Chains of theories

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This is work with Özcan Kasal. There is some parallel work by Alice Medvedev (presented in Oléron, June, 2011) concerning ACFA.

Suppose

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots ,$$

all theories, closed under entailment, so their signatures also form a chain:

$$\mathscr{S}_0 \subseteq \mathscr{S}_1 \subseteq \mathscr{S}_2 \subseteq \cdots$$

In one example of interest, T_m is m-DF, the theory of fields with m commuting derivations $\partial_0, \dots, \partial_{m-1}$; their union is ω-DF.

In general, what properties are preserved in $\bigcup_{k\in\omega} T_k$? Compare:

Theorem (Chang, Łoś–Suszko)**.** *For a fixed theory T, the following are equivalent:*

1. *T is $\forall\exists$-axiomatizable.*
2. $\mathrm{Mod}(T)$ *is closed under taking unions of chains*

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots .$$

Again, if $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$, then among possible properties of the theories T_k,

1) Preserved by $\bigcup_{k\in\omega} T_k$ are: (a) consistency, (b) completeness, (c) quantifier elimination, (d) model-completeness, (e) stability, (f) ...;
2) not preserved (but this is not obvious) are: companionability, ω-stability, superstability, ...

(a) Consistency is preserved, by compactness.

(b) Completeness is preserved, because every sentence of the union $\bigcup_{k\in\omega} \mathscr{S}_k$ is a sentence of some $\mathscr{S}_k$.

(c) Likewise for quantifier-elimination.

(d) **Model-completeness** of a theory T may be usually remembered as

$$\mathfrak{A} \subseteq \mathfrak{B} \implies \mathfrak{A} \preccurlyeq \mathfrak{B}$$

within $\mathrm{Mod}(T)$. Equivalently (the theory axiomatized by)

$$T \cup \mathrm{diag}(\mathfrak{A})$$

is always complete when $\mathfrak{A} \models T$, where

$$\mathrm{diag}(\mathfrak{A}) = \{\sigma \in \mathrm{Th}(\mathfrak{A}_A) \colon \sigma \text{ is quanfifier-free}\}$$

the theory of the structures in which $\mathfrak{A}$ embeds. A sufficient (and obviously necessary) condition is (Abraham) **Robinson's Condition,**

$$\mathfrak{A} \subseteq \mathfrak{B} \implies \mathfrak{A} \preccurlyeq_1 \mathfrak{B}$$

where the conclusion means every quantifier-free formula over $\mathfrak{A}$ soluble in $\mathfrak{B}$ is soluble in $\mathfrak{A}$. Robinson's Condition is, equivalently,

$$\mathfrak{A} \models_{\mathrm{ec}} T$$

—every model of T is an **existentially closed** model. For this it is sufficient (and in fact necessary) that T admit quantifier-elimination down to $\exists$ formulas. Therefore model-completeness is preserved in unions of chains.

(e) A *complete* theory T is

- **κ-stable,** if $\kappa \geqslant |T|$ and
$$|A| \leqslant \kappa \implies |\mathrm{S}(A)| \leqslant \kappa$$
for all parameter-sets A of models of T;
- **superstable,** if κ-stable for κ large enough;
- **stable,** if κ-stable for some κ.

When $|T| = \omega$, then

- superstability implies κ-stability when $\kappa \geqslant 2^{\omega}$;
- stability implies κ-stability when $\kappa = \kappa^{\omega}$.

(Note that if $\operatorname{cof}(\kappa) = \omega$, as when $\kappa = \aleph_{\omega}$, then $\kappa < \kappa^{\omega}$.)

In fact *in*stability of T is equivalent to the presence of a formula $\varphi(\vec{x}, \vec{y})$ defining an infinite linear order in some model of T, so that, for all n in ω,

$$T \vdash \exists(\vec{x}_0, \dots, \vec{x}_n) \left(\bigwedge_{0 \leqslant i \leqslant j \leqslant n} \varphi(\vec{x}_i, \vec{x}_j) \land \bigwedge_{0 \leqslant j < i \leqslant n} \neg\varphi(\vec{x}_i, \vec{x}_j) \right).$$

If $T = \bigcup_{k \in \omega} T_k$, then these sentences are all in some $\mathscr{S}_k$, and then (assuming T_k is complete) T_k will be instable.

