Chains of structures and of theories

David Pierce

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Mathematics Department
Mimar Sinan Fine Arts University
Istanbul
dpierce@msgsu.edu.tr
http://mat.msgsu.edu.tr/~dpierce/

These are my notes for a 50-minute talk at Sabancı University on the date above. I did not discuss the theorem of Łoś and Tarski on page 5, or the proof of my theorem on page 6, or the details of §5. (Today's date is April 4, 2013.)

Abstract

The union of a chain of fields is a field. The union of a chain of vector-spaces with their scalar-fields is still a vector-space, but it may have strictly lower dimension than the spaces in the chain. A model-theoretic result of the 1950s called the Chang–Los-Suszko Theorem relates these observations to the logical form of the theories of the structures in the chains.

Instead of looking at chains of models of a fixed theory, one may fruitfully look at chains of theories themselves. Such a

chain might consist of the theories of fields equipped with finite numbers of commuting derivations; or of the theories of vector-spaces with predicates for linear dependence of finite numbers of vectors. I shall discuss some results concerning these and other examples.

Contents

1 Chains of structures

Given a chain

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$$

of fields, we know that the union

$$\bigcup_{n\in\omega} K_n$$

is also a field. Likewise for ordered fields, or groups, or vector spaces (given with their scalar fields).

However, in the last case, *dimension* need not be preserved in the union. Indeed, suppose a field-extension L/K has transcendence-basis $(a_1, a_2, a_3, \dots)$. Fixing n in ω, we let

$$K_0 = K, \qquad K_1 = K(a_1), \qquad K_2 = K(a_1, a_2),$$

and in general, for each j in ω,

$$K_j = K(a_1, \dots, a_j);$$

we also let

$$\begin{aligned} V_0 &= \operatorname{span}_K(1, a_1, \dots, a_n), \\ V_1 &= \operatorname{span}_{K_1}(1, a_1, \dots, a_{n+1}) = \operatorname{span}_{K_1}(1, a_2, \dots, a_{n+1}), \end{aligned}$$

and in general, for each j in ω,

$$V_j = \operatorname{span}_{K_j}(1, a_1, \dots, a_{n+j}) = \operatorname{span}_{K_j}(1, a_{j+1}, \dots, a_{n+j})$$

Then V_j is a vector-space over K_j, and

$$\begin{gathered} \dim_{K_j}(V_j) = n + 1, \\ (V_0, K_0) \subseteq (V_1, K_1) \subseteq (V_2, K_2) \subseteq \cdots, \\ L = \bigcup_{j \in \omega} K_j, \\ \dim_L\left(\bigcup_{j \in \omega} V_j\right) = 1. \end{gathered}$$

2 Logic of chains of structures

A **field** is just a **model** of the **theory** of fields in the **signature**

$$\{0, 1, -, +, \cdot\}.$$

All but one of the **axioms** of this theory are **universal,** for example

$$\forall(x, y, z)\ x(yz) = (xy)z.$$

The remaining axiom is **universal-existential,** or $\forall\exists$:

$$\forall x\ \exists y\ (x = 0 \lor xy = 1).$$

The axioms for vector-spaces are no more complex; but the axiom requiring dimension at least 2 *is* more complex, or at least differently complex; it is $\exists\forall$:

$$\exists(\boldsymbol{u}, \boldsymbol{v})\ \forall(x, y)\ (\boldsymbol{u} \cdot x + \boldsymbol{v} \cdot y = \mathbf{0} \Rightarrow x = 0 \wedge y = 0).$$

This is why dimension need not be preserved in unions:

Theorem (Chang [1959], Łoś & Suszko [1957]). *Unions of chains of models of a theory are always models too, if and only if the theory can be axiomatized by $\forall\exists$ sentences.*

But consider fields now in the signature

$$\{0, -, +, \cdot\},$$

without a symbol for 1. An embedding of *rings* in this signature need not preserve 1. For example, the field $\mathbb{Q}$ embeds in the product ring $\mathbb{Q} \times \mathbb{Q}$ under $x \mapsto (x, 0)$; but $(1, 0)$ is not the 1 of $\mathbb{Q} \times \mathbb{Q}$ (it is $(1, 1)$).

However, an embedding of rings that happen to be fields must preserve 1. Therefore, in the signature $\{0, -, +, \cdot\}$, the union of a chain of fields is still a field. Now, the axiom saying that there *is* a 1 would seem to take the form

$$\exists x\ \forall y\ (xy = y).$$

However, this complexity is not required, because of the Chang–Łoś-Suszko Theorem. The axioms for integral domains are universal, the most complex being

$$\forall(x, y)\ (xy = 0 \Rightarrow x = 0 \vee y = 0).$$

Replacing the axiom $\forall x\ 1 \cdot x = x$ with the $\forall\exists$ sentence

$$\forall(x, y)\ \exists z\ xzy = y$$

results in axioms for the theory of fields.

