Model-theory of differential fields

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After some generalities about model-theory, I give a specific result about differential fields: for every natural number, the theory of fields with that number of commuting derivations has a model-companion. This is so because, if a system of partial differential equations is given, there is a way to tell in finitely many steps whether the system is soluble, and moreover the number of steps is independent of the parameters of the system.

1 Model-theory

I consider model-theory to be the study of *structures* as *models* of *theories.* This definition has three terms that need explanation. For the moment, a *theory* is just a set of sentences of a formal logic. If each of those sentences is true in a structure, that structure is a *model* of the theory. The logic is usually first-order, and will always be so here; this means variables stand for individuals, never for sets as such, and moreover sentences are finite: only finitely many variables appear in a given sentence, and all conjunctions and disjunctions are finitary.

A *structure* then consists of one or more sets, together with various relations and operations on those sets, along with distinguished individual elements of those sets. None of these additional features is actually required to be present: a bare set is a structure. But groups, rings, ordered fields, and vector-spaces are also structures. One non-example is a topological space, considered as a set Ω with the closure operation $X \mapsto \overline{X}$: the problem here is that X ranges, not over elements of Ω, but over sets of elements of Ω. However, certain topological spaces are essential to model theory: these are the Stone spaces of Boolean algebras of equivalent formulas in a given number of free variables.

One reason to use first-order logic is that it has a compactness theorem: if every finite subset of a theory has a model, then so does the whole theory. This might be seen <u>this</u> as a restriction: it implies for example that there is no theory whose models are precisely the torsion groups. But compactness can also be used to ground non-standard analysis, making rigorous the intuitive approach of Newton and Leibniz to calculus.

A structure has a *signature,* namely a set of symbols for the distinguished relations, operations, and individuals of the structure. Then a sentence σ can be true or false in a structure $\mathfrak{M}$ only if the non-logical symbols in σ come from the signature of $\mathfrak{M}$. A set Γ of sentences *entails* a sentence σ, and σ is a *logical consequence* of Γ, if σ is true in every model of Γ. Now we can say that Γ is a *theory* if (and only if) it contains all of its logical consequences. In any case, if T is the set of logical consequences of Γ, then Γ is a set of *axioms* for T. The theory T is *complete* if, for every sentence σ of its signature, T entails either σ or its negation.

Every structure has a theory, namely the set of sentences that are true in the structure. This theory is automatically complete. However, by Gödel's Incompleteness Theorem of 1931, there is no method for writing down a set of axioms for the theory of $\mathbb{N}$ in the signature $(\mathbb{N}, +, \times)$. Nonetheless, by slightly earlier work of Tarski's student Presburger, there *is* such a method in the smaller signature $\{+\}$. In a word, the structure $(\mathbb{N}, +)$ is *'tame'*, but $(\mathbb{N}, +, \times)$ is not tame. An early theme of model-theory is just the identification of tame structures [5]. Further examples of such structures are $(\mathbb{R}, +, \times, \leqslant)$, $(\mathbb{C}, +, \times)$, and $(\mathbb{Q}, \leqslant)$.

By the compactness theorem, every theory with infinite models has infinitely many non-isomorphic models. The theory ACF_0 of $(\mathbb{C}, +, \times)$ has countably many non-isomorphic countable models, but just one model of each uncountable cardinality. The theory LO^* of $(\mathbb{Q}, \leqslant)$ has one countable model, but 2^κ models in each uncountable cardinality κ. An ongoing task of model-theory is to understand the combinatorial properties that effect distinctions between the classes of models of theories such as the one just described. For example, LO^* is *unstable,* because of the ordering; no ordering can be defined in models ACF_0, so this theory is *stable*; in fact it is ω-stable, because the Stone spaces of its formulas in countably many parameters are themselves countable.

Abraham Robinson investigated a variant of completeness that he called *model completeness.* To define this, we first define the *diagram* of a structure: this is the set of *quantifier-free* sentences, with parameters from the structure, that are true in the structure. For example, the diagram of $(\mathbb{N}, +, \times)$ is generated by the usual addition and multiplication tables learned at school. Then a theory T is *model-complete* if the set $T \cup \mathrm{diag}(\mathfrak{M})$ axiomatizes a complete theory whenever $\mathfrak{M}$ is a model of T. For example, the theory ACF of algebraically closed fields is model-complete. The theory ACF is not itself complete, but it becomes complete when an axiom specifying the characteristic of a model is added. Indeed, the new theory is complete because it is model-complete and it has a model that embeds in all other models.

