Differential fields

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Model-theory:

- is 'foundational', like category theory or set theory;
- is a kind of mathematics done self-consciously;
- pays attention to the language of mathematics;
- is a study of **structures** as **models** of **theories**;
- is a study of the relation of **truth** between structures and sentences.

The truth-relation is usually symbolized by

$$\models \tag{1}$$

(which is `\models` in LaTeX).

Examples of the truth-relation.

$$\mathbb{C} \vDash \exists x\ x^2 + 1 = 0, \tag{2}$$

but

$$\mathbb{R} \nvDash \exists x\ x^2 + 1 = 0, \tag{3a}$$

that is,

$$\mathbb{R} \vDash \neg\exists x\ x^2 + 1 = 0, \tag{3b}$$

or equivalently

$$\mathbb{R} \vDash \forall x\ x^2 + 1 \neq 0. \tag{3c}$$

Also,

$$\mathbb{Q} \vDash \forall x\ \forall y\ (x < y \to \exists z\ (x < z \wedge z < y)), \tag{4}$$

but

$$\mathbb{Z} \vDash \exists x\ \exists y\ (x < y \wedge \forall z\ (x \not< z \vee z \not< y)). \tag{5}$$

Let $\mathscr{S}$ be a **signature,** such as the signature $\{+, -, \cdot, 0, 1, \leqslant\}$ of ordered fields: it is a set of

- **operation-symbols** (or *function-symbols,* here $+$, $-$, and $\cdot$),
- **constants** (or *constant-symbols,* here 0 and 1),
- **predicates** (or *relation-symbols,* here $\leqslant$).

There is a class

$$\mathrm{Mod}(\mathscr{S}) \tag{6}$$

of all **structures** $\mathfrak{A}$ with signature $\mathscr{S}$. Then $\mathfrak{A}$ consists of:

- a set A, called the **universe** of the structure (or there may be several 'universes', as in a vector-space; these are **sorts**);
- an **interpretation** of $\mathscr{S}$: a function $s \mapsto s^{\mathfrak{A}}$ on $\mathscr{S}$ assigning to each operation-symbol in $\mathscr{S}$ an operation on A, and so on.

This is just a formalization of the idea that, for example, the same symbol $+$ denotes a different operation in different abelian groups.

Again $\mathscr{S}$ is a signature. There is a set

$$\mathrm{Sn}(\mathscr{S}) \tag{7}$$

of **sentences** of first-order logic in the signature $\mathscr{S}$. A sentence is a **formula** with no free variables. Symbols used in formulas of $\mathscr{S}$ are:

- the symbols in $\mathscr{S}$;
- **variables,** as x and y, standing for *individuals* (not sets as such);
- **Boolean connectives,** as $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$;
- **quantification symbols,** as $\exists$ and $\forall$;
- **brackets** (parentheses), as needed.

Then $\models$ is a relation between $\mathrm{Mod}(\mathscr{S})$ and $\mathrm{Sn}(\mathscr{S})$, namely

$$\{(\mathfrak{A}, \sigma) \in \mathrm{Mod}(\mathscr{S}) \times \mathrm{Sn}(\mathscr{S}) \colon \sigma \text{ is true in } \mathfrak{A}\}. \tag{8}$$

Non-examples. The following are *not* sentences of first-order logic:

- the formula

$$\exists y \; x \cdot y = 1 \tag{9}$$

  defining the group of units of a commutative ring (there is a free variable, x);

- the induction axiom for $\mathbb{N}$,

$$\forall X \; (1 \in X \wedge \forall y \; (y \in X \to y + 1 \in X) \to \forall y \; y \in X) \tag{10}$$

  (here X ranges over sets, not individuals);

- the axiom distinguishing *torsion* groups among abelian groups,

$$\forall x \; (x = 0 \vee 2x = 0 \vee 3x = 0 \vee \cdots) \tag{11}$$

  (sentences must be finite).

We restrict to first-order logic for **compactness:**

The relation $\vDash$ gives us a Galois correspondence, as do:

- $\{(x, \sigma) \in K \times \mathrm{Aut}(K) \colon x^\sigma = x\}$ in field-theory,
- $\{(f, \vec{a}) \in K[\vec{X}] \times K^n \colon f(\vec{a}) = 0\}$ in algebraic geometry.

