Mathematics as Philosophy

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`http://www.math.metu.edu.tr/~dpierce/talks/`

ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗ ΕΙΣΙΤΩ
(Geometri yapamayan girmesin):

Motto of the American Mathematical Society, presumably based on Platonic tradition

Mathematics is justified by its service to philosophy!

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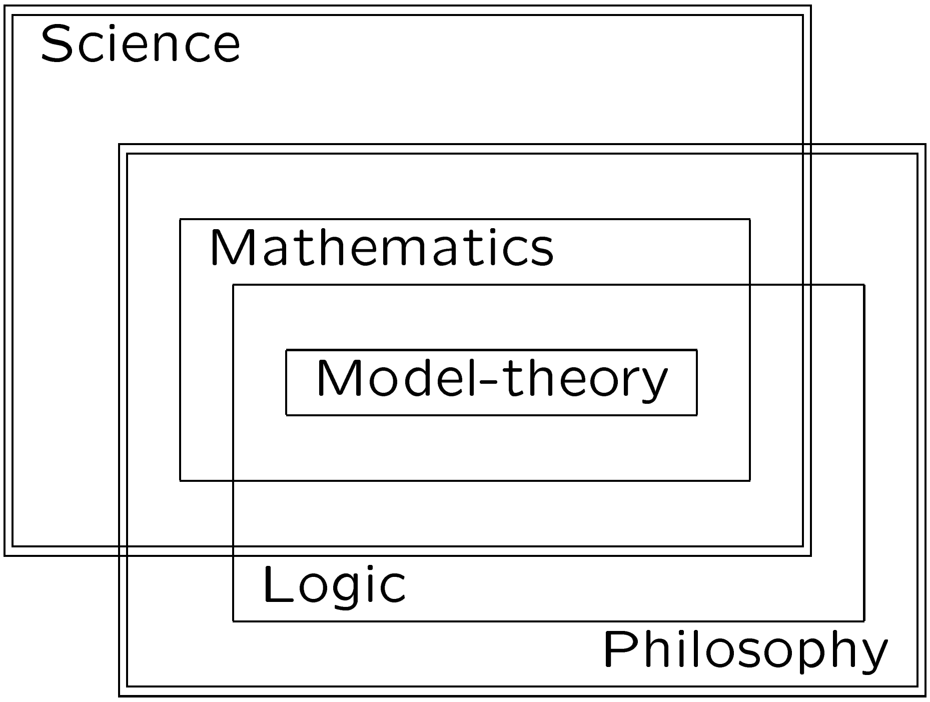
(Teraziyle ve kitaplarla çocuklardan özgür insanlar yaparım):

Motto of St John's College, Annapolis, Maryland, and Santa Fe, New Mexico, USA.

"The following teachers will return to St John's College next year:

"Homer, Æschylus, Herodotus, Plato, Sophocles, Aristophanes, Thucydides, Aristotle, Euripides, Lucretius, the Bible, Plutarch, Virgil, Tacitus, Epictetus, Plotinus, Augustine, Anselm, Aquinas, Dante, Chaucer, Shakespeare, Machiavelli, Montaigne, Bacon, Descartes, Cervantes, Pascal, Milton, Hobbes, Locke, Rousseau, Swift, Leibniz, Berkeley, Hume, Kant, Wordsworth, Austen, Smith, Twain, Tolstoy, Goethe, Hegel, Tocqueville, Kierkegaard, Dostoevski, Marx, Nietzsche, Freud. . ."



Mathematical truths are:

like philosophical truths, **personal**: not needing verification by experiment or by the agreement of a multitude; but,

like scientific truths, **universal**: one expects them to be agreed on by all who take the trouble to understand them—and the agreement generally happens.

Unwilling math-student's complaint:
"There's only one right answer!"

In **model-theory**, a model is a **structure**, considered with respect to some **theory** or *theories* of which it is a model.

(Likewise a person is a son or daughter with respect to his or her parents.)

Model-theory is:

mathematics done with an awareness of the language with which one does it;

the study of the construction and classification of structures within specified classes of structures (Wilfred Hodges, *Model Theory*);

algebraic geometry without fields (Hodges, *A Shorter Model Theory*);

the geography of **tame mathematics** (Lou van den Dries);

the study of structures *quâ* models of theories.

Bourbaki on the role of logic: "In other words, logic, so far as we mathematicians are concerned, is no more and no less than the grammar of the language which we use, a language which had to exist before the grammar could be constructed. . .

"The primary task of the logician is thus the analysis of the body of existing mathematical texts, particularly of those which by common agreement are regarded as the most correct ones, or, as one formerly used to say, the most 'rigorous.'" ("Foundations of Mathematics for the Working Mathematician", *Journal of Symbolic Logic* **14**, 1949)

Is mathematics one?

Some possible divisions:

pure/applied

geometry/algebra/arithmetic/analysis/. . .

Euclid gives all mathematical facts in geometric terms, even those facts that we should call arithmetic or algebraic

Descartes: Euclid must have used non-geometrical means to discover some of those facts

Euclid, *Elements* II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments."

Descartes, *Geometry: By geometric means,* the squares and rectangles can be replaced by straight lines, once we have chosen a straight line to represent unity.

Thus, all arithmetic operations can be referred to geometry—though there may be no practical need to do so.

Old-style geometry is not abandoned: see Lobachevski.

A possible analogy:

Algebra, geometry &c. are to the mathematical world as the senses are to the physical world.

We have a **common sense** whereby the things that we see, hear and touch are known to be part of one world.

The visible, audible and tangible worlds are the same in principle, but it would be foolish to treat the world as *simply* one of these.

We might even hope to develop additional senses, the better to understand the world:

THERE is NO NATURAL RELIGION
(William Blake, c. 1788):

The Argument: Man has no notion of moral fitness but from Education. Naturally he is only a natural organ subject to Sense.

I Man cannot naturally Percieve, but through his natural or bodily organs

II Man by his reasoning power. can only compare & judge of what he has already perciev'd.

III *From a perception of only 3 senses or 3 elements none could deduce a fourth or fifth*

IV None could have other than natural or organic thoughts if he had none but organic perceptions

V Mans desires are limited by his perceptions. none can desire what he has not perciev'd

VI The desires & perceptions of man untaught by any thing but organs of sense, must be limited to objects of sense.

"The essential business of **language** is to assert or deny facts." (Bertrand Russell, introduction to Wittgenstein's *Tractatus*, 1922)

"*Bodily actions expressing certain emotions,* in so far as they come under our control and are conceived by us, in our awareness of controlling them, as our way of expressing these emotions, *are language*. . . The grammatical and logical articulations of intellectualized language are no more fundamental to language as such than the articulations of bone and limb are fundamental to living tissue" (R.G. Collingwood, *The Principles of Art*, 1937).

What *is* an assertion of a fact?

Say it is an utterance, in a certain manner, of a *statement* (or *proposition*).

What is a statement?

Some authors say it is a sentence that is either true or false.

I think a statement is a sentence distinguished from other sentences by its *form*: it is not a question, but its verb is declarative, not subjunctive or imperative, . . .

Then a statement *becomes* true or false when placed in an appropriate *context*.

A true statement (in context) need not be a *correct* answer to the question it is intended to answer.

Bourbaki: What are the *structures* at the heart of mathematics? They can be:

algebraic, like:
the *fields* $(\mathbb{R}, +, \times)$ and $(\mathbb{C}, +, \times)$,
the *vector-spaces* $\mathbb{R}^n$ and $\mathbb{C}^n$,
the *group* of rigid motions of the Euclidean plane . . . ;

ordered:—
totally, like $(\mathbb{Z}, \leqslant)$ and $(\mathbb{R}, \leqslant)$, or
partially, like $(\mathcal{P}(\Omega), \subseteq)$, . . . ;

topological: the real number x is *in the interior of* the set A if: for some positive ε, if $|y - x| < \varepsilon$, then y is in A; symbolically,

$$\exists\varepsilon > 0\ \forall y\ (|y - x| < \varepsilon \Rightarrow y \in A)$$

Bourbaki aims to characterize structures by means of *axioms.*

Model-theory treats directly of the first two kinds of structures and examines how axioms *fail* to characterize any one structure.

Let ω be the set of **natural numbers:** the smallest of sets Ω of sets such that $\varnothing \in \Omega$ and, for all sets A, if $A \in \Omega$, then $A \cup \{A\} \in \Omega$. Let j and n range over ω.

We write 0 for $\emptyset$, and 1 for $\{0\}$, and so forth; also $n + 1$ for $n \cup \{n\}$. If $j \subset n$, we write $j < n$. Then $n = \{j : j < n\}$.

Let I be a finite subset of ω. The **Cartesian power** M^I is the set of functions from I to M; a typical element a of M^I can be written $(a_j : j \in I)$.

Since 0 is empty, M^0 consists of the empty function, 0; so $M^0 = \{0\} = 1$.

M^n consists of $(a_0, \dots, a_{n-1})$, where $a_j \in M$.

