Numbers and sets

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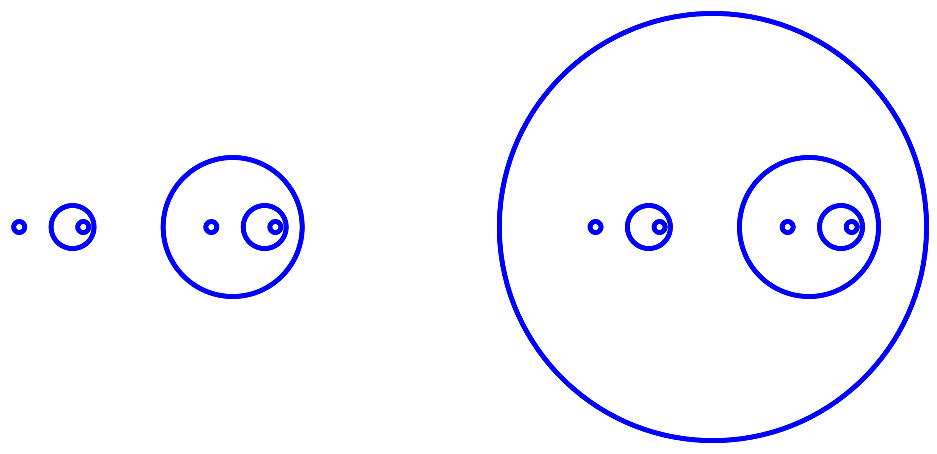
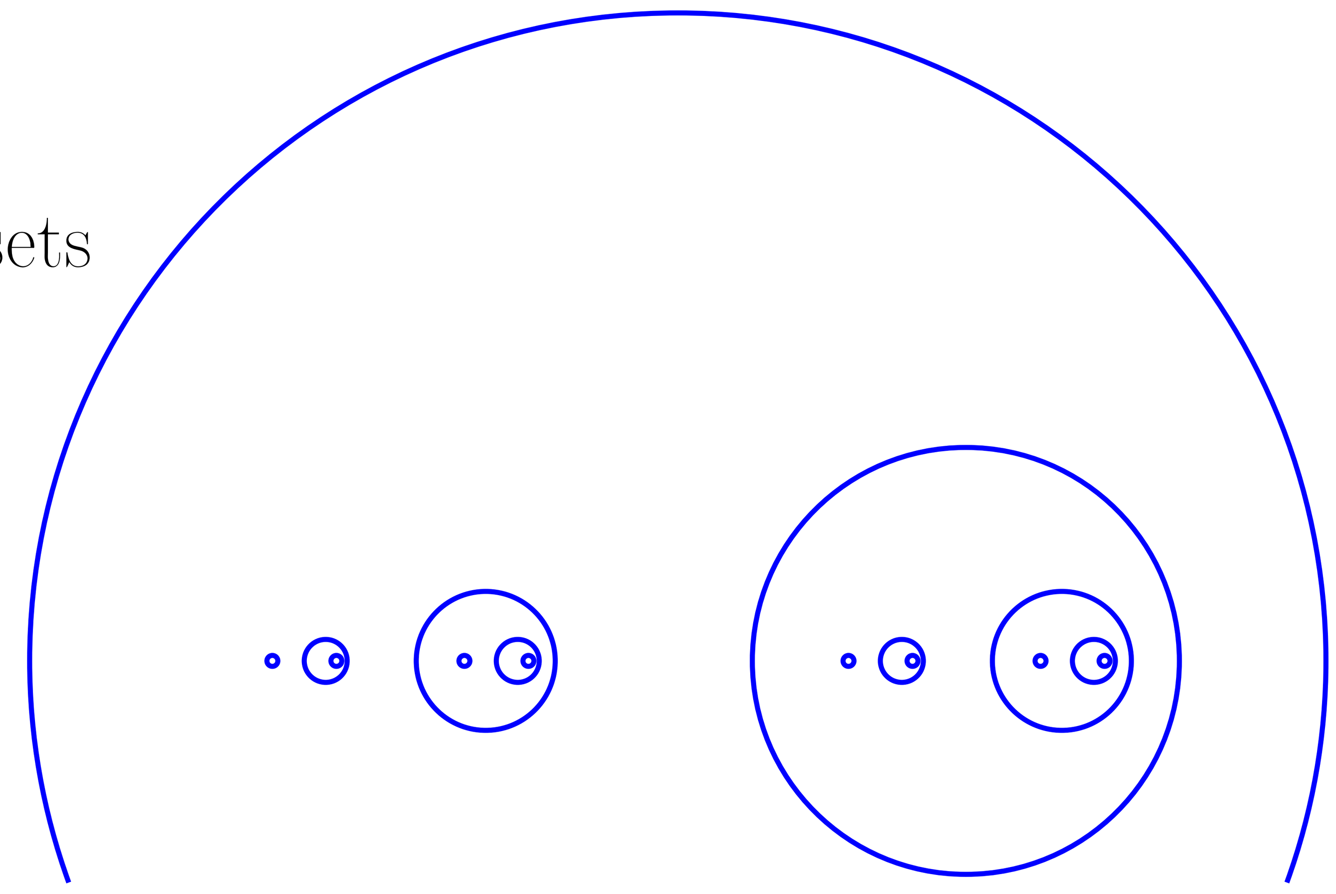
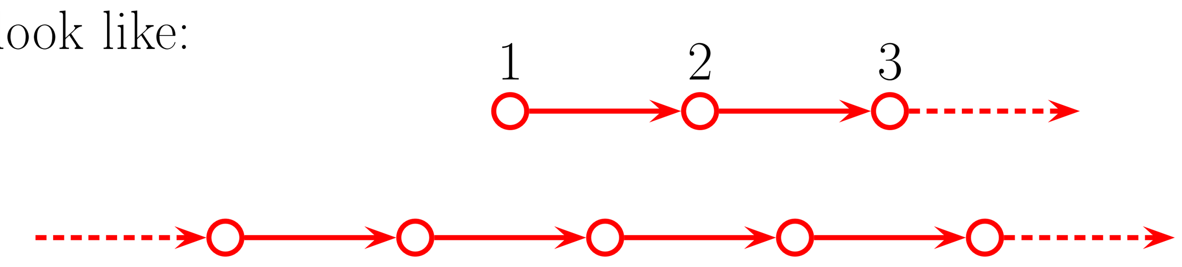




