Free groups

Notes from the Istanbul model theory seminar

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These notes are usually based on my handwritten notes taken during the seminar. The schedule of the talks and speakers is in an appendix. I have made adjustments and supplements according to my understanding and preferences.

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1. Free groups

Our source is Baumslag, *Combinatorial Group Theory.*

Let G be a group, and let H be a subgroup of G:

$$H \leqslant G.$$

We consider the set

$$H \backslash G$$

of right cosets of H in G. The quotient map $g \mapsto Hg$ from G to $H \backslash G$ is surjective, so it has a right inverse, which we may denote by

$$Hg \mapsto \overline{g}.$$

(Here we use the Axiom of Choice if the index of H in G is infinite.) Then

$$H\overline{g} = Hg.$$

We may assume $\overline{1} = 1$. We denote the range of the function $Hg \mapsto \overline{g}$ by R. Then R is a complete set of right-coset representatives of H in G, and R contains 1. In a word, R is a right **transversal**[1] of H in G (and R contains 1). The situation is depicted in Figure 1.1.

For every g in G, the element $\overline{g}$ of R is such that

$$g \in H\overline{g}.$$

Then there is a unique element h of H such that

$$g = h \cdot \overline{g}.$$

We define a function $(r, g) \mapsto \delta(r, g)$ from $R \times G$ to H by

$$(*) \qquad rg = \delta(r, g) \cdot \overline{rg}.$$

[1] This word was introduced the following week.

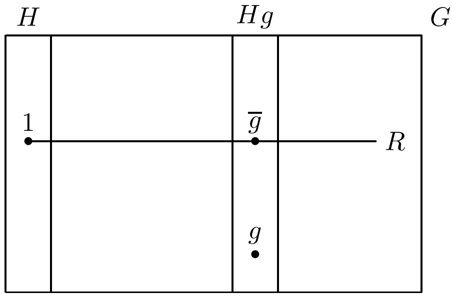

Figure 1.1. Transversal of a subgroup

In particular then,

$$g = \delta(1, g) \cdot \overline{g}.$$

Note also that $\delta(r, g)$ is just another name for $rg \cdot (\overline{rg})^{-1}$.

Now suppose that G is generated by X:

$$G = \langle X \rangle.$$

Then we have:

Theorem 1. *The subgroup H of G is generated by the image of $R \times X$ under δ:*

$$H = \langle \delta(r, x) \colon r \in R \wedge x \in X \rangle.$$

Proof. By definition, each $\delta(r, x)$ is in H, so we have

$$\langle \delta(r, x) \colon r \in R \wedge x \in X \rangle \leqslant H.$$

To show the reverse inclusion, we let $h \in H$. Then we can write h as ${x_1}^{\varepsilon_1} \cdots {x_n}^{\varepsilon_n}$, where $x_i \in X$ and $\varepsilon_i = \pm 1$. Then we have

$$\begin{aligned} h &= {x_1}^{\varepsilon_1} \cdots {x_n}^{\varepsilon_n} \\ &= 1 \cdot {x_1}^{\varepsilon_1} \cdots {x_n}^{\varepsilon_n} \\ &= \delta(1, {x_1}^{\varepsilon_1}) \cdot \overline{{x_1}^{\varepsilon_1}} \cdot {x_2}^{\varepsilon_2} \cdots {x_n}^{\varepsilon_n} \\ &= \delta(1, {x_1}^{\varepsilon_1}) \cdot \delta(\overline{{x_1}^{\varepsilon_1}}, {x_2}^{\varepsilon_2}) \cdot \overline{\overline{{x_1}^{\varepsilon_1}} \cdot {x_2}^{\varepsilon_2}} \cdot {x_3}^{\varepsilon_3} \cdots {x_n}^{\varepsilon_n} \end{aligned}$$

and so on; ultimately,

$$h = \underbrace{\delta(1, {x_1}^{\varepsilon_1}) \cdot \delta(r_2, {x_2}^{\varepsilon_2}) \cdots \delta(r_n, {x_n}^{\varepsilon_n})}_{\in H} \cdot r$$

for some r_i and r in R. Then $r = \overline{h}$, so $r = 1$, and therefore

$$h = \delta(1, {x_1}^{\varepsilon_1}) \cdot \delta(r_2, {x_2}^{\varepsilon_2}) \cdots \delta(r_n, {x_n}^{\varepsilon_n}).$$

If we can move the ε_i outside the parentheses (perhaps by adjusting the r_i within R), we are done.

