}} \colon (f(x_0), \dots, f(x_n)) \in R^{\mathfrak{A}}\},$$

and

$$\begin{aligned}&(Fx_0 \cdots x_{n-1} = x_n)^{\mathfrak{A}}\\ &\qquad = \{f \in A^{\{x_0, \dots, x_n\}} \colon F^{\mathfrak{A}}(f(x_0), \dots, f(x_{n-1})) = f(x_n)\}.\end{aligned}$$

For arbitrary formulas,

$$\begin{aligned}(t_0 = t_1)^{\mathfrak{A}} = \{f \in A^{\mathrm{v}(t_0) \cup \mathrm{v}(t_1)} \colon&\\ &{t_0}^{\mathfrak{A}}(f \restriction \mathrm{v}(t_0)) = {t_1}^{\mathfrak{A}}(f \restriction \mathrm{v}(t_1))\},\end{aligned}$$

$$\begin{aligned}(Rt_0 \cdots t_n)^{\mathfrak{A}} = \{f \in A^{\mathrm{v}(t_0) \cup \cdots \cup \mathrm{v}(t_n)} \colon&\\ &({t_0}^{\mathfrak{A}}(f \restriction \mathrm{v}(t_0)), \dots, {t_n}^{\mathfrak{A}}(f \restriction \mathrm{v}(t_n))) \in R^{\mathfrak{A}}\}.\end{aligned}$$

1. For negations:
$$(\neg\varphi)^{\mathfrak{A}} = A^{\mathrm{fv}(\varphi)} \smallsetminus \varphi^{\mathfrak{A}}.$$

2. For disjunctions, if $\mathrm{fv}(\varphi) = \mathrm{fv}(\psi)$, then
$$(\varphi \lor \psi)^{\mathfrak{A}} = \varphi^{\mathfrak{A}} \cup \psi^{\mathfrak{A}};$$
but in general we have to say
$$\begin{aligned}(\varphi \lor \psi)^{\mathfrak{A}} = \{f \in A^{\mathrm{fv}(\varphi)\cup\mathrm{fv}(\psi)} \colon f \restriction \mathrm{fv}(\varphi) \in \varphi^{\mathfrak{A}}\}\\ \cup \{f \in A^{\mathrm{fv}(\varphi)\cup\mathrm{fv}(\psi)} \colon f \restriction \mathrm{fv}(\psi) \in \psi^{\mathfrak{A}}\}.\end{aligned}$$

3. Finally, for instantiations,
$$(\exists x\ \varphi)^{\mathfrak{A}} = \{f \restriction (\mathrm{fv}(\varphi) \smallsetminus \{x\}) \colon f \in \varphi^{\mathfrak{A}}\}.$$

2.3.2 Constants

Alternatively, if $\mathfrak{A} \in \mathbf{Str}_{\mathscr{S}}$, we can create the *expanded* signature $\mathscr{S}(A)$, which is $\mathscr{S}$ together with a new **constant** (nullary operation symbol) for each element of A. Then we expand $\mathfrak{A}$ to an element $\mathfrak{A}_A$ of $\mathbf{Str}_{\mathscr{S}(A)}$, where the constant associated with an element of A is interpreted as that element. Normally we denote the constant and the element by the same symbol.

A **closed term** is a term with no variables. Every closed term of $\mathscr{S}(A)$ has an obvious interpretation in $\mathfrak{A}$:
$$(Ft_0 \cdots t_{n-1})^{\mathfrak{A}} = F^{\mathfrak{A}}({t_0}^{\mathfrak{A}}, \dots, {t_{n-1}}^{\mathfrak{A}}).$$

Now we can define truth as follows.

0. For atomic sentences we have
$$\begin{aligned}\mathfrak{A} \vDash t_0 = t_1 &\iff {t_0}^{\mathfrak{A}} = {t_1}^{\mathfrak{A}},\\ \mathfrak{A} \vDash Rt_0 \cdots t_n &\iff ({t_0}^{\mathfrak{A}}, \dots, {t_n}^{\mathfrak{A}}) \in R^{\mathfrak{A}}.\end{aligned}$$

1. For negations,
$$\mathfrak{A} \models \neg\sigma \iff \mathfrak{A} \not\models \sigma.$$

2. For disjunctions,
$$\mathfrak{A} \models (\sigma \vee \tau) \iff \mathfrak{A} \models \sigma \text{ OR } \mathfrak{A} \models \tau.$$

3. For instantiations, we introduce yet another notation: by
$$\varphi^x_a$$
we mean the result of replacing each *free occurrence* of x in φ with a. The formal definition is recursive:
   a) If φ is atomic, then φ^x_a is just φ;
   b) $\neg\varphi^x_a$ is $\neg\psi$, where ψ is φ^x_a;
   c) $(\varphi \vee \psi)^x_a$ is $\varphi^x_a \vee \theta$, where θ is ψ^x_a;
   d) $\exists y\ \varphi^x_a$ is just $\exists y\ \varphi$, if x is the same variable as y; otherwise it is $\exists y\ \psi$, where ψ is φ^x_a.