An arbitrary theory T is **companionable** if, for some theory T^* of its signature,

- $T_\forall = T^*{}_\forall$,
- T is model-complete.

In this case, T^* is the **model-companion** of T. If

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots,$$

and each T_k has the model-companion $T_k{}^*$, *and*

$$T_0{}^* \subseteq T_1{}^* \subseteq T_2{}^* \subseteq \cdots, \tag{$*$}$$

then $\bigcup_{k\in\omega} T_k{}^*$ is the model-companion of $\bigcup_{k\in\omega} T_k$. However $(*)$ may fail.

Theorem (McGrail)**.** *m-DF_0 (in characteristic 0) has a model-companion, m-DCF_0, which admits quantifier-elimination and is ω-stable.*

Theorem (P.)**.** *m-DF has a model-companion, m-DCF. Nevertheless, $\bigcup_{m\in\omega} m$-DF is not companionable.*

For the last part (non-companionability), if $j \in \omega$, let K_j be an e.c. (*existentially closed*) model of ω-DF (that is, $\bigcup_{m\in\omega} m$-DF), with

$$\mathbb{F}_p(\alpha) \subseteq K_j, \qquad \alpha \notin \mathbb{F}_p{}^{\mathrm{alg}}, \qquad \partial_i\alpha = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then α has no p-th root in K_j, since

$$\partial_j\alpha = 1, \qquad\qquad \partial_j(x^p) = p \cdot x^{p-1} \cdot \partial_j x = 0.$$

Therefore α has no p-th root in a nonprincipal ultraproduct

$$\prod_{j\in\omega} K_j/\mathfrak{p},$$

even though, in this, $\partial_i\alpha = 0$ for all i in ω, so α has a p-th root in some extension. Thus the ultraproduct is not e.c.. Therefore the class of e.c. models of ω-DF is not elementary.

Theorem (P.)**.**

$$m\text{-DCF}_0 \subseteq (m+1)\text{-DCF}_0,$$

and therefore ω-DF_0 *has a model-companion,* ω-DCF_0*, which is stable, but not superstable.*

This is established by means of:

Theorem (folklore, P.)**.** *Assuming* $T_0 \subseteq T_1$*, each* T_k *having signature* $\mathscr{S}_k$*, consider:*

A. *If*

$$\mathfrak{A} \models T_1, \qquad \mathfrak{B} \models T_0, \qquad \mathfrak{A} \upharpoonright \mathscr{S}_0 \subseteq \mathfrak{B},$$

then there is $\mathfrak{C}$ *such that*

$$\mathfrak{C} \models T_1, \qquad \mathfrak{A} \subseteq \mathfrak{C}, \qquad \mathfrak{B} \subseteq \mathfrak{C} \upharpoonright \mathscr{S}_0.$$

B. *For all* $\mathfrak{A}$*,*

$$\mathfrak{A} \models_{\mathrm{ec}} T_1 \implies \mathfrak{A} \upharpoonright \mathscr{S}_0 \models_{\mathrm{ec}} T_0.$$

C. T_0 *has the* Amalgamation Property: *if one model embeds in two others, then those two in turn embed in a fourth model, compatibly with the original embeddings.*

D. T_1 *is* $\forall\exists$ *(so that every model embeds in an e.c. model).*

We have the two implications

$$A \implies B, \qquad B \;\&\; C \;\&\; D \implies A,$$

but there is no implication among the four conditions that does not follow from these. This is true, even if T_1 *is required to be a* conservative *extension of* T_0*, so that* $T_1 \upharpoonright \mathscr{S}_0 = T_0$*.*

Proof. (Can be left as exercise.) Suppose A holds. Let

$$\mathfrak{A} \models_{\mathrm{ec}} T_1, \qquad \mathfrak{B} \models T_0, \qquad \mathfrak{A} \upharpoonright \mathscr{S}_0 \subseteq \mathfrak{B}.$$

We show

$$\mathfrak{A} \restriction \mathscr{S}_0 \preccurlyeq_1 \mathfrak{B}$$

(*i.e.* every existential formula over $\mathfrak{A} \restriction \mathscr{S}_0$ soluble in $\mathfrak{B}$ is soluble in $\mathfrak{A} \restriction \mathscr{S}_0$). By hypothesis, there is a model $\mathfrak{C}$ of T_1 such that

$$\mathfrak{A} \subseteq \mathfrak{C}, \qquad \mathfrak{B} \subseteq \mathfrak{C} \restriction \mathscr{S}_0.$$

Then

$$\mathfrak{A} \preccurlyeq_1 \mathfrak{C},$$
$$\mathfrak{A} \restriction \mathscr{S}_0 \preccurlyeq_1 \mathfrak{C} \restriction \mathscr{S}_0,$$
$$\mathfrak{A} \restriction \mathscr{S}_0 \preccurlyeq_1 \mathfrak{B}.$$

Therefore $\mathfrak{A} \restriction \mathscr{S}_0$ must be an e.c. model of T_0. Thus B holds.