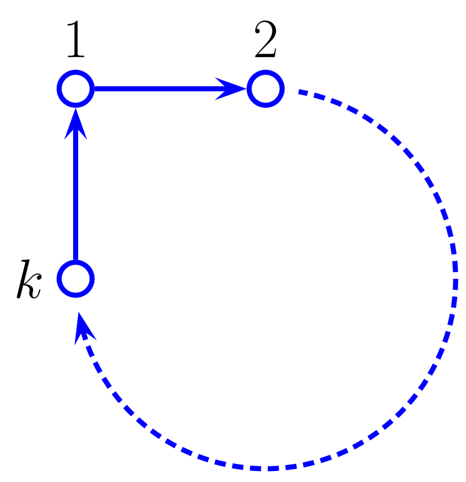
By an **iterative algebra,** I mean an ordered triple $(A, 1, S)$, or $\mathfrak{A}$, where
(1) A is a set; (2) $1 \in A$; (3) $S\colon A \to A$.

Example. $A = \mathbb{N} = \{1, 2, 3, \dots\}$ and $S(n) = n + 1$.

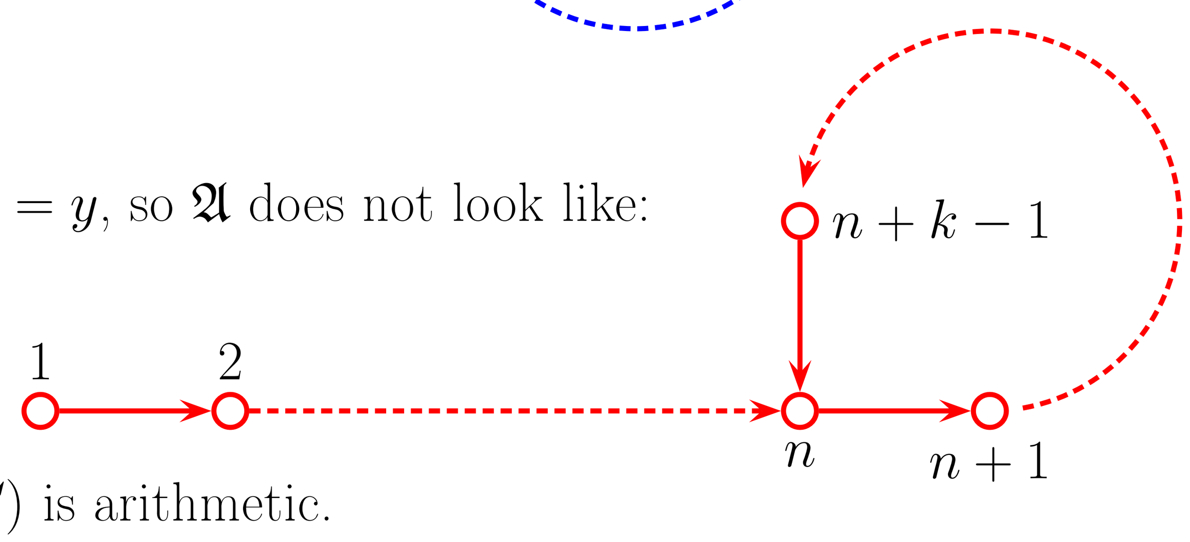
An iterative algebra $\mathfrak{A}$ can be conceived as a **directed graph,** where
(1) A is the set of **nodes,** (2) 1 is a particular node, and
(3) each pair $(x, S(x))$ is an **arrow** from x to $S(x)$.
The iterative algebra $\mathfrak{A}$ **admits induction** if it has no proper subalgebra, so it does *not* look like:



Then $\mathfrak{A}$ is **arithmetic** if
(1) it admits induction,
(2) $1 \neq S(x)$, so $\mathfrak{A}$ does *not* look like:



(3) $S(x) = S(y) \Rightarrow x = y$, so $\mathfrak{A}$ does not look like:



In particular, $(\mathbb{N}, 1, S)$ is arithmetic.
In general, $\mathfrak{A}$ **admits recursion** if, for every iterative algebra $\mathfrak{B}$, there is a unique homomorphism h from $\mathfrak{A}$ to $\mathfrak{B}$:

$$\begin{tikzcd}
1^{\mathfrak{A}} \arrow[r, "\in"] \arrow[d, "h"'] & A \arrow[r, "S^{\mathfrak{A}}"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
1^{\mathfrak{B}} \arrow[r, "\in"'] & B \arrow[r, "S^{\mathfrak{B}}"'] & B
\end{tikzcd}$$