By the way, another **preservation theorem** is:

Theorem (Łoś [1955], Tarski [1954]). *Substructures of models of a theory are always models too, if and only if the theory can be axiomatized by universal sentences.*

So fields cannot be given universal axioms in the usual signature $\{0, 1, -, +, \cdot\}$, since the substructures of fields (in this signature) are just the integral domains, and not every integral domain is a field. For example,

$$(\mathbb{Z}, 0, 1, -, +, \cdot\,) \subseteq (\mathbb{Q}, 0, 1, -, +, \cdot\,).$$

Similarly in the signature $\{1, \cdot\}$, groups cannot be given universal axioms, since for example

$$(\mathbb{N}, 1, \cdot\,) \subseteq (\mathbb{Z}, 1, \cdot\,),$$

and the former is not a group.

3 Chains of theories of vector-spaces

Also by the Chang–Łoś-Suszko Theorem, the axioms for vector-spaces of dimension at least two *cannot* be simplified—unless we enlarge the signature, as by including the **predicate** $\parallel$ for parallelism. This will be defined by the axiom

$$\forall(\boldsymbol{u}, \boldsymbol{v}) \Big(\boldsymbol{u} \parallel \boldsymbol{v} \Leftrightarrow \exists(x, y) \big(\boldsymbol{u} \cdot x + \boldsymbol{v} \cdot v = \mathbf{0} \land \neg(x = 0 \land y = 0)\big)\Big),$$

which has the form

$$\forall \boldsymbol{x}\ (\varphi \Leftrightarrow \exists \boldsymbol{y}\ \theta),$$

which is equivalent to the $\forall\exists$ sentences

$$\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ (\varphi \Rightarrow \theta), \qquad \forall(\boldsymbol{x}, \boldsymbol{y})\ (\theta \Rightarrow \varphi).$$

Then having dimension at least 2 is given by the $\forall\exists$ axiom

$$\exists(\boldsymbol{u}, \boldsymbol{v})\ \boldsymbol{u} \parallel \boldsymbol{v}.$$

In the larger signature, every vector-space embeds in a space of dimension exactly 2. Indeed, given L/K with $[L:K] \geqslant 3$, we may suppose $(1, a, b)$ in L^3 is linearly independent over K. Then the vector-space (K^3, K) embeds in (L^2, L) under

$$(t, x, y) \mapsto (x - at, y - bt).$$

Every embedding of vector-spaces preserves parallelism. The present embedding preserves *non*-parallelism: this is a special case of:

Theorem (P. [2009]). *If $K \subseteq L$, and $(1, a_1, \dots, a_n)$ from L^{n+1} is linearly independent over K, then the embedding*

$$(t, x_1, \dots, x_n) \mapsto (x_1 - a_1 t, \dots, x_n - a_n t)$$

of (K^{n+1}, K) in (L^n, L) preserves n-ary linear independence.

Proof. Consider $(a_1, \dots, a_n)$ as a row-vector $\boldsymbol{a}$. Then we can write the given embedding as

$$\left(\begin{array}{c|c} t & \boldsymbol{x} \end{array}\right) \mapsto \boldsymbol{x} - t \cdot \boldsymbol{a},$$

or—if the $n \times n$ identity matrix is I_n—as

$$\left(\begin{array}{c|c} t & \boldsymbol{x} \end{array}\right) \mapsto \left(\begin{array}{c|c} t & \boldsymbol{x} \end{array}\right) \cdot \left(\begin{array}{c} -\boldsymbol{a} \\ \hline I_n \end{array}\right).$$

This embedding takes the rows of an $n \times (n+1)$ matrix $\left(\begin{array}{c|c} \boldsymbol{t} & X \end{array}\right)$ over K to the rows of the $n \times n$ matrix

$$X - \boldsymbol{t} \cdot \boldsymbol{a}.$$

Moreover

$$\begin{aligned} \det(X - \boldsymbol{t} \cdot \boldsymbol{a}) &= \det \left(\begin{array}{c|c} 1 & \boldsymbol{0} \\ \hline \boldsymbol{t} & X - \boldsymbol{t} \cdot \boldsymbol{a} \end{array}\right) \\ &= \det \left(\left(\begin{array}{c|c} 1 & \boldsymbol{a} \\ \hline \boldsymbol{t} & X \end{array}\right) \left(\begin{array}{c|c} 1 & -\boldsymbol{a} \\ \hline \boldsymbol{0} & I_n \end{array}\right) \right) \\ &= \det \left(\begin{array}{c|c} 1 & \boldsymbol{a} \\ \hline \boldsymbol{t} & X \end{array}\right), \end{aligned}$$

so that

$$\det(X - \boldsymbol{t} \cdot \boldsymbol{a}) \neq 0 \iff \det \left(\begin{array}{c|c} 1 & \boldsymbol{a} \\ \hline \boldsymbol{t} & X \end{array} \right) \neq 0$$
$$\implies \operatorname{rank}\left(\begin{array}{c|c} \boldsymbol{t} & X \end{array} \right) = n;$$

the converse holds too since the entries in X and $\boldsymbol{t}$ are from K. $\square$