If $\mathcal{K} \subseteq \mathrm{Mod}(\mathscr{S})$, we define the **theory** of $\mathcal{K}$:

$$\mathrm{Th}(\mathcal{K}) = \bigcap_{\mathfrak{A} \in \mathcal{K}} \{\sigma \in \mathrm{Sn}(\mathscr{S}) \colon \mathfrak{A} \vDash \sigma\}, \tag{12}$$

the set of sentences true in every structure in $\mathcal{K}$.

If $\Gamma \subseteq \mathrm{Sn}(\mathscr{S})$, we define the class of **models** of Γ:

$$\mathrm{Mod}(\Gamma) = \bigcap_{\sigma \in \Gamma} \{\mathfrak{A} \in \mathrm{Mod}(\mathscr{S}) \colon \mathfrak{A} \vDash \sigma\}. \tag{13}$$

Such a class is called **elementary.** So there is a one-to-one correspondence between theories and elementary classes.

The **Compactness Theorem** is that for all subsets Γ of $\mathrm{Sn}(\mathscr{S})$, if every finite subset of Γ has a model, then so does Γ itself.

If $\sigma \in \mathrm{Th}(\mathrm{Mod}(\Gamma))$, that is, Γ **entails** σ, we may write

$$\Gamma \vdash \sigma. \tag{14}$$

The **filter** F of $\mathrm{Sn}(\mathscr{S})$ generated by Γ is the smallest subset of $\mathrm{Sn}(\mathscr{S})$ that

- includes Γ,
- is closed under entailment,
- is closed under conjunction ($\wedge$).

Then

$$F \subseteq \mathrm{Th}(\mathrm{Mod}(\Gamma)). \tag{15}$$

The Compactness Theorem is

$$F = \mathrm{Th}(\mathrm{Mod}(\Gamma)). \tag{16}$$

The Compactness Theorem fails in second-order logic: The sentences

$$\forall X\,(1 \in X \wedge \forall y\,(y \in X \to y+1 \in X) \to \forall y\; y \in X), \tag{17a}$$

$$\forall x\; x+1 \neq 1, \tag{17b}$$

$$\forall x\,\forall y\,(x+1 = y+1 \to x = y), \tag{17c}$$

$$c \neq 1, \quad c \neq 1+1, \quad c \neq 1+1+1, \quad \cdots \tag{17d}$$

say there is an infinite natural number (denoted by c). They have no model, but every finite subset of them does.

Also by Compactness, the class of torsion groups is not elementary, since for every k in $\mathbb{N}$, the sentence

$$\exists x\,(x \neq 0 \wedge 2x \neq 0 \wedge \cdots \wedge kx \neq 0) \tag{18}$$

is true in the torsion group $\mathbb{Z}/(k+1)\mathbb{Z}$.

In the following table, $\mathcal{K}$ and $\mathcal{K}^*$ are subclasses of $\mathrm{Mod}(\mathscr{S})$:

$\mathscr{S}$	$\mathcal{K}$	$\mathcal{K}^*$
$\varnothing$	sets	infinite sets
$\{<\}$	linear orders	dense linear orders w/o endpoints
$\{+,-,0\}\cup\mathbb{F}_p$	vector-spaces/$\mathbb{F}_p$	infinite vector-spaces/$\mathbb{F}_p$
$\{+,-,\cdot,0,1\}$	fields	algebraically closed fields

In each case, $\mathcal{K}^*$ is the class of **existentially closed** members of $\mathcal{K}$. That is, if $\mathfrak{A}\in\mathcal{K}^*$, and $\mathfrak{B}\in\mathcal{K}$, and

$$\mathfrak{A}\subseteq\mathfrak{B}, \tag{19}$$

then every *existential* sentence true in $\mathfrak{B}$ is already true in $\mathfrak{A}$.