Theorem (Dedekind)**.** *An iterative algebra admits recursion if and only if it is arithmetic; in particular, all such iterative algebras are isomorphic.*

Also $\mathbb{N}$ is **well-ordered** by the relation $<$, defined recursively by

$$x \not< 1, \qquad x < n + 1 \Leftrightarrow x \leqslant n.$$

Functions can be defined on $\mathbb{N}$ by **well-ordered recursion:** the simplest example is h, given by

$$h(n) = \{h(x)\colon x < n\}.$$

Then h is a bijection from $\mathbb{N}$ onto $\boldsymbol{\omega}$, the set of **von Neumann** natural numbers. The first five of these—0, 1, 2, 3, and 4—are illustrated above.
The class $\mathbf{ON}$ of von Neumann **ordinals** comprises each set that
(1) is well-ordered by membership ($\in$),
(2) is **transitive** (its members are also subsets).
Then $\mathbf{ON}$ itself is well-ordered by membership and is transitive; it is an iterative algebra with respect to $\varnothing$ and $x \mapsto x \cup \{x\}$; and $\boldsymbol{\omega}$ is a subalgebra of $\mathbf{ON}$ and is a **free algebra.**

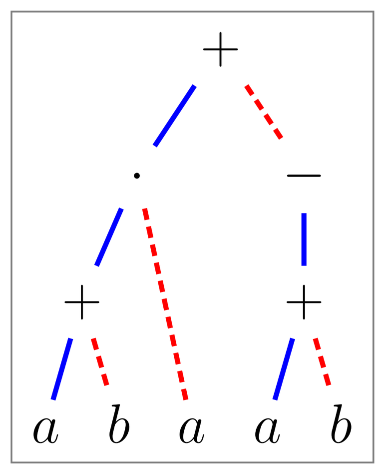
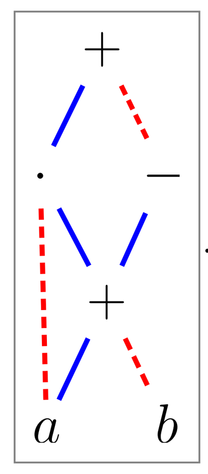
All of the foregoing generalizes to an arbitrary algebraic signature, $\mathscr{S}$.
One free algebra in $\mathscr{S}$ is the **term algebra:** the smallest set of strings that, for each n in $\boldsymbol{\omega}$, for each n-ary symbol F in $\mathscr{S}$, is closed under the concatenation