We do this. For arbitrary r in R and x in X, we have

$$rx^{-1} = \delta(r, x^{-1}) \cdot \overline{rx^{-1}},$$

$$(\dagger) \qquad r = \delta(r, x^{-1}) \cdot \overline{rx^{-1}} \cdot x = \underbrace{\delta(r, x^{-1}) \cdot \delta(\overline{rx^{-1}}, x)}_{\in H} \cdot \underbrace{\overline{\overline{rx^{-1}} \cdot x}}_{\in R}$$

so $\delta(r, x^{-1}) \cdot \delta(\overline{rx^{-1}}, x) = 1$ and hence

$$(\ddagger) \qquad \delta(r, x^{-1}) = \delta(\overline{rx^{-1}}, x)^{-1}. \qquad \square$$

Corollary 2. *A subgroup of finite index of a finitely generated group is finitely generated. Indeed, if G is n-generated, and H has index k in G, then H is at most kn-generated.*

We aim now to prove that every subgroup of a free group is free. To this end, we consider the group G above as the free group F on the set X. Still $H \leqslant F$. A set S of right-coset representatives of H in F is called a (right) **Schreier set** (with respect to X and H) if, whenever ${x_1}^{\varepsilon_1} \cdots {x_n}^{\varepsilon_n}$ is a reduced word on X that belongs to S, then each initial segment ${x_1}^{\varepsilon_1} \cdots {x_i}^{\varepsilon_i}$ belongs to S. A Schreier set is **complete,** or is a **transversal,** if it is complete as a set of (right-) coset representatives.

We shall be able to find a Schreier transversal by recursion on the *lengths* of cosets. In case X is not otherwise known to be well-orderable, we shall have to use the Axiom of Choice.

If $g \in F$, then its **length,** $\ell(g)$, is the length of the reduced word on X that is equal to g. Then a coset Hg has a **length,** $\ell(Hg)$, where

$$\ell(Hg) = \min\{\ell(hg) \colon h \in H\}.$$

That is, $\ell(Hg)$ is the minimum length of a representative of Hg.

Lemma 3. *There is a Schreier transversal (with respect to X and H).*

Proof. We assign to the unique coset of length 0 the representive 1. This representative itself has length 0.

Suppose now that, for each coset of length at most n, a representative of the same length has been chosen, and moreover, when this representative is written out as a reduced word on X, then every initial segment is also one of the chosen representatives. Say $\ell(Hg) = n + 1$. Then Hg has an element $y_1 \cdots y_n \cdot y_{n+1}$, where $y_i \in X \sqcup X^{-1}$. We have

$$\ell(Hy_1 \cdots y_n) = m \leqslant n$$

for some m, and so, by hypothesis, $Hy_1 \cdots y_n$ has a chosen representative z of length m. Then

$$Hg = Hy_1 \cdots y_n \cdot y_{n+1} = Hz \cdot y_{n+1},$$

and we may choose $z \cdot y_{n+1}$ as a representative of Hg. We have

$$n + 1 \leqslant \ell(z \cdot y_{n+1}) \leqslant m + 1 \leqslant n + 1,$$

so $m = n$. Therefore the recursion can indeed be continued. Strictly, we need to be able to choose *one* such y_{n+1}. But, assuming X has been well-ordered, we can well-order $X \sqcup X^{-1}$, and then we can well-order the reduced words in X by a lexigraphic ordering; finally, we can let $y_1 \cdots y_n \cