The Compactness Theorem

Abstract of a course to be given at the 5th World Congress and School on Universal Logic June 20–30, 2015, Istanbul

David Pierce

September 18, 2014
Mimar Sinan Fine Arts University
Istanbul
http://mat.msgsu.edu.tr/~dpierce/

Let us say that a logic has the *compactness property* if a set of sentences of the logic has a model whenever every finite subset of the set has a model. For present purposes, **the Compactness Theorem** is that first-order logic has the compactness property. This theorem is fundamental to model theory. However, Hodges's comprehensive twelve-chapter 1993 volume *Model Theory* finds no need to state and prove the theorem until Chapter 6. It is worthwhile to think about what needs Compactness and what does not.

One consequence of the Compactness Theorem is that a set of sentences with arbitrarily large finite models must have an infinite model. A more purely mathematical consequence is the **Prime Ideal Theorem:** *every nontrivial commutative ring has a prime ideal.* One can prove this by noting first that every *maximal* ideal is prime. Moreover, every *countable* ring has a maximal ideal; for we can obtain a

generating set of such an ideal by considering the elements of the ring one by one. In particular then, every finitely generated subring of a given ring has a maximal ideal, because every finitely generated ring is countable. By the Compactness Theorem then, the original ring must have an ideal that is at least prime, although it might not be maximal. The point here is that primeness is a "local" property, while maximality is not.

It is usually understood that every nontrivial commutative ring has, not just a prime ideal, but a maximal ideal. To make it easy to prove such results, Zorn stated in 1935 the result now known by his name. However, Zorn's Lemma relies on the Axiom of Choice. The Compactness Theorem is strictly weaker than this, with respect to ZF (Zermelo–Fraenkel set theory without Choice). For, Compactness is also a *consequence* of the Prime Ideal Theorem, even the Boolean Prime Ideal Theorem; and *this* is strictly weaker than the Axiom of Choice (as shown by Halpern and Lévy in 1971).

The Compactness Theorem for *countable* sets of sentences needs nothing beyond ZF. Skolem showed this implicitly in 1922 when he established the paradox that Zermelo's axioms for set theory must have a countable model, if they have a model at all. In 1930, Gödel proved countable Compactness explicitly, though not by that name. Mal′tsev stated the full Compactness Theorem as the General Local Theorem in 1941, having proved it implicitly in 1936; he used it to prove algebraic results in the way we proved the Prime Ideal Theorem above.

In his 1950 address to the International Congress of Mathematicians, Tarski gave the Compactness Theorem its current name and noted its topological meaning. But this meaning is not generally well expressed in today's textbooks of model theory.

The class of structures having a given signature can be given a topology, although the closed "sets" in this topology are not sets, but proper classes (except for the empty set): they are the classes of models of sets of sentences. The space of all structures has a Kolmogorov (T_0) quotient that is a set: it is the space of complete theories of structures. If one replaces sentences with their logical equivalence classes, then the

set of sentences becomes a Boolean algebra, called a Lindenbaum algebra; and the complete theories of structures become ultrafilters of the Lindenbaum algebra. By means of the Boolean Prime Ideal Theorem, the Stone space consisting of *all* ultrafilters of the Lindenbaum algebra is easily shown to be compact. Or one could look instead at the spectrum, consisting of the prime ideals of the corresponding Boolean ring; the spectrum of every ring is compact. The Compactness Theorem says more: every ultrafilter of the Lindenbaum algebra is derived from the complete theory of a structure.

The compactness theorem for *propositional* logic can be seen as a version of the theorem that the product of two-element discrete spaces (or indeed any compact Hausdorff spaces) is compact. The Compactness Theorem for first-order logic does not follow so readily, though it can be seen to result from a kind of reduction of first-order logic to propositional logic. Then Lindström's Theorem is roughly that there is no such reduction for certain more expressive logics—but see that tutorial for more. Sometimes the Compactness Theorem is derived from the Completeness Theorem: see *that* tutorial for more. Meanwhile, the present tutorial is intended to fill out the foregoing sketch of the Compactness Theorem as such.

References

[1] John W. Dawson, Jr. The compactness of first-order logic: from Gödel to Lindström. *Hist. Philos. Logic*, 14(1):15–37, 1993.

[2] Kurt Gödel. The completeness of the axioms of the functional calculus of logic. In van Heijenoort [8], pages 582–91. First published 1930.

[3] J. D. Halpern and A. Lévy. The Boolean prime ideal theorem does not imply the axiom of choice. In *Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los An-*

geles, Calif., 1967), pages 83–134. Amer. Math. Soc., Providence, R.I., 1971.

[4] Wilfrid Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[5] Anatoliĭ Ivanovič Mal′cev. *The metamathematics of algebraic systems. Collected papers: 1936–1967*. North-Holland Publishing Co., Amsterdam-London, 1971. Translated, edited, and provided with supplementary notes by Benjamin Franklin Wells, III, Studies in Logic and the Foundations of Mathematics, Vol. 66.

[6] Thoralf Skolem. Some remarks on axiomatized set theory. In van Heijenoort [8], pages 290–301. First published 1923.

[7] Alfred Tarski. Some notions and methods on the borderline of algebra and metamathematics. In *Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1*, pages 705–720. Amer. Math. Soc., Providence, R. I., 1952.

[8] Jean van Heijenoort, editor. *From Frege to Gödel: A source book in mathematical logic, 1879–1931*. Harvard University Press, Cambridge, MA, 2002.

[9] Ernst Zermelo. Investigations in the foundations of set theory I. In van Heijenoort [8], pages 199–215. First published 1908.

[10] Max Zorn. A remark on method in transfinite algebra. *Bull. Amer. Math. Soc.*, 41(10):667–670, 1935.