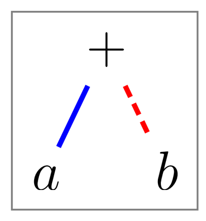
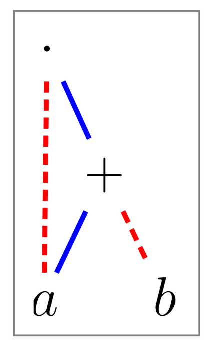
$$(t_0, \dots, t_{n-1}) \longmapsto F t_0 \cdots t_{n-1}.$$

Terms can be written as **labelled trees** or (refinements of) **Hasse diagrams:**

In $\{a, b, -, +, \cdot\}$, the term $+ \cdot + a\, b\, a - + ab$, or more conventionally

$$((a + b) \cdot a) + -(a + b), \qquad (*)$$

corresponds to either of  and . Now let

 $= (\{a\}, \{b\}, +)$,  $= \Big(\big\{a, b, (\{a\}, \{b\}, +)\big\}, \{a\}, \cdot\Big)$,

and so on; then $(*)$ is as depicted below. If F is n-ary in $\mathscr{S}$ as before, X_k is a set when $k < n$, and $X = (X_0, \dots, X_{n-1}, F)$, define

$$\operatorname{pred}_k(X) = X_k, \qquad \operatorname{pred}(X) = X_0 \cup \dots \cup X_{n-1},$$
$$Y \in' X \Leftrightarrow Y \in \operatorname{pred}(X)$$

(here 'pred' is for *predecessor*); say that X is k-**transitive** if

$$Y \in \operatorname{pred}_k(X) \Rightarrow \operatorname{pred}(Y) \subseteq \operatorname{pred}_k(X).$$

Let $\mathbf{ON}_{\mathscr{S}}$ comprise those X such that
(1) $X = (X_0, \dots, X_{n-1}, F)$ for some n-ary F in $\mathscr{S}$, for some n in $\boldsymbol{\omega}$; and each X_k is nonempty;
(2) each element Y of $\operatorname{pred}(X)$ is $(Y_0, \dots, Y_{m-1}, G)$ for some m-ary G in $\mathscr{S}$, for some m in $\boldsymbol{\omega}$; and each Y_ℓ is nonempty;
(3) X is k-transitive for each k;
(4) each element of $\operatorname{pred}(X)$ is ℓ-transitive for each ℓ;
(5) $\in'$ **directs** each set $\operatorname{pred}_k(X)$ (finite subsets have upper bounds);
(6) $\in'$ directs each set $\operatorname{pred}_\ell(Y)$ for each Y in $\operatorname{pred}(X)$;
(7) $\in'$ is **well-founded** on $\operatorname{pred}(X)$ (nonempty subsets have minimal elements).

Call an element X of $\mathbf{ON}_{\mathscr{S}}$ a **limit** if some $\operatorname{pred}_k(X)$ has no maximal element with respect to $\in'$. Let $\boldsymbol{\omega}_{\mathscr{S}}$ consist of those X in $\mathbf{ON}_{\mathscr{S}}$ such that neither X nor any element of $\operatorname{pred}(X)$ is a limit.

Theorem. *The relation $\in'$ is well-founded on $\mathbf{ON}_{\mathscr{S}}$, and*

$$X \in \mathbf{ON}_{\mathscr{S}} \Rightarrow \operatorname{pred}(X) \subseteq \mathbf{ON}_{\mathscr{S}}.$$

The class $\mathbf{ON}_{\mathscr{S}}$ is an $\mathscr{S}$-algebra with respect to the operations

$$F(X_0, \dots, X_{n-1}) = (\operatorname{pred}(X_0) \cup \{X_0\}, \dots, \operatorname{pred}(X_{n-1}) \cup \{X_{n-1}\}, F);$$

and $\boldsymbol{\omega}_{\mathscr{S}}$ is a free subalgebra.