Descartes as model theorist

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Istanbul Model Theory Seminar, May 16, 2013
Edited September 20, 2013

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Abstract

In his *Geometry* of 1637, Rene Descartes gave a geometric justification of algebraic manipulations of symbols. He did this by interpreting a field in a vector-space with a notion of parallelism. At least this is how we might describe it today. I alluded to this in the abstract for my February 28 seminar, but did not actually talk about it. Now I want to talk about it.

By fixing a unit, Descartes defines the product of two line segments as another segment. He relies on a theory of proportion for this. Presumably this is the theory developed in Book V of Euclid's *Elements*—the theory that inspired Dedekind's definition of real numbers as "cuts" of rational numbers.

But this theory has an "Archimedean" assumption: for any two given segments, some multiple of the smaller exceeds the larger.

In fact this assumption is not needed, as Hilbert observed in Foundations of Geometry. Hilbert uses instead Pappus's Theorem. This work may be known now as “interpreting a field in a projective plane”.

I tracked down Pappus's original argument (from the 4th century), and [on May 13] I wrote an account of it on Wikipedia [article *Pappus's hexagon theorem,* section “Origins”].

As for model theory, another result that comes out of these considerations is that there are model-complete theories of Lie-rings equipped with an endomorphism of the abelian-group-structure.

Contents

1 Geometry

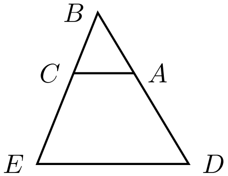
Fixing a line segment in the Euclidean plane as a unit, Descartes [2] defines multiplication of segments.

Thus he justifies algebra by interpreting it in geometry.

Hilbert will go the other way, using algebra to produce models of his geometric axioms.

Descartes needs Proposition VI.2 of Euclid's *Elements* [3], that a line parallel to the base of a triangle divides the sides proportionally, as in Figure 1. Descartes himself uses single minuscule letters for line segments. He uses the reverse of our $\propto$, instead of $=$. Strictly we should probably consider these minuscule letters as *lengths* of line

Figure 1 Descartes's definition of multiplication

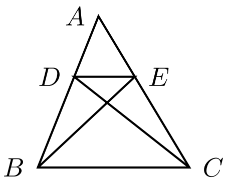


$$AB = 1,$$
$$BD = a,$$
$$BC = b,$$
$$BE = ab.$$

segments; and the **length** of a segment should be understood as the class of all segments that are *congruent* to it. (In Euclid, equality *means* congruence; *sameness* is a different notion.)

The proof of VI.2 uses the auxiliary lines in Figure 2, and VI.1,

Figure 2 Euclid's Proposition VI.2



$$\begin{aligned} \triangle BDE &= \triangle CDE, \\ BD : DA &:: \triangle BDE : \triangle ADE \\ &:: \triangle CDE : \triangle ADE \\ &:: CE : EA. \end{aligned}$$

that triangles (and parallelograms) with the same height are to one another as their bases.

This follows easily from the definition of proportion in Book V of the *Elements*. This definition uses an "Archimedean" assumption: for any two magnitudes of the same kind (as line segments, or areas), some multiple of the smaller exceeds the larger. If A, B, C, and D are magnitudes, A and B being of the same kind, and likewise C and D, then

$$A : B :: C : D$$

means for all positive integers k and m,

$$\begin{aligned} kA > mB &\iff kC > mD, \\ kA = mB &\iff kC = mD, \\ kA < mB &\iff kC < mD. \end{aligned}$$

We then might understand

$$(A : B) = \{m/k\colon kA < mB\}.$$

Thus a ratio corresponds to a *Dedekind cut* of positive rational numbers.

Dedekind [1] does not say his definition (discovered November 24, 1858) of the real numbers is inspired by Euclid. But apparently he read Euclid in school [8, p. 47].

Dedekind does not show explicitly that the real numbers defined by him satisfy the field axioms; but he says it can be done. His idea seems to be this: the operations of $+$ and $\times$ are *continuous* in each coordinate, and therefore every equation, like

$$(x + y) \cdot z = x \cdot z + y \cdot z,$$

that is satisfied by all rationals is satisfied by all reals as well.

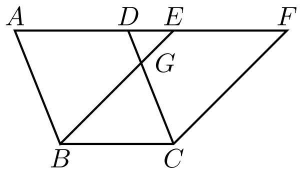
The works that I know—Landau [5, Thm 144, p. 55] and Spivak [9, pp. 563–4]—do not take this approach, but work directly with the definitions of $+$ and $\times$ as cuts.