Now, if $1 \leqslant n < \omega$, let

- VS_n be the theory of vector-spaces with predicates for k-ary linear dependence when $2 \leqslant k \leqslant n$;
- ${\mathrm{VS}_n}^*$ be axiomatized by VS_n, along with
  - the space is n-dimensional,
  - the scalar-field is algebraically closed.

In addition, let

- $\mathrm{VS}_\omega = \bigcup_{1 \leqslant n < \omega} \mathrm{VS}_n$,
- ${\mathrm{VS}_\omega}^*$ be axiomatized by VS_ω, along with
  - the space is infinite-dimensional,
  - the scalar-field is algebraically closed.

Note then

$${\mathrm{VS}_\omega}^* \neq \bigcup_{1 \leqslant n < \omega} {\mathrm{VS}_n}^*$$

(the latter is inconsistent). However:

Theorem (P. [2009]). *If $1 \leqslant n \leqslant \omega$, the models of ${\mathrm{VS}_n}^*$ are precisely the **existentially closed** models of VS_n.*

The existentially closed models of a theory T are just those models $\mathfrak{M}$ such that every quantifier-free formula over $\mathfrak{M}$ soluble in some extension (which is also a model of T) is already soluble in $\mathfrak{M}$ itself.

The existentially closed fields are the algebraically closed fields.

By the next-to-last theorem, for every model of VS_n, every equation

$$\boldsymbol{a}_0 \cdot x_0 + \cdots + \boldsymbol{a}_n \cdot x_n = 0$$

over the model (*i.e.* with the $\boldsymbol{a}_i$ belonging to the model) has a solution in some extension.

In general, if T and T^* are two theories, in the same signature, such that

1) T has $\forall\exists$ axioms,
2) the models of T^* are precisely the existentially closed models of T,

then T^* is the **model-companion** of T.

So each $\mathrm{VS}_n{}^*$ is the model-companion of VS_n (if $1 \leqslant n \leqslant \omega$), and ACF is the model-companion of the theory of fields.

4 Chains of theories of differential fields

But model-companions need not exist. For example, let m-DF be the theory of fields equipped with m commuting **derivations** ∂_0, …, ∂_{m-1}, so that

$$\begin{gathered}\partial_i(x+y) = \partial_i x + \partial_i y,\\ \partial_i(xy) = x \cdot \partial_i y + y \cdot \partial_i x,\end{gathered}$$

and let

$$\omega\text{-DF} = \bigcup_{m \in \omega} m\text{-DF}.$$

If we require also that the fields have characteristic 0, the theories become m-DF_0 and ω-DF_0.

Theorem (A. Robinson [1959]). *The theory* 1-DF_0 *has a model-companion,* 1-DCF_0*, the theory of* ***differentially closed fields*** *of characteristic* 0*.*

Theorem (McGrail [2000]). *For each m in ω, the theory m-DF_0 has a model-companion, m-DCF_0.*

Theorem (P. [2013?]). *For each m in ω, the theory m-DF has a model-companion, m-DCF.*

Theorem (Kasal & P. [2013?]).

1. *The theory ω-DF has no model-companion.*
2. *The theory ω-DF_0 has a model-companion, which is*

$$\bigcup_{m\in\omega} m\text{-}\mathrm{DCF}_0.$$

Proof. 1. For each j in ω, the theory ω-DF has an existentially closed model such that

$$\mathbb{F}_p(\alpha) \subseteq K_j, \qquad \alpha \notin \mathbb{F}_p{}^{\mathrm{alg}}, \qquad \partial_i\alpha = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then α cannot have a pth root (since derivatives of pth powers are 0). In a **non-principal ultraproduct** of the K_j, we have $\partial_i\alpha = 0$ for all i in ω; but α still has no pth root; so the ultraproduct is not existentially closed as a model of ω-DF.