An I**-ary operation** on M is a function from M^I to M.

Atn I**-ary relation** is a subset of M^I.

A **structure** on M is a set of operations and relations on M, each of them n-ary for some n in ω.

In the background are three "primordial" structures:

- the structure of the **natural numbers:**

$$(\omega, ', 0),$$

where $'$ is the unary operation $x \mapsto x + 1$;

- the **Boolean algebra** of subsets of a "universal" set U:

$$(\mathcal{P}(U), \cap, {}^{\mathsf{c}}, \cup, \varnothing, U, \subseteq);$$

- **propositional logic:**

$$(\mathbb{B}, \wedge, \neg, \vee, 0, 1, \Rightarrow),$$

where $\mathbb{B} = \{0, 1\} = \mathcal{P}(\{\varnothing\})$ and can be understood as $\{\text{false}, \text{true}\}$.

A structure on M may be denoted $\mathfrak{M}$. This has a **signature**: a set $\mathcal{L}$ of symbols for the specified operations and relations.

The symbols in $\mathcal{L}$ are primary; say they are f and R, symbolizing $n(f)$-ary operations $f^{\mathfrak{M}}$ and $n(R)$-ary relations $R^{\mathfrak{M}}$ on M.

Any element a of M can be understood as a nullary operation-symbol: a **constant-symbol**. For any subset A of M, for the signature $\mathcal{L} \cup A$ there is a first-order language, $\mathcal{L}_{\omega\omega}(A)$.

For an I-ary formula ϕ of $\mathcal{L}_{\omega\omega}(A)$, there is an I-ary relation $\phi^{\mathfrak{M}}$ on M.

Such relations are the A-**definable sets** of $\mathfrak{M}$; they compose Boolean sub-algebras $\mathcal{D}^I(A)$ of the $\mathcal{P}(M^I)$.

SYMBOL SYNTAX	INTERPRETATION SEMANTICS
signature $\mathcal{L}$	$\mathfrak{M}$, an $\mathcal{L}$-structure
	OPERATIONS ON M:
variable x_j	$a \mapsto a_j : M^I \to M$
constant-symb. c	$c^{\mathfrak{M}}$, an element of M
function-symb. f	$f^{\mathfrak{M}} : M^{n(f)} \to M$
term t	$t^{\mathfrak{M}}$, a composition
LOGICAL:	FUNCTIONS ON $\mathcal{P}(M^I)$
connectives:	operations:
$\wedge$	$\cap$
$\neg$	$A \mapsto A^{\mathsf{c}}$
$\vee$	$\cup$
$\to$	$(A, B) \mapsto A^{\mathsf{c}} \cup B$
quantifiers:	projections:
$\exists x_j$	$\pi^I_j\colon\ a \mapsto (a_i : i \in I \setminus \{j\})$
$\forall x_j$	$A \mapsto (\pi^I_j(A^{\mathsf{c}}))^{\mathsf{c}}$
	RELATIONS ON M:
equals-sign $=$	equality
relation-symb. R	$R^{\mathfrak{M}}$, a subset of $M^{n(R)}$
I-ary formula ϕ	$\phi^{\mathfrak{M}}$, a subset of M^I
nullary formula or *sentence* σ	1 (true) or 0 (false)

One may care only about the definable sets:

Let also J be a finite subset of ω. If

$$\sigma : I \to J,$$

then $\sigma^* : M^J \to M^I$, whence

$$\sigma^* : \mathcal{P}(M^J) \to \mathcal{P}(M^I),$$
$$\sigma_* : \mathcal{P}(M^I) \to \mathcal{P}(M^J),$$

where $\sigma^*(c) = (c_{\sigma(i)} : i \in J)$ if $c \in M^J$, and

$$\sigma^* A = \{\sigma^*(c) : c \in A\},$$
$$\sigma_* B = \{c : \sigma^*(c) \in B\}.$$

We can re-define a **structure** on M to be $\mathcal{D}$, which comprises, for each I, a Boolean sub-algebra $\mathcal{D}^I$ of $\mathcal{P}(M^I)$ such that

$$\{(c,c) \in M^2\} \in \mathcal{D}^2;$$
$$A \in \mathcal{D}^J \Rightarrow \sigma^* A \in \mathcal{D}^I;$$
$$B \in \mathcal{D}^I \Rightarrow \sigma_* B \in \mathcal{D}^J.$$

From $\mathcal{D}$ we can recover a structure on M in the original sense whose 0-definable sets are just the sets in the $\mathcal{D}^I$.