Two more examples:

1. $\mathscr{S} = \{+, -, \cdot, 0, 1, F\}$,
   $\mathcal{K}$ is the class of fields with algebraically closed subfield F, and
   $\mathcal{K}^*$ is the class of algebraically closed extensions of transcendence-degree 1 of an algebraically closed field F.
2. $\mathscr{S} = \{+, -, \mathbf{0}\} \cup \{+, -, \cdot, 0, 1\} \cup \{*\}$,
   $\mathcal{K}$ is the class of vector-spaces, and
   $\mathcal{K}^*$ is the class of vector-spaces of dimension 1 over an algebraically closed field.

As the ground field grows larger, dimension can only go down.

If $T = \mathrm{Th}(\mathrm{Mod}(\Gamma))$, then T is the theory **axiomatized** by Γ.

The classes of groups, rings, and fields are **closed under unions of chains.** Their theories are axiomatized by sets of $\forall\exists$ sentences (such as the field axiom $\forall x\ \exists y\ (x = 0 \vee xy = 1)$).

Theorem (Chang, Łoś–Suszko, 1950s)**.** Let T be a theory. TFAE:

1. $\mathrm{Mod}(T)$ is closed under unions of chains.
2. T is axiomatized by a set of $\forall\exists$ sentences.

In general,

$$T_{\forall\exists} = \mathrm{Th}(\mathrm{Mod}(\{\sigma \in T\colon \sigma \text{ is } \forall\exists\})). \tag{20}$$

The equivalent conditions of the theorem can be written as

$$T = T_{\forall\exists}. \tag{21}$$

In this case, the class of existentially closed models of T is nonempty. If it is elementary, then its theory is the **model-companion** of T.

The theory of fields with an algebraically closed subfield has no model-companion: having transcendence-degree 1 is not an 'elementary' or first-order property.

A stronger notion than model-companion is **model-completion.**

Theorem (P.*)**.** The theory of vector-spaces of dimension 1 over an algebraically closed field is:

1) the model-companion of the theory of vector-spaces,
2) not the model-completion of the theory of vector-spaces,
3) the model-completion of the theory of vector-spaces of dimension at most 1.

*D. Pierce, *Model-theory of vector-spaces over unspecified fields* (Arch. Math. Logic [2009] 48:421–436).

Technical details: A subset Γ of $\mathrm{Sn}(\mathscr{S})$ is **complete** if

$$\Gamma \nvdash \sigma \iff \Gamma \vdash \neg\sigma \tag{22}$$

for all σ in $\mathrm{Sn}(\mathscr{S})$, that is,

$$\mathrm{Th}(\mathrm{Mod}(\Gamma)) = \mathrm{Th}(\{\mathfrak{A}\}) = \mathrm{Th}(\mathfrak{A}) \tag{23}$$

for some $\mathfrak{A}$ in $\mathrm{Mod}(\mathscr{S})$. By

$$\mathfrak{A}_A \tag{24}$$

we mean $\mathfrak{A}$, considered as an element of $\mathscr{S} \cup A$. Then we define

$$\mathrm{diag}(\mathfrak{A}) = \{\sigma \in \mathrm{Th}(\mathfrak{A}_A) \colon \sigma \text{ is quantifier-free}\}. \tag{25}$$

Theorem. $\mathrm{Mod}(\mathrm{diag}(\mathfrak{A}))$ is the class of structures of $\mathscr{S} \cup A$ in which $\mathfrak{A}_A$ embeds.

If T is a theory of $\mathscr{S}$, then

$$T_\forall = \mathrm{Th}(\mathrm{Mod}(\{\sigma \in T\colon \sigma \text{ is universal}\})). \tag{26}$$

Theorem. The class of substructures of models of T is elementary, and its theory is $T_\forall$.

For example, if T is field-theory, then $T_\forall$ is the theory of integral domains.

Let T and T^* be theories of $\mathscr{S}$ such that $T_\forall = {T^*}_\forall$. Then T^*:

1) is the **model-companion** of T if $T \cup \mathrm{diag}(\mathfrak{A})$ is complete for all models $\mathfrak{A}$ of T^*;
2) is the **model-completion** of T if $T \cup \mathrm{diag}(\mathfrak{A})$ is complete for all models $\mathfrak{A}$ of T;
3) admits **quantifier-elimination** if $T \cup \mathrm{diag}(\mathfrak{A})$ is complete for all substructures $\mathfrak{A}$ of models of T^* (or T).