   Now we can define
$$\mathfrak{A} \models \exists x\ \varphi \iff \text{ for some } a \text{ in } A,\ \mathfrak{A} \models \varphi^x_a.$$

The recursive definition of truth relies on a recursive definition of $\mathrm{Sn}_{\mathscr{S}}$ (and then on their unique readability, as before): starting with atomic *sentences*, we close under
1) negation,
2) disjunction, and
3) the "multivalued operations" $\varphi^x_a \mapsto \exists x\ \varphi$, that is, for each a in A, the operation that, given a sentence, finds all of the (finitely numerous) formulas φ such that the sentence is φ^x_a, and then forms the sentences $\exists x\ \varphi$.

2.4 Abstract logic

The class $\mathbf{Str}_{\mathscr{S}}$, the set $\mathrm{Sn}_{\mathscr{S}}$, and the relation $\vDash$ between them make up the **first order logic** of $\mathscr{S}$. For an instance of an **abstract logic** of $\mathscr{S}$, we replace $\mathrm{Sn}_{\mathscr{S}}$ with $\mathrm{Sn}^*_{\mathscr{S}}$, which has the same closure properties as $\mathrm{Sn}_{\mathscr{S}}$ just mentioned: $\mathrm{Sn}^*_{\mathscr{S}}$ must contain all atomic sentences and be closed under (1) negation, (2) disjunction, and (3) the operations $\varphi^x_a \mapsto \exists x\ \varphi$. But $\mathrm{Sn}^*_{\mathscr{S}}$ may have some other closure properties as well. Then the definition of truth must be enlarged to deal with the new closure properties; but the existing parts of the definition stand. We make the following additional requirements:

Occurrence: In every element σ of $\mathrm{Sn}^*_{\mathscr{S}}$, only finitely many symbols of $\mathscr{S}$ may occur.

Reduction: Of the symbols in $\mathscr{S}$, the definition of truth of σ in $\mathfrak{A}$ involves only those that actually occur in σ.

Renaming: All that matters about these symbols is whether they denote relations or operations, and how many arguments they take.

Isomorphism: Truth must be preserved under isomorphism of structures.

Relativization: Sentences can be *relativized* in the way to be described on page 16.

As we shall use it, Relativization will subsume Renaming and Isomorphism.

2.5 The Galois correspondence

If $\sigma \in \mathrm{Sn}_{\mathscr{S}}$, we let

$$\mathbf{Mod}(\sigma) = \{\mathfrak{A} \in \mathbf{Str}_{\mathscr{S}} : \mathfrak{A} \vDash \sigma\}; \tag{¶}$$

this is the class of **models** of σ. Then

$$\left.\begin{aligned} \mathbf{Mod}(\sigma \vee \tau) = \mathbf{Mod}(\sigma) \cup \mathbf{Mod}(\tau), \\ \mathbf{Mod}(\exists x\ x \neq x) = \varnothing, \end{aligned}\right\} \tag{‖}$$

and so the sets $\mathbf{Mod}(\sigma)$ are a basis of closed *classes* of a topology on $\mathbf{Str}_{\mathscr{S}}$. We may call this topology the **first order topology.**

- The **Compactness Theorem** is that this topology is compact.
- The most basic form of the **Löwenheim–Skolem Theorem** is that every sentence with model has a **countable model** (that is, a model with a countable universe); in other words, every nonempty basic closed set has a countable element.