Descartes does not recognize a need to prove associativity and commutativity of his multiplication.

Note that addition is "obviously" commutative and associative, since equality of parts implies equality of the whole. Consider for example Euclid's I.35 in Figure 3, that parallelograms on the same base and in the same parallels are equal, because they can be divided into parts that are respectively equal.

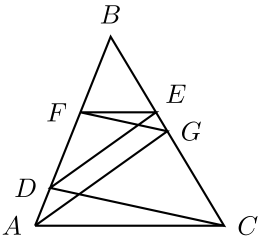
We can prove commutativity of multiplication using Figure 4. Here

Figure 3 Euclid's Proposition I.35



$$\begin{aligned} ABE &= DCF, \\ ABGD &= GCFE, \\ ABCD &= ABGD + GBC \\ &= GCFE + GBC \\ &= EBCF. \end{aligned}$$

Figure 4 Pappus's Theorem (parallel case)



$$\begin{gathered} DC \parallel FG \;\;\&\;\; DE \parallel AG \\ \implies AC \parallel FE. \end{gathered}$$

let am

$$BF = 1, \qquad BE = 1, \qquad BD = a, \qquad BG = b.$$

Assume

$$DC \parallel FG, \qquad DE \parallel AG.$$

Then $BA = ab$ and $BC = ba$. These are equal, provided $AC \parallel FE$. To prove this, we may note that

$$\begin{aligned} BF : BD &:: BG : BC, \\ BD : BA &:: BE : BG, \end{aligned}$$

and therefore, by Euclid's Proposition V.23, *ex aequali,*

$$BF : BA :: BE : BC.$$

The proof of V.23 does *not* use commutativity of multiplication of integers, but uses the Archimedean property. In fact it uses a bit more than this: of two unequal magnitudes of the same kind, their *difference* is also of the same kind. The argument can be made as follows where capital letters are now magnitudes.

V.8. Suppose $A > B$, and C is of the same kind as these. Then for some k we have $k(A - B) > C$, and so for some m we have

$$kA > mC > kB,$$

and thus $A : C > B : C$ (hence also $C : A < C : B$).

V.14. If $A : B :: C : D$ and $A > C$, then

$$C : D :: A : B > C : B,$$

so $B > D$.

V.21. Suppose

$$\begin{aligned} A : B &:: E : F, \\ B : C &:: D : E. \end{aligned}$$

If $A > C$, then $E : F :: A : B > C : B :: E : D$, so $D > F$.

V.23. Same supposition as V.21. Then for all k and m,

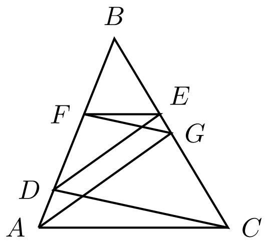
$$\begin{aligned} kA : kB &:: mE : mF, \\ kB : mC &:: kD : mE, \end{aligned}$$

and so $kA > mC \implies kD > mF$. Thus

$$A : C :: D : F.$$

For associativity, in the same Figure 4, repeated now as Figure 5, suppose

Figure 5 Pappus's Theorem (parallel case) again



$$DC \parallel FG \ \& \ DE \parallel AG \implies AC \parallel FE.$$

$$BF = 1, \quad BD = ab, \quad BA = ac, \qquad BE = b, \quad BG = c.$$

Then $DE \parallel AG$. Also

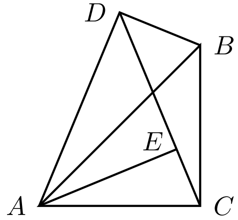
$$\begin{aligned} DC \parallel FG &\implies BC = c(ab), \\ AC \parallel FE &\implies BC = b(ac). \end{aligned}$$

The theorem we have used is that if the vertices of a hexagon lie alternately on two straight lines, and two pairs of opposite sides are parallel, then so is the third pair. More generally, the intersection points of the pairs of opposite sides lie on the same straight line—in our case, this is the "line at infinity". Pappus proved the finite case [6, VII.138–9].

Pascal's Theorem is the case where the vertices of the hexagon lie on a conic section. It is enough to prove the case of a circle, since non-degenerate conic sections are projections of a circle.