Theorem (Robinson, $\leqslant$1963)**.** Let T be a theory. TFAE:

- T has a model-completion.
- $T = T_{\forall\exists}$, and there is $\phi \mapsto \hat{\phi}$ on existential (or just primitive) formulas such that, if $\mathfrak{M} \vDash T$ and $\vec{a} \in M^n$.

$$\mathfrak{M} \vDash \hat{\phi}(\vec{a}) \iff \mathfrak{M} \subseteq \mathfrak{N} \vDash T \cup \{\phi(\vec{a})\} \text{ for some } \mathfrak{N}, \tag{27}$$

- the model-completion of T is axiomatized by

$$T \cup \{\forall \vec{x}\, (\hat{\phi}(\vec{x}) \to \phi(\vec{x})) \colon \phi \text{ existential}\}. \tag{28}$$

The immediate example is the theory DF of **differential fields:**

A **differential field** is a pair (K, D), where

- K is a field (for simplicity, of characteristic 0 here),
- $D\colon K \to K$, with

$$D(x+y) = Dx + Dy, \qquad D(xy) = Dx \cdot y + x \cdot Dy. \tag{29}$$

Example. $(\mathbb{C}(X), f \mapsto f')$.

Theorem (Seidenberg)**.** $\phi \mapsto \hat{\phi}$ as in Robinson's Theorem exists when T is DF.

Corollary (Robinson, $\leqslant$1963)**.** DF has a model-completion.

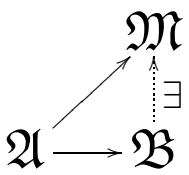
The model-completion of DF is called

$$\text{DCF}\,. \tag{30}$$

For more comprehensible axioms for DCF, one can use:

Theorem (Blum, $\leqslant$1977)**.** TFAE:

- T^* is the model-completion of T,
- If $\mathfrak{A}, \mathfrak{B} \vDash T$; $\mathfrak{M} \vDash T^*$; $\mathfrak{M}$ is $|B|^+$-saturated:



(31)

If, further, $T = T_\forall$, so substructures of models are models, then the embedding of $\mathfrak{A}$ in $\mathfrak{B}$ can be analyzed as

$$\mathfrak{A} \to \mathfrak{A}(a_1) \to \mathfrak{A}(a_1, a_2) \to \cdots \to \mathfrak{B} \tag{32}$$

where each structure is a model of T; so $\mathfrak{B} = \mathfrak{A}(a)$ suffices *(Blum's Criterion)*.

Since $\mathrm{DF}_\forall$ is 'close enough' to DF is universal, Blum gets nice axioms for DCF:

Theorem (Blum, $\leqslant$1977)**.** $(K, D) \in \mathrm{Mod}(\mathrm{DCF})$ if and only if:

- $(K, D) \in \mathrm{Mod}(\mathrm{DF})$,
- $K = K^{\mathrm{alg}}$,
- for all ordinary polynomials f and g over K, if $g \neq 0$ and $\partial_{n+1} f \neq 0$, then

$$(K, D) \vDash \exists x \, (f(x, Dx, \ldots, D^{n+1}x) = 0 \wedge \\ \wedge \, g(x, Dx, \ldots, D^{n}x) \neq 0). \quad (33)$$

There is an alternative approach to simplifying the axioms of DCF:

We want to understand when systems of differential polynomial equations and inequations have solutions in some extension.

We can eliminate inequalities by the usual trick:

$$x \neq 0 \iff \exists y\ xy = 1. \tag{34}$$

Over a model (K, D) of DF, TFAE:

$$\exists \vec{x} \bigwedge_f f(\vec{x}, D\vec{x}, \ldots, D^n\vec{x}) = 0, \tag{35}$$

$$\exists (\vec{x}_0, \ldots, \vec{x}_n)\ (\bigwedge_f f(\vec{x}_0, \ldots, \vec{x}_n) = 0 \wedge \bigwedge_{i<n} D\vec{x}_i = \vec{x}_{i+1}). \tag{36}$$

The latter is an instance of

$$\exists \vec{x}\ (\bigwedge_f f(\vec{x}) = 0 \wedge \bigwedge_{i<k} Dx_i = g_i(\vec{x})), \tag{37}$$

where $\vec{x} = (x_0, \ldots, x_{n-1})$ and $k \leqslant n$.