If one is bothered by a topological space that is a proper class, one can pass to a *Kolmogorov quotient* of $\mathbf{Str}_{\mathscr{S}}$, namely

$$\{\mathrm{Th}(\mathfrak{A})\colon \mathfrak{A} \in \mathbf{Str}_{\mathscr{S}}\},$$

where

$$\mathrm{Th}(\mathfrak{A}) = \{\sigma \in \mathrm{Sn}_{\mathscr{S}} : \mathfrak{A} \vDash \sigma\}.$$

However, we do not want to do this, since we shall be interested in refinements of the first order topology, as induced by abstract logics. In any case, let us note that two structures $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathbf{Str}_{\mathscr{S}}$ are **elementarily equivalent,** and we write

$$\mathfrak{A} \equiv \mathfrak{B},$$

if they are indistinguishable in the first order topology, that is,

$$\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B}).$$

Now, if we replace $\mathrm{Sn}_{\mathscr{S}}$ above with $\mathrm{Sn}^*_{\mathscr{S}}$ from an abstract logic of $\mathscr{S}$, we can still make the definition (¶) when $\sigma \in \mathrm{Sn}^*_{\mathscr{S}}$, and then (‖) still holds for arbitrary σ and τ in $\mathrm{Sn}^*_{\mathscr{S}}$. Thus passing to $\mathrm{Sn}^*_{\mathscr{S}}$ gives us a finer topology on $\mathbf{Str}_{\mathscr{S}}$ than the first order topology. However, because of our requirements on abstract logics, this topology still does not distinguish isomorphic structures.

It is going to be convenient to allow *multi-sorted* structures, that is, structures (like vector spaces) with more than one universe, each of these universes being called a **sort.** Then variables fall into different sorts, and there may be functions (like scalar multiplication) from a product of some of the sorts (possibly with repeated sorts) to one of them. We shall need only structures with finitely many sorts, and in this case, we can consider the structures as one-sorted, by means of a dodge: we replace the sorts with their disjoint union, and we introduce new relation symbols to distinguish the sorts. (We shall also have to deal with the fact that, in the formal definition, operations are total functions on a power of the universe.)

Two sentences σ and τ are **logically equivalent,** and we may write

$$\sigma \sim \tau,$$

if $\mathbf{Mod}(\sigma) = \mathbf{Mod}(\tau)$.

Theorem 1. *Every first order sentence is logically equivalent to an unnested sentence.*

Proof. First note that

$$t_0 = t_1 \sim \exists x\ (t_0 = x \wedge t_1 = x),$$
$$Rt_0 \cdots t_n \sim \exists x_0 \cdots \exists x_n\ (t_0 = x_0 \wedge \cdots \wedge t_n = x_n \wedge Rx_0 \cdots x_n)$$

(where $(\varphi \wedge \psi)$ is an abbrevation for $\neg(\neg\varphi \wedge \neg\psi)$, and $(\varphi \wedge \psi \wedge \theta)$ means $(\varphi \wedge (\psi \wedge \theta))$, and so on). The formulas $t_k = x$ and $t_k = x_k$ may not be unnested. If they are not, we apply the equivalence

$$Ft_0 \cdots t_{n-1} = y \sim \exists x_0 \cdots \exists x_{n-1}$$
$$(t_0 = x_0 \wedge \cdots \wedge t_n = x_n \wedge Fx_0 \cdots x_{n-1} = y),$$

repeatedly as necessary, in order to conclude that every atomic sentence is equivalent to an unnested sentence. □

3 Local isomorphisms

An n-**ary** formula is a formula whose free variables belong to the set $\{v_k : k < n\}$. Thus if $m < n$, then m-ary formulas are also n-ary. If $\mathfrak{A}$

is a structure of $\mathscr{S}$, and φ is an n-ary formula of $\mathscr{S}(A)$, and $\boldsymbol{a} \in A^n$, then the expression denoted by

$$\varphi(\boldsymbol{a})$$

is the result of replacing each free occurrence of v_k in φ with a_k, for each k in n. Suppose also $\mathfrak{B} \in \mathbf{Str}_{\mathscr{S}}$. We denote by

$$I_0$$

the relation from $A^{<\omega}$ to $B^{<\omega}$ consisting of those pairs $(\boldsymbol{a}, \boldsymbol{b})$ such that, for some n in ω, both $\boldsymbol{a}$ and $\boldsymbol{b}$ are n-tuples, and for all n-ary *unnested* formulas φ of $\mathscr{S}$,

$$\mathfrak{A} \vDash \varphi(\boldsymbol{a}) \iff \mathfrak{B} \vDash \varphi(\boldsymbol{b}).$$