Suppose conversely $B \ \& \ C \ \& \ D$ holds. Let

$$\mathfrak{A} \models T_1, \qquad \mathfrak{B} \models T_0, \qquad \mathfrak{A} \restriction \mathscr{S}_0 \subseteq \mathfrak{B}.$$

We establish the consistency of

$$T_1 \cup \operatorname{diag}(\mathfrak{A}) \cup \operatorname{diag}(\mathfrak{B}).$$

It is enough to show the consistency of

$$T_1 \cup \operatorname{diag}(\mathfrak{A}) \cup \{\exists \vec{x}\ \varphi(\vec{x})\},$$

where φ is an arbitrary quantifier-free formula of $\mathscr{S}_0(A)$ such that

$$\mathfrak{B} \models \exists \vec{x}\ \varphi(\vec{x}).$$

By D, there is $\mathfrak{C}$ such that

$$\mathfrak{C} \models_{\mathrm{ec}} T_1, \qquad \mathfrak{A} \subseteq \mathfrak{C}.$$

By B then,

$$\mathfrak{C} \restriction \mathscr{S}_0 \models_{\mathrm{ec}} T_0, \qquad \mathfrak{A} \restriction \mathscr{S}_0 \subseteq \mathfrak{C} \restriction \mathscr{S}_0.$$

By C, both $\mathfrak{B}$ and $\mathfrak{C} \upharpoonright \mathscr{S}_0$ embed over $\mathfrak{A} \upharpoonright \mathscr{S}_0$ in a model $\mathfrak{D}$ of T_0. In particular,

$$\mathfrak{D} \models \exists \vec{x}\, \varphi(\vec{x}).$$

Therefore φ is already soluble in $\mathfrak{C} \upharpoonright \mathscr{S}_0$ itself. Thus

$$\mathfrak{C} \models T_1 \cup \operatorname{diag}(\mathfrak{A}) \cup \{\exists \vec{x}\, \varphi(\vec{x})\}.$$

Therefore A holds.

For the rest, 11 (counter-)examples are found... $\square$

Now suppose

$$\begin{gathered}(L, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}_0,\\ K \subseteq L,\\ (K, \partial_0 \upharpoonright K, \dots, \partial_{m-1} \upharpoonright K, \partial_m) \models (m+1)\text{-DF}_0,\\ a \in L \smallsetminus K.\end{gathered}$$

We shall define a differential field

$$(K\langle a\rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_m),$$

where $a \in K\langle a\rangle$, and for each i in m,

$$\tilde{\partial}_i \upharpoonright K\langle a\rangle \cap L = \partial_i \upharpoonright K\langle a\rangle \cap L, \tag{†}$$

and $\tilde{\partial}_m \upharpoonright K = \partial_m$.

Considering ω^{m+1} as the set of $(m+1)$-tuples of natural numbers, we shall have

$$K\langle a\rangle = K(a^\sigma \colon \sigma \in \omega^{m+1}),$$

where

$$a^\sigma = \tilde{\partial}_0{}^{\sigma(0)} \cdots \tilde{\partial}_m{}^{\sigma(m)} a. \tag{‡}$$

In particular then, by (†), we must have

$$\sigma(m) = 0 \implies a^\sigma = \partial_0{}^{\sigma(0)} \cdots \partial_{m-1}{}^{\sigma(m-1)} a. \tag{§}$$

Using this rule, we make the definition

$$K_1 = K(a^\sigma \colon \sigma(m) = 0).$$

Recursively, we can define

$$K_j = K(a^\sigma \colon \sigma(m) < j)$$

as desired. If $L \smallsetminus K\langle a\rangle \neq \varnothing$, we can repeat, as necessary. It may not be possible to make L itself closed under $\tilde{\partial}_m$.