Theorem (P.–Pillay 1998, P. 2004)**.** $(K, D) \in \mathrm{Mod}(\mathrm{DCF})$ if and only if:

- $(K, D) \in \mathrm{Mod}(\mathrm{DF})$,
- $K = K^{\mathrm{alg}}$,
- for all f in $K[X_0, \ldots, X_{n-1}]$ and g_i in $K(\vec{X})$,

$$\exists \vec{x} \, (\bigwedge_f f(\vec{x}) = 0 \wedge \bigwedge_{i<k} Dx_i = g_i(\vec{x})), \tag{38}$$

  provided the f impose no algebraic condition on $(x_0, \ldots, x_{n-1})$.

In the last condition, it is enough that the f define an irreducible variety with generic point $\vec{a}$ such that $(a_0, \ldots, a_{k-1})$ is a transcendence-basis of $K(\vec{a})/K$.

How do these ideas work in case of several derivations?

Let DF^m be the theory of fields (of characteristic zero) with m commuting derivations.

Example. $(\mathbb{C}(X_0, \ldots, X_{m-1}), \partial_0, \ldots, \partial_{m-1})$.

So DF^m is the theory of $(m+1)$-tuples $(K, \partial_0, \ldots, \partial_{m-1})$, where

- $\partial_i \in \mathrm{Der}(K)$, that is, $(D, \partial_i) \in \mathrm{Mod}(\mathrm{DF})$;
- $[\partial_i, \partial_j] = 0$.

Theorem (McGrail, 2000)**.** DF^m has a model-completion, called DCF^m.

To prove this, McGrail uses Blum's Criterion (and so considers only systems in one variable).

Let

$$\omega = \{0, 1, 2, \ldots\}; \tag{39}$$

if $\sigma \in \omega^m$, this means

$$\sigma = (\sigma(0), \ldots, \sigma(m-1)). \tag{40}$$

For simplicity, we assume $m = 2$. If $\sigma \in \boldsymbol{\omega}^2$, we write

$$\partial^\sigma x = {\partial_0}^{\sigma(0)} {\partial_1}^{\sigma(1)} x. \tag{41}$$

Suppose

$$(K, \partial_0, \partial_1) \subseteq (L, \tilde{\partial}_0, \tilde{\partial}_1), \tag{42}$$

both being models of DF, and $a \in L$. We define

$$K\{a\} = K[\partial^\sigma a \colon \sigma \in \boldsymbol{\omega}^2], \tag{43}$$

the differential ring generated by a over K. In particular, there is a differential polynomial ring $K\{X\}$. Then we can define

$$I(a) = \{f \in K\{X\} \colon f(a) = 0\}; \tag{44}$$

this is a prime differential ideal. Kolchin identifies a kind of finite subset Λ of this ideal, called a **characteristic set** of the ideal, that determines the ideal in the following sense.

Again if $\sigma \in \omega^2$, let

$$|\sigma| = \sigma(0) + \sigma(1). \tag{45}$$

Since the characteristic set Λ of $I(a)$ is finite, for some n in ω,

$$\Lambda \subseteq K[\partial^\sigma X \colon |\sigma| \leqslant n]. \tag{46}$$

For each f in this ring there is $\tilde{f}$ in $K[X_\sigma \colon |\sigma| \leqslant n]$ such that

$$f = \tilde{f}(\partial^\sigma X \colon |\sigma| \leqslant n). \tag{47}$$

If $(b_\sigma \colon |\sigma| \leqslant n)$ is a generic zero of $\{\tilde{f} \colon f \in \Lambda\}$ from some extension of K, then there are b_σ for all additional σ in ω^2 such that we can extend the ∂_i to derivations $\tilde{\partial}_i$ on $K(b_\sigma \colon \sigma \in \omega^2)$ such that

$$\tilde{\partial}_0 b_\sigma = b_{\sigma+(1,0)}, \qquad \tilde{\partial}_1 b_\sigma = b_{\sigma+(0,1)}, \qquad I(b_{(0,0)}) = I(a). \tag{48}$$

This is the sense in which Λ determines $I(a)$; but Kolchin gives an algebraic description.