Since there are no unnested sentences, we have

$$0 \; I_0 \; 0.$$

For any k in ω, if I_k has been defined as a subset of I_0, we define I_{k+1} to comprise those $(\boldsymbol{a}, \boldsymbol{b})$ in I_0 such that, for all c in A, for some d in B, and also, for all d in B, for some c in A,

$$(\boldsymbol{a}, c) \; I_k \; (\boldsymbol{b}, d).$$

Finally, let

$$I = \bigcap_{k \in \omega} I_k.$$

If $\boldsymbol{a} \; I \; \boldsymbol{b}$, then, for all c in A, for some d in B, and also, for all d in B, for some c in A,

$$(\boldsymbol{a}, c) \; I \; (\boldsymbol{b}, d).$$

If $I \neq 0$, then $\mathfrak{A}$ and $\mathfrak{B}$ are said to be **locally isomorphic.**

Theorem 2. *Countable locally isomorphic structures are isomorphic.*

Proof. If $A = \{a_k : k \in \omega\}$ and $B = \{b_k : k \in \omega\}$, then by recursion there are $k \mapsto b'_k$ and $k \mapsto a'_k$ from ω to B and A respectively such that, for each n in ω,

$$(a_0, a'_0, \dots, a_n, a'_n)\ I\ (b'_0, b_0, \dots, b'_n, b_n).$$

Then $a_k \mapsto b'_k$ is a well-defined isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, whose inverse is $b_k \mapsto a'_k$. □

Theorem 3. *Locally isomorphic structures are elementarily equivalent.*

Proof. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are locally isomorphic. We shall show that, for all *unnested* formulas φ of $\mathscr{S}$,

$$\boldsymbol{a}\ I\ \boldsymbol{b} \implies \big(\mathfrak{A} \vDash \varphi(\boldsymbol{a}) \iff \mathfrak{B} \vDash \varphi(\boldsymbol{b})\big). \tag{$**$}$$

We use induction on the complexity of formulas.

1. The claim is true by definition if φ is unnested atomic.
2. If $(**)$ holds when φ is ψ, then it holds when φ is $\neg\psi$.
3. If $(**)$ holds when φ is ψ or θ, then it holds when φ is $\psi \lor \theta$.
4. Suppose $(**)$ holds when φ is ψ. Say $\boldsymbol{a}\ I\ \boldsymbol{b}$. If $\mathfrak{A} \vDash (\exists x\ \psi)(\boldsymbol{a})$, then for some c in A, we have $\mathfrak{A} \vDash \psi^x_c(\boldsymbol{a})$. By hypothesis, for some d in B, we have $(\boldsymbol{a}, c)\ I\ (\boldsymbol{b}, d)$, and then $\mathfrak{B} \vDash \psi^x_d(\boldsymbol{b})$, so $\mathfrak{B} \vDash (\exists x\ \psi)(\boldsymbol{b})$. By symmetry, we are done. □

4 Refinements of the first order topology

There is an alternative proof of the last theorem that does not rely directly on the recursive definition of formulas. This proof turns out to establish a more general result, namely the next theorem, where, given the set $\mathrm{Sn}^*_{\mathscr{S}}$ of sentences of an abstract logic of $\mathscr{S}$, we define

$$\mathrm{Th}^*(\mathfrak{A}) = \{\sigma \in \mathrm{Sn}^*_{\mathscr{S}} : \mathfrak{A} \vDash \sigma\}.$$

Theorem 4. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are locally isomorphic, and a given abstract logic of* $\mathscr{S}$ *satisfies the Löwenheim–Skolem Theorem, then*

$$\mathrm{Th}^*(\mathfrak{A}) = \mathrm{Th}^*(\mathfrak{B}).$$

Proof. Suppose there is σ in $\mathrm{Sn}^*_{\mathscr{S}}$ that is true in $\mathfrak{A}$, but not in $\mathfrak{B}$. We shall obtain a contradiction to the assumption that $\mathfrak{A}$ and $\mathfrak{B}$ are locally isomorphic. Let $\mathscr{S}_0$ consist of the symbols of $\mathscr{S}$ that actually occur in σ; by the Occurrence property, $\mathscr{S}_0$ is finite. We can **reduce** $\mathfrak{A}$ to a structure $\mathfrak{A} \upharpoonright \mathscr{S}_0$ having signature $\mathscr{S}_0$; but still, by the Reduction property,

$$\mathfrak{A} \vDash \sigma \iff \mathfrak{A} \upharpoonright \mathscr{S}_0 \vDash \sigma.$$