Because then $\mathrm{ACF} \cup \mathrm{diag}(K)$ is complete for every field K, the theory ACF is called the *model-completion* of the theory of fields. A slightly more general notion is that of *model-companion*: a theory T^* is a model-companion of T if every model of one of the theories embeds in a model of the other, and moreover T^* is model-complete. If LO is the theory of linear orders, then LO^* as defined above is indeed its model-companion. The theory of groups has no model-companion; neither does the theory of fields with a distinguished algebraically closed subfield.

Suppose T is a theory such that the union of an increasing chain of models is itself a model; equivalently, like most theories studied in algebra, T is axiomatized by $\forall\exists$ sentences. The *existentially closed* models of T are those models $\mathfrak{M}$ such that, for every quantifier-free formula $\varphi(\mathbf{x})$ in the signature of $\mathfrak{M}$ with parameters, if $\varphi(\mathbf{x})$ has a solution in some extension of $\mathfrak{M}$ that is a model of T, then $\varphi(\mathbf{x})$ already has a solution in $\mathfrak{M}$ itself. If (and only if) there is a theory whose models are precisely the existentially closed models of T, this theory is the model-companion of T [1]. The theory of fields with a distinguished algebraically closed subfield has no model-companion, because the existentially closed models of this theory are the algebraically closed fields *of transcendence-degree one* over an algebraically closed subfield, and transcendence-degree can be given no first-order characterization.

2 Differential fields

High-school algebra and calculus combine in *differential fields:* fields equipped with one or more *derivations,* namely operations D with the algebraic properties of 'taking the derivative': $D(x+y) = Dx + Dy$, and $D(x{\cdot}y) = Dx{\cdot}y + x{\cdot}Dy$. Let DF be the theory of fields with a single derivation. A required characteristic can be indicated by a subscript. Using an elimination result of Seidenberg, Robinson found a model-companion of DF_0, but its axioms were not illuminating. The model-companion of field-theory needs only axioms saying that every non-constant polynomial *in one variable* has a root. Lenore Blum showed that a similar result was possible for DF_0. Meanwhile Carol Wood found a model-companion for DF_p when p is positive. Combining these results yields the following:

Theorem (Robinson, Blum, Wood)**.** *A model* (K, D) *of* DF *is existentially closed if and only if each of the following conditions holds.*

1. *K is separably closed.*
2. *(K, D) is* differentially perfect *(in positive characteristic p, if $Dx = 0$, then x has a pth root).*
3. *For all ordinary polynomials f and g over K in $(x^0, \dots, x^{n+1})$ and $(x^0, \dots, x^n)$ respectively such that $\partial f/\partial x^{n+1} \neq 0$ and $g \neq 0$, the formula*

$$f(x, Dx, \dots, D^{n+1}x) = 0 \wedge g(x, Dx, \dots, D^n x) \neq 0$$

has a solution in K.

Hence DF *has a model-companion.*

Following Robinson, we may call this model-companion DCF. Its *completions*—the complete theories that include it—are obtained by specifying a characteristic. The completion DCF_0 is ω-stable; when p is positive, DCF_p is not ω-stable, but is stable [6].

An alternative way to simplify the axioms of DCF is to consider, not systems in one variable, but first-order systems—first-order, not in the sense of logic, but in the sense of having only single applications of the derivation. Then the models of DCF can be described geometrically [4, 2]:

Theorem. *A differential field (K, D) is existentially closed if and only if each of the following conditions holds.*

1. *K is separably closed.*
2. *(K, D) is differentially perfect.*
3. *For every variety V over K, if there are rational maps φ and ψ from V to $\mathbb{A}^n$ for some n, where φ is dominant and separable, then V has a K-rational point P such that φ and ψ are regular at P, and $D \circ \varphi(P) = \psi(P)$.*

In the last condition, it is sufficient to assume $n = \dim V$.

3 Several derivations

Given a positive integer m, we may let m-DF be the theory of fields with m commuting derivations. So in this theory, equations are so-called partial differential equations. As usual, a required characteristic can be given by a subscript. The existence of a model-companion of m-DF_0—call it m-DCF_0—was established by Tracey McGrail; alternative characterizations and generalizations (still in characteristic 0) were given by Yaffe and by Tressl. However, as with Robinson's original account of DCF_0, none of these descriptions of m-DCF_0 is perspicuous.

It appears that neither of the methods described above for simplifying the axioms of DCF is useful for m-DCF. Nonetheless, we have the theorem below (in which no characteristic is specified) [3]. Notation is as follows.