Again Λ is a characteristic set of $I(a)$. Suppose $g \in K\{X\} \setminus I(a)$. Then Λ and g involve parameters from K. We can put these in a finite list $\vec{c}$ and write Λ as $\Lambda_{\vec{c}}$ and g as $g_{\vec{c}}$.

McGrail shows there is a formula $\phi(\vec{y})$ such that

$$(K, \partial_0, \partial_1) \vDash \phi(\vec{c}), \tag{49}$$

and whenever $(L, \partial_0, \partial_1) \in \mathrm{Mod(DF)}$ and $(L, \partial_0, \partial_1) \vDash \phi(\vec{d})$, then the differential system

$$\bigwedge_{f \in \Lambda_{\vec{d}}} f = 0 \wedge g_{\vec{d}} \neq 0 \tag{50}$$

is soluble in some extension of $(L, \partial_0, \partial_1)$. Then one of the axioms of DCF is

$$\forall \vec{y} \left(\phi(\vec{y}) \to \exists x \, \Big(\bigwedge_{f \in \Lambda_{\vec{y}}} f(x) = 0 \wedge g_{\vec{y}}(x) \neq 0 \Big) \right). \tag{51}$$

We can see what is going on as follows. We ask whether the set of differential polynomials

$$\partial^{(2,2)}X - X, \qquad\qquad \partial^{(1,1)}X - \partial^{(0,2)}X \tag{52}$$

has a zero. These polynomials belong to $K[\partial^\sigma X : |\sigma| \leqslant 4]$. We depict a potential zero as

$$\begin{array}{ccccc} \partial^{(0,0)}x & \partial^{(1,0)}x & \partial^{(2,0)}x & \partial^{(3,0)}x & \partial^{(4,0)}x \\ \partial^{(0,1)}x & \partial^{(1,1)}x & \partial^{(2,1)}x & \partial^{(3,1)}x & \\ \partial^{(0,2)}x & \partial^{(1,2)}x & \partial^{(2,2)}x & & \\ \partial^{(0,3)}x & \partial^{(1,3)}x & & & \\ \partial^{(0,4)}x & & & & \end{array} \tag{53}$$

A generic zero of the polynomials

$$X_{(2,2)} - X_{(0,0)}, \qquad\qquad X_{(1,1)} - X_{(0,2)} \tag{54}$$

in $K[X_\sigma \colon |\sigma| \leqslant 4]$ can be depicted as

$$\begin{array}{ccccc} a & * & b & * & * \\ * & b & * & * & \\ * & * & a & & \\ * & * & & & \\ * & & & & \end{array} \tag{55}$$

Closing under differentiation imposes further conditions:

$$\begin{array}{ccccc} a & * & b & c & \underline{a} \\ * & \underline{b} & c & a & \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \tag{56}$$

But the underlined entries should have a common derivative—say d—outside the triangle:

$$\begin{array}{ccccc} a & * & b & c & \underline{a} \\ * & \underline{b} & c & a & d \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \tag{57}$$

This imposes further conditions *inside* the triangle:

$$\begin{array}{ccccc} a & d & b & c & a \\ d & b & c & a & d \\ b & c & a & & \\ c & a & & & \\ a & & & & \end{array} \tag{58}$$

That's fine, we can extend this diagram indefinitely; so the original differential polynomials have a zero.