So we may assume that $\mathscr{S}$ is the finite signature $\mathscr{S}_0$. We shall form a multi-sorted structure, to be denoted by $(\mathfrak{A}, \mathfrak{B})$, with a signature $\mathscr{S}^*$. There will be a sentence τ of $\mathscr{S}^*$ such that $(\mathfrak{A}, \mathfrak{B}) \vDash \tau$, and if $(\mathfrak{C}, \mathfrak{D}) \vDash \tau$, then $\mathfrak{C}$ and $\mathfrak{D}$ are locally isomorphic, and σ is true in $\mathfrak{C}$, but not in $\mathfrak{D}$. By the Löwenheim–Skolem Theorem, countable such $\mathfrak{C}$ and $\mathfrak{D}$ exist, and then they are isomorphic by Theorem 2, which is absurd.

The structure $(\mathfrak{A}, \mathfrak{B})$ will have four sorts, namely A, $A^{<\omega}$, B, and $B^{<\omega}$. The relation I from $A^{<\omega}$ to $B^{<\omega}$ will be a relation of the structure. There will also be operations as in (§) on page 3, and likewise for B in place of A. There will be constants for 0 in $A^{<\omega}$ and $B^{<\omega}$ respectively. Then τ will have, as conjuncts,

$$\begin{gathered} 0 \; I \; 0, \\ \forall \boldsymbol{x} \; \forall \boldsymbol{y} \; \forall z \; \exists w \; (\boldsymbol{x} \; I \; \boldsymbol{y} \Rightarrow (\boldsymbol{x}, z) \; I \; (\boldsymbol{y}, w)), \\ \forall \boldsymbol{x} \; \forall \boldsymbol{y} \; \forall w \; \exists z \; (\boldsymbol{x} \; I \; \boldsymbol{y} \Rightarrow (\boldsymbol{x}, z) \; I \; (\boldsymbol{y}, w)), \\ \forall \boldsymbol{x} \; \forall \boldsymbol{y} \; \forall \boldsymbol{z} \; \forall \boldsymbol{w} \; (\boldsymbol{x} \; I \; \boldsymbol{y} \wedge \boldsymbol{z} \; I \; \boldsymbol{w} \Leftrightarrow (\boldsymbol{x}, \boldsymbol{z}) \; I \; (\boldsymbol{y}, \boldsymbol{w})). \end{gathered}$$

Also, let Γ consist of the unnested atomic formulas of $\mathscr{S}$ of one of the precise forms

$$v_0 = v_1, \qquad R v_0 \cdots v_n, \qquad F v_0 \cdots v_{n-1} = v_n.$$

Since $\mathscr{S}$ is assumed to be finite, Γ is also finite, and there is some n such that each formula in Γ is $(n+1)$-ary. Then τ can have, as a conjunct,

$$\forall x_0 \cdots \forall x_n \, \forall y_0 \cdots \forall y_n \\ \left(\boldsymbol{x} \; I \; \boldsymbol{y} \Rightarrow \bigwedge_{\varphi \in \Gamma} \bigl(\varphi(x_0, \dots, x_n) \Leftrightarrow \psi(y_0, \dots, y_n) \bigr) \right).$$

Here the x_k are understood to range over A, while $\boldsymbol{x}$ is the image of $(x_0, \dots, x_n)$ in $A^{<\omega}$; likewise with y for x and B for A; and the two instances of φ are also appropriately **relativized,** that is, their variables are assigned to the sorts A and B respectively. Now, in any model of τ, the symbol I will be interpreted as a nonempty subset of I (though not necessarily as I itself). Finally, by the Relativization property, we let τ have, as conjuncts, σ relativized to A and $\neg\sigma$ relativized to B. Then we have τ as desired. $\square$

If $I_k \neq 0$, we write
$$\mathfrak{A} \equiv_k \mathfrak{B}.$$

Theorem 5. *If $\mathscr{S}$ is finite, then for all n in ω there is a finite set Γ_n of sentences of $\mathscr{S}$ such that*

$$\mathfrak{A} \equiv_n \mathfrak{B} \iff \{\sigma \in \Gamma_n \colon \mathfrak{A} \vDash \sigma\} = \{\sigma \in \Gamma_n \colon \mathfrak{B} \vDash \sigma\}.$$