2. It is enough to show m-$\mathrm{DCF}_0 \subseteq (m+1)$-$\mathrm{DCF}_0$, so that the theory $\bigcup_{m\in\omega} m$-DCF_0 is consistent. This is by a general result noted also by Medvedev (2013?). If

$$K \models (m+1)\text{-}\mathrm{DF}_0, \qquad K \subseteq L, \qquad L \models m\text{-}\mathrm{DF}_0,$$

it is enough to find M so that

$$M \models (m+1)\text{-}\mathrm{DF}_0, \qquad L \subseteq M, \qquad K \subseteq M.$$

This can be done… □

5 Chains of theories

Suppose

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots ,$$

each T_k being a theory with signature $\mathscr{S}_k$, so that

$$\mathscr{S}_0 \subseteq \mathscr{S}_1 \subseteq \mathscr{S}_2 \subseteq \cdots$$

Medvedev notes that properties of the T_k that are preserved in $\bigcup_{k\in\omega} T_k$ include:

1. **completeness** (containing either σ or $\neg\sigma$, for all sentences σ of the signature)
2. **consistency** (having a model),
3. **model-completeness** (being one's own model-companion),
4. **stability.**

Not preserved are

5. **companionability** (having a model-companion),
6. **ω-stability,**
7. **superstability.**

1. *Completeness* is preserved, because sentences have finite length, so that every sentence of $\bigcup_{k\in\omega} \sigma_k$ is a sentence of some σ_k.

2. That *consistency* is preserved is precisely the Compactness Theorem of first-order logic. This fails in second-order logic. For example, let DP (for Dedekind and Peano) be the *second-order* theory of $(\mathbb{N}, 1, +)$. Add a new constant c to the signature, and let DP_k be axiomatized by

$$\mathrm{DP} \cup \{c \neq 1, c \neq 1+1, \dots, c \neq \underbrace{1+\cdots+1}_{k}\}.$$

Then $\bigcup_{k\in\omega} \mathrm{DP}_k$ has no model.

3. *Model-completeness* is preserved, because (by means of Compactness) it is equivalent to every formula's being equivalent (*modulo* the theory in question) to an existential formula.

4. *Stability* is a possible property of complete theories. *In*stability of T is equivalent to the presence of a formula $\varphi(\boldsymbol{x}, \boldsymbol{y})$ defining an infinite linear order in some model of T, so that, for all n in ω,

$$T \vdash \exists(\boldsymbol{x}_0, \dots, \boldsymbol{x}_n) \left(\bigwedge_{0 \leqslant i \leqslant j \leqslant n} \varphi(\boldsymbol{x}_i, \boldsymbol{x}_j) \land \bigwedge_{0 \leqslant j < i \leqslant n} \neg\varphi(\boldsymbol{x}_i, \boldsymbol{x}_j) \right).$$

If $T = \bigcup_{k \in \omega} T_k$, then these sentences are all in some $\mathscr{S}_k$, and then (assuming T_k is complete) T_k will be instable.

5. We have already seen that ω-DF is not *companionable,* although it is the union of the companionable theories m-DF.

6. Fix a complete theory T in a countable signature $\mathscr{S}$. For each model $\mathfrak{M}$ of T, for each set A of parameters from $\mathfrak{M}$, we let

- $\mathrm{LT}(A)$ be the Boolean algebra, called a **Lindenbaum–Tarski algebra,** of formulas in $\mathscr{S}$ with parameters from A, considered *modulo* (equivalence in) T;
- $\mathrm{S}(A)$ be the Stone space of $\mathrm{LT}(A)$ (*i.e.* the set of maximal ideals, or equivalently of ultrafilters).

If κ is an infinite cardinal, and for all $\mathfrak{M}$ and A as above,

$$|A| \leqslant \kappa \implies |\mathrm{S}(A)| \leqslant \kappa,$$

then T is **κ-stable.** For example, the theory ACF of algebraically closed fields is κ-stable for all κ, since, if $K \models \mathrm{ACF}$, there is a continuous bijection from $\mathrm{S}(K)$ to the spectrum of $K[X]$.

In fact ω-stability implies κ-stability for all κ.

McGrail shows that each m-DCF_0 is complete and ω-stable. However, for each set A of *differential* constants in a model of ω-DCF_0, for each element σ of A^ω, the subset

$$\{\partial_k x = \sigma(k) \colon k \in \omega\}$$

of $\mathrm{LT}(A)$ belongs to a different element of $\mathrm{S}(A)$, so that the latter has size $|A|^\omega$.

7. This shows ω-DCF_0 is not even *superstable,* that is, not always κ-stable when $\kappa \geqslant 2^\omega$, that is, $\kappa \geqslant \beth_1$. For, ω-DCF_0 is not $\beth_\omega$-stable, since ${\beth_\omega}^\omega > \beth_\omega$.

In fact, being stable is equivalent to being κ-stable for some κ.