But suppose we started instead with

$$\partial^{(2,2)}X - X, \qquad \partial^{(1,1)}X - \partial^{(0,2)}X - 1. \tag{59}$$

This gives us (writing b' for $b+1$)

$$\begin{array}{ccccc} a & * & b & * & * \\ * & b' & * & * & \\ * & * & a & & \\ * & * & & & \\ * & & & & \end{array} \qquad \begin{array}{ccccc} a & * & b & c & a \\ * & b' & c & a & \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \qquad \begin{array}{ccccc} a & d & b & c & a \\ d & b' & c & a & d \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \tag{60}$$

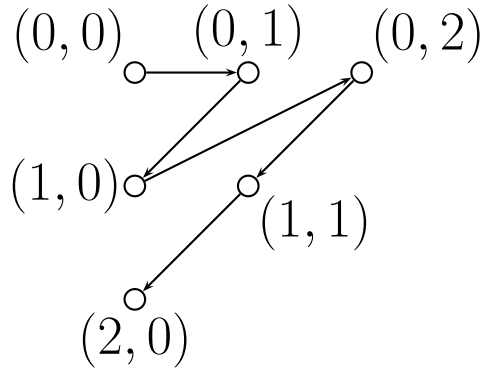
so $\partial_1 d$ must be both b and b', which is absurd.

This test works generally as follows.

We define a strict partial ordering $\lessdot$ on $\boldsymbol{\omega}^2$ by

$$\sigma \lessdot \tau \iff (|\sigma|, \sigma(0)) <_\ell (|\tau|, \tau(0)) \tag{61}$$

where $<_\ell$ is the left lexicographic ordering on $\boldsymbol{\omega}^2$. So $\lessdot$ is thus:



Let $\leqslant$ be the product ordering of $\boldsymbol{\omega}^2$, so

$$\sigma \leqslant \tau \iff \sigma(0) \leqslant \tau(0) \wedge \sigma(1) \leqslant \tau(1). \tag{62}$$

Then we always have

$$\sigma \leqslant \sigma + \tau. \tag{63}$$

Suppose again

$$(K, \partial_0, \ldots, \partial_{m-1}) \subseteq (L, \tilde{\partial}_0, \ldots, \tilde{\partial}_{m-1}), \tag{64}$$

both being models of DF, and $a \in L$. Then

$$\tilde{\partial}^\sigma a \in K(\tilde{\partial}^\rho a \colon \rho \lessdot \sigma)^{\mathrm{alg}} \implies \tilde{\partial}^{\sigma+\tau} a \in K(\tilde{\partial}^\rho a \colon \rho \lessdot \sigma+\tau)^{\mathrm{alg}}. \tag{65}$$

A tuple $(a_\sigma \colon |\sigma| \leqslant n)$ of elements of some field-extension of K will be called **soluble** if the a_σ belong to some L as above so that

$$\tilde{\partial}_0 a_\sigma = a_{\sigma+(1,0)}, \qquad\qquad \tilde{\partial}_1 a_\sigma = a_{\sigma+(0,1)} \tag{66}$$

when these make sense. For solubility, it is *necessary* that, for all f in $K[\partial^\sigma \colon |\sigma| < n]$,

$$\tilde{f}(a_\sigma \colon |\sigma| < n) = 0 \implies \widetilde{\partial_i f}(a_\sigma \colon |\sigma| \leqslant n) = 0. \tag{67}$$

Call this the **differential condition.** The last example shows it is not *sufficient* for solubility.

Again we have the tuple $(a_\sigma \colon |\sigma| \leqslant n)$ is a tuple of elements of some field-extension of K. We ask what is sufficient for solubility.

Call σ a **leader** if it is $\leqslant$-minimal among those τ such that

$$a_\tau \in K(a_\rho \colon \rho \lessdot \tau)^{\mathrm{alg}}. \tag{68}$$

Theorem (P.)**.** For $(a_\sigma \colon |\sigma| \leqslant n)$ to be soluble, it is sufficient that

1) it meet the differential condition, and

2) for every leader a_σ, we have $|\sigma| \leqslant n/2$.

Moreover, if $(a_\sigma \colon |\sigma| \leqslant n)$ is soluble, then n can be chosen large enough so that the foregoing conditions are met; and the new n depends only on the original n.

We can make adjustments to allow each a_σ to be a *tuple* $(a_\sigma^j \colon j < k)$ and to allow K to have arbitrary characteristic.

This leads to a model-completion of the theory of fields (of unspecified characteristic) with m commuting derivations.