Proof. Assuming $\mathscr{S}$ is finite, for each n in ω, for each k in $n+1$, we shall define a finite set Γ^k_{n-k} of k-ary formulas such that,

$$\text{for all } \boldsymbol{a} \text{ in } A^k \text{ and } \boldsymbol{b} \text{ in } B^k, \; \boldsymbol{a} \; I_{n-k} \; \boldsymbol{b} \iff \\ \{\varphi \in \Gamma^k_{n-k} \colon \mathfrak{A} \vDash \varphi(\boldsymbol{a})\} = \{\varphi \in \Gamma^k_{n-k} \colon \mathfrak{B} \vDash \varphi(\boldsymbol{b})\}. \quad (\dagger\dagger)$$

We can let Γ^n_0 be the set of all n-ary unnested atomic formulas of $\mathscr{S}$. If $\ell < n$, and we have a finite set $\Gamma^{\ell+1}_{n-\ell-1}$ of $(\ell+1)$-ary formulas such that (††) holds when $k = \ell + 1$, then we can let

$$\Gamma^{\ell}_{n-\ell} = \left\{ \forall v_\ell \bigvee_{\varphi \in X} \varphi \wedge \bigwedge_{\varphi \in X} \exists v_\ell\, \varphi \colon X \subseteq \Gamma^{\ell}_{n-\ell} \right\}.$$

So we have Γ^k_{n-k} as desired for all k in $n+1$. Now let $\Gamma_n = \Gamma^0_n$. □

Thus each equivalence relation $\equiv_n$ partitions $\mathbf{Str}_{\mathscr{S}}$ into finitely many subclasses. Moreover, each of those subclasses is a basic closed set in the first order topology:

Corollary. *If $\mathscr{S}$ is finite, then for all n in ω, there is a finite set Σ_n of sentences of $\mathscr{S}$ such that, for all structures $\mathfrak{A}$ of $\mathscr{S}$, there is $\sigma_n(\mathfrak{A})$ in Σ_n such that for all structures $\mathfrak{B}$ of $\mathscr{S}$,*

$$\mathfrak{A} \equiv_n \mathfrak{B} \iff \mathfrak{B} \vDash \sigma_n(\mathfrak{A}).$$

Proof. Let $\sigma_n(\mathfrak{A})$ be

$$\bigwedge\{\sigma \in \Gamma_n \colon \mathfrak{A} \vDash \sigma\} \wedge \bigwedge\{\neg\sigma \colon \sigma \in \Gamma_n \ \ \& \ \ \mathfrak{A} \nvDash \sigma\}.$$ □

Theorem 6 (Lindström)**.** *In every abstract logic of $\mathscr{S}$ that admits the Löwenheim–Skolem and Compactness Theorems, every sentence is logically equivalent to a first order sentence.*

Proof. Suppose σ is a sentence of such a logic, and $\mathscr{S}$ consists of the symbols in σ, so $\mathscr{S}$ is finite. We shall show that, for some n in ω, for all $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathbf{Str}_{\mathscr{S}}$,

$$\mathfrak{A} \equiv_n \mathfrak{B} \implies (\mathfrak{A} \vDash \sigma \iff \mathfrak{B} \vDash \sigma). \tag{‡‡}$$

If this does hold, then $\mathbf{Mod}(\sigma)$ is $\bigcup_{\tau \in X} \mathbf{Mod}(\tau)$ for some subset X of Σ_n, namely

$$\{\sigma_n(\mathfrak{A}) \colon \mathfrak{A} \in \mathbf{Mod}(\sigma)\};$$

this means $\sigma \sim \bigvee X$. Suppose however (‡‡) fails for all n. Then for all n there are $\mathfrak{A}_n$ and $\mathfrak{B}_n$ such that

$$\mathfrak{A}_n \equiv_n \mathfrak{B}_n, \qquad \mathfrak{A}_n \vDash \sigma, \qquad \mathfrak{B}_n \vDash \neg\sigma.$$

We define a sequence $(\sigma_n : n \in \omega)$ recursively so that

- each σ_n is in Σ_n;
- $\{k \in \omega : \mathfrak{A}_k \vDash \sigma_n\}$ is infinite;
- $\mathbf{Mod}(\sigma_{n+1}) \subseteq \mathbf{Mod}(\sigma_n)$.

By Compactness (and the Relativization property of $\mathrm{Sn}^*_{\mathscr{S}}$), there is a structure $(\mathfrak{A}, \mathfrak{B})$ such that $\mathfrak{A} \vDash \sigma$ and $\mathfrak{B} \vDash \neg\sigma$, but both $\mathfrak{A}$ and $\mathfrak{B}$ are models of the σ_n. By Theorem 4, the abstract logic cannot admit the Löwenheim–Skolem Theorem. $\square$

References

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