One can prove Pappus's Theorem without using proportions (or the Archimedian property in any way). See Hilbert's *Foundations of Geometry* [4, §14, pp. 24–29], where the theorem is named for Pascal. Hilbert argues as follows. In Figure 6, the angles ACB and ADB are right, so the points $ABCD$ lie on a circle, and therefore the angles

Figure 6 Hilbert's lemma



$$\angle ACB = \angle ADB = \text{right}$$
$$\implies \angle BAD = \angle CAE$$

ABD and ACD are equal, so their complements BAD and CAE are equal. Considering now how AE is the result of two projections, in two different ways, from AB, we can write this as

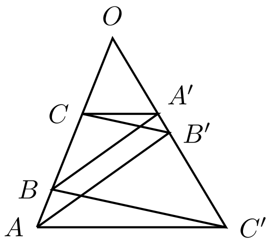
$$c \cos \alpha \cos \beta = c \cos \beta \cos \alpha,$$

where $c = AB$, and $\alpha = \angle BAC$, and $\beta = \angle CAE$. Hilbert writes the conclusion as

$$\beta\alpha c = \alpha\beta c;$$

here αc just means the length of AC. Now apply this to Figure 7, which is just Figures 4 and 5 relettered. We assume $CB' \parallel BC'$

Figure 7 Hilbert's proof



and $CA' \parallel AC'$. Perpendiculars dropped from O to CB', CA', and BA' make angles λ', μ', and ν' with OA, and angles λ, μ, and ν with OC', respectively. Then with distances from O lettered in the obvious way, we have (in Hilbert's notation, as above)

$$\lambda b' = \lambda' c, \qquad \mu a' = \mu' c, \qquad \nu a' = \nu' b,$$
$$\lambda' b = \lambda c', \qquad \mu' a = \mu c',$$

and therefore, since we can permute the angles,[1] we apply these equations in order to get

$$\nu'\mu'\lambda a = \nu'\mu\lambda c' = \nu'\mu\lambda' b = \nu\mu\lambda' a' = \nu\mu'\lambda' c = \nu\mu'\lambda b',$$
$$\nu' a = \nu b',$$

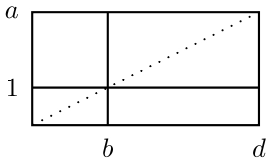
and therefore $BA' \parallel AB'$.

Hilbert gives also another argument, from an unnamed source. He then develops an "algebra of segments", more or less along the lines of Descartes. In short, he interprets a field.

There is an alternative approach to interpreting a field, using only Book I of the *Elements.* Fix a unit segment. By Propositions I.42 and I.44, in effect, every rectangle is equal to a rectangle with unit side. The other side of this rectangle can then be defined as the product of the two sides of the first rectangle. This multiplication is immediately commutative, as well as distributive over addition.

More precisely, multiplication is effected as in Figure 8, where points

Figure 8 Multiplication



$$d = ab$$

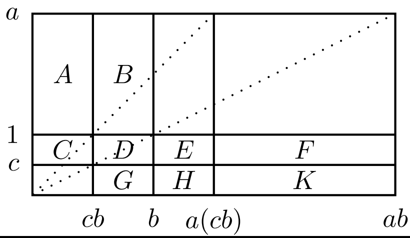
are labelled with their distances from the lower left vertex. Then

[1] We proved commutativity. Associativity is automatic, since the angles represent functions; but Hilbert does not seem to say this explicitly.

(by I.43 and its converse) $d = ab$ if and only if the diagonal passes through the intersection of the vertical and horizontal lines.

Associativity can be established by means of Figure 9. Again points

Figure 9 Associativity of multiplication



are labelled with their distances from the lower left vertex. The longer diagonal gives us both cb and ab. Then the shorter diagonal then gives us $a(cb)$. This is equal to $c(ab)$, provided

$$C + D + E = K.$$

The longer diagonal gives us

$$A + B = E + F + H + K,$$
$$B = E + F,$$

and therefore

$$A = H + K.$$

The shorter diagonal gives us

$$A = D + E + G + H,$$

and therefore

$$D + E + G = K.$$

We finish by noting (from the longer diagonal)

$$C = G.$$

Therefore $c(ab) = a(cb)$. We have assumed $c < 1 < a$ and $b < a(cb)$. Strictly we should consider four more cases:

(1) $c < 1 < a$, but $a(cb) = b$;

(2) $c < 1 < a$, but $a(cb) < b$;

(3) $c < a < 1$; and

(4) $1 < c < a$.