- ω is the set of von-Neumann natural numbers, so that an element n is the set $\{0, \dots, n-1\}$.

- If $\xi \in \omega^m$, that is $\xi = (\xi(0), \dots, \xi(m-1))$, then

$$|\xi| = \xi(0) + \dots + \xi(m-1), \qquad \partial^\xi = {\partial_0}^{\xi(0)} \cdots {\partial_{m-1}}^{\xi(m-1)}.$$

- If n is a positive integer, then $\trianglelefteqslant$ is the total ordering of $\omega^m \times n$ that is taken from the left lexicographic ordering of ω^{m+1} by means of the embedding

$$(\xi, k) \longmapsto (|\xi|, k, \xi(0), \dots, \xi(m-2))$$

of $\omega^m \times n$ in ω^{m+1}.
- If $(\sigma, k) \in \omega^m \times n$, and D is a derivation of a field K, and $\mathbf{x} = (x_h^\xi \colon (\xi, h) \trianglelefteqslant (\sigma, k))$, and $f \in K(\mathbf{x})$, then f has a derivative Df, which is the linear function over $K(\mathbf{x})$ in new variables y_h^ξ given by

$$Df = \sum_{(\xi,h) \trianglelefteqslant (\sigma,k)} \frac{\partial f}{\partial x_h^\xi} \cdot y_h^\xi + f^D,$$

where $f \mapsto f^D$ is the derivation of $K(\mathbf{x})$ that extends D and takes each x_h^ξ to 0.
- If $i < m$, then $\mathbf{i} = (e(0), \dots, e(m-1))$, where $e(j) = 0$ if $j \neq i$, and $e(i) = 1$.
- $\leqslant$ is the product ordering (a partial ordering) of ω^m.

Suppose now $(K, \partial_0, \dots, \partial_{m-1})$ is a model of m-DF, and n and r are positive integers. Let us say that an extension $K(a_h^\xi \colon |\xi| \leqslant 2r \wedge h < n)$ of K is *nice* if

1) for all f in $K(x_h^\xi \colon |\xi| < 2r \wedge h < n)$ such that $f(a_h^\xi \colon |\xi| < 2r \wedge h < n) = 0$, for each i in m,

$$\partial_i f(a_h^\xi, a_h^{\xi+\mathbf{i}} \colon |\xi| < 2r \wedge h < n) = 0;$$

2) for each k in n, each $<$-minimal element ρ of $\{\sigma \in \omega^m \colon a_k^\sigma \in K(a_h^\xi \colon (\xi, h) \vartriangleleft (\sigma, k))^{\mathrm{sep}}\}$ has $|\rho| \leqslant r$.

The first condition of niceness here is that each ∂_i extends to a derivation from $K(a_h^\xi \colon |\xi| < 2r \wedge h < n)$ to $K(a_h^\xi \colon |\xi| \leqslant 2r \wedge h < n)$ such that

$$\partial_i a_h^\xi = a_h^{\xi+\mathbf{i}}.$$

The second condition of niceness is that if both ρ and σ are $<$-minimal elements of the indicated set, then, under the extensions of the derivations ∂_i just described, a_k^ρ and a_k^σ have a common derivative a_k^τ, where $|\tau| \leqslant 2r$. This ensures that there will be no obstacle to extending the ∂_i indefinitely as commuting derivations:

Theorem. *A model $(K, \partial_0, \dots, \partial_{m-1})$ of m-DF is existentially closed if and only if, for all positive integers r and n, for every nice extension $K(a_h^\xi\colon |\xi| \leqslant 2r \wedge h < n)$ of K, for some tuple $(b_h\colon h < n)$ of elements of K, the tuple $(a_h^\xi\colon |\xi| < 2r \wedge h < n)$ has a specialization $(\partial^\xi b_h\colon |\xi| < 2r \wedge h < n)$.*

Every system of equations over $(K, \partial_0, \dots, \partial_{m-1})$ can be understood first as a system of equations of ordinary polynomials belonging to $K(x_h^\xi\colon |\xi| \leqslant r \wedge h < n)$ for some r and n. Suppose we formally differentiate these polynomials with respect to the ∂_i, using the rule $\partial_i x_h^\xi = x_h^{\xi+\mathbf{i}}$. We may introduce new variables x_h^ξ, as long as $|\xi| \leqslant 2r$. If no new algebraic condition on $(x_h^\xi\colon |\xi| \leqslant r \wedge h < n)$ is introduced in this way, then by the theorem, the original system of differential equations has a solution.

References

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