2 Vector spaces

Descartes's idea does let us interpret the scalar field in a vector space (of dimension at least two) by means of *parallelism.* This is worked out in my paper [7]. Given two parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we define $[\boldsymbol{a} : \boldsymbol{b}]$ as the class of pairs $(\boldsymbol{c}, \boldsymbol{d})$ of parallel vectors such that

$$\boldsymbol{a} - \boldsymbol{c} \parallel \boldsymbol{b} - \boldsymbol{d}$$

—assuming $\boldsymbol{a} \nparallel \boldsymbol{c}$; otherwise we must be able to find a third pair of parallel vectors with the same relation to the first two pairs. Then the field is the set of these classes $[\boldsymbol{a} : \boldsymbol{b}]$, where $\boldsymbol{b} \neq \mathbf{0}$. Equality and inequality of these, and addition and multiplication of these, are defined by *existential* formulas. Hence we obtain an equivalence of the categories of:

1) vector spaces with scalar field as a separate sort,
2) vector spaces with scalar field only as interpreted above,

where in each case the morphisms are merely *embeddings,* not just elementary embeddings.

3 Lie rings

Suppose K is a field. Let $\mathrm{Der}(K)$ be the set of derivations of K. Then this is both

1. a vector space over K, and
2. a *Lie ring,* the multiplication being the Lie bracket, $(X, Y) \mapsto [X, Y]$, where
$$[X, Y] = X \circ Y - Y \circ X.$$

For example,

$$K = \mathbb{Q}(x^0, \dots, x^{m-1}), \qquad \operatorname{Der}(K) = \langle \partial_0, \dots, \partial_{m-1} \rangle_K,$$

where $\partial_i = \partial/\partial x^i$. Suppose V is both a subspace and sub-ring of $\operatorname{Der}(K)$. Then (K, V) is a **Lie–Rinehart pair.**

Since V is a vector space over K, we may suppose $K \subseteq \operatorname{End}(V, +)$. In particular, we have

$$(f, D) \mapsto fD \colon K \times V \to V.$$

Since $V \subseteq \operatorname{Der}(K)$, we have

$$(D, f) \mapsto Df \colon V \times K \to K.$$

Two compatibility conditions are satisfied. First, if $f, g \in K$ and $D \in V$, then

$$(fD)g = f(Dg).$$

Thus the expression

$$fDg$$

is unambiguous. Next, if $f, g \in K$ and $D, E \in V$, then

$$\begin{aligned}[D, fE]g &= D(fEg) - fE(Dg) \\ &= (Df)(Eg) + fD(Eg) - fE(Dg) \\ &= ((Df)E)g + f[D, E]g,\end{aligned}$$

so

$$[D, fE] = (Df)E + f[D, E].$$

Suppose $D \in V$ and $t \in K$ and $Dt \neq 0$. For every f in K, we have

$$\Big(\frac{f}{Dt} D\Big) t = f.$$

Thus

$$K = \{Dt \colon D \in V\}.$$

Let b denote the Lie bracket operation. I propose to call the structure $(V, +, -, 0, \mathsf{b}, t)$ a **Lie ring of vectors.** The class of these is elementary. For, first of all, there are axioms as follows:

1. $(V, +, -, 0)$ is an abelian group.
2. b makes this a lie ring: b distributes over $+$, and

$$X \mathbin{\mathsf{b}} X = 0, \quad X \mathbin{\mathsf{b}} (Y \mathbin{\mathsf{b}} Z) + Y \mathbin{\mathsf{b}} (Z \mathbin{\mathsf{b}} X) + Z \mathbin{\mathsf{b}} (X \mathbin{\mathsf{b}} Y) = 0.$$

3. t is an endomorphism of the group:

$$t(X + Y) = tX + tY.$$

Next, rearranging the second compatibility condition above, we obtain

$$(Df)E = D \mathbin{\mathsf{b}} (fE) - t(D \mathbin{\mathsf{b}} E).$$

If f is replaced by t, then the right hand side is a term in our signature. We then take the left hand side as an abbreviation of this. By the axioms so far, each operation $X \mapsto (Dt)X$ or Dt is an endomorphism of $(V, +)$. Let

$$K = \{Xt\colon X \in V\}.$$

Then this is a group under

$$Xt + Yt = (X + Y)t.$$

The map $X \mapsto Xt$ is a group homomorphism from K to $\mathrm{End}(V, +)$. We want it to be a *ring monomorphism.* So the axioms say further:

4. The action is faithful:

$$XtY = 0 \to Y = 0 \lor XtZ = 0.$$

5. K is closed under multiplication:

$$\exists W \ (Xt)((Yt)Z) = (Wt)Z.$$

   (Here the outer universal quantifiers are suppressed.) Then multiplication is associative and distributes over addition, by what we already have; so K is an associative ring. Expressions like

$$(Xt)(Yt)Z$$

   are now unambiguous.

6. K is commutative:
$$(Xt)(Yt)Z = (Yt)(Xt)Z.$$
7. K has inverses:
$$\exists Z\ (Zt)(Xt)Y = Y.$$
In particular, K contains 1, which is different from 0, since the action is faithful.

We also need K to be closed under the action of V. Again by the second compatibility condition, we have

$$(DFt)E = D \mathsf{b} (FtE) - (Ft)(D \mathsf{b} E),$$

the right hand being a term of the signature; we use the left as an abbreviation. So we now require:

8. K is closed under $x \mapsto Dx$, for all D in V:
$$\exists W\ (XYt)Z = (Wt)Z.$$
9. The first compatibility condition holds:
$$(((Xt)Y)(Zt))W = ((Xt)(YZt))W.$$

This is it. We have not shown that V acts on K as a Lie ring of derivations, but this is automatic from the definition of the action, since K is now established as a sub-ring of $\mathrm{End}(V, +)$.

If $0 < m < \omega$, let LV^m be the theory of m-dimensional Lie rings of vectors. Let $(V, +, \mathsf{b}, t)$ be a model, with scalar field, K. Then V has a basis of *commuting* derivations $\partial_0, \dots, \partial_{m-1}$ of K, so

$$(K, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}.$$

This structure has a one-dimensional interpretation in $(V, +, \mathsf{b}, t)$ with coordinate map $X \mapsto Xt$ (from V to K). To show this, we need, for certain formulas ϕ of the signature $\{+, \cdot, \partial_0, \dots, \partial_{m-1}\}$, formulas ϕ^* of the signature $\{+, \mathsf{b}, t\}$ such that

$$(V, +, t) \models \phi^*(X, \dots) \iff (K, +, \cdot, \partial_0, \dots, \partial_{m-1}) \models \phi(Xt, \dots).$$

These are as follows.

ϕ	ϕ^*
$x = y$	$(Xt)\partial_0 = (Yt)\partial_0$
$x + y = z$	$(Xt)\partial_0 + (Yt)\partial_0 = (Zt)\partial_0$
$x \cdot y = z$	$(Xt)(Yt)\partial_0 = (Zt)\partial_0$
$\partial_i x = y$	$(\partial_i Xt)\partial_0 = (Yt)\partial_0$
$x \neq y$	$(Xt)\partial_0 \neq (Yt)\partial_0$

The existence of all but the last ϕ^* ensure the interpretation. That *all* of the ϕ^* are quantifier-free (existential would be enough) ensures that, if $(V, +, \mathsf{b}, t)$ is existentially closed, then so is $(K, +, \cdot, \partial_i \colon i < m)$.

To prove this, we need only note that, if

$$(K, +, \cdot, \partial_0, \dots, \partial_{m-1}) \subseteq (L, +, \cdot, \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1}),$$

then, letting

$$\tilde{V} = \langle \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1} \rangle_L,$$

and letting $\tilde{\mathsf{b}}$ be the Lie bracket, we have an embedding $\partial_i \mapsto \tilde{\partial}_i$ of $(V, +, \mathsf{b}, t)$ in $(\tilde{V}, +, \tilde{\mathsf{b}}, t)$.

Similarly, there is an m-dimensional interpretation of $(V, +, \mathsf{b}, t)$ in $(K, +, \cdot, \partial_0, \dots, \partial_{m-1})$, with coordinate map

$$(x^0, \dots, x^{m-1}) \mapsto \sum_{i<m} x^i \partial_i.$$

As before, if $(K, +, \cdot, \partial_0, \dots, \partial_{m-1})$ is existentially closed, so must $(V, +, \mathsf{b}, t)$ be.

The theory LV^m has a model companion, whose axioms say that if $\{\partial_0, \dots, \partial_{m-1}\}$ is a commuting spanning set—that is, if

$$\partial_i a^j = \delta_i^j$$

for all i and j in m, for some $(a^j : j < m)$ in K^m—then the structure $(K, +, \cdot, \partial_i \colon i < m)$ is a model of m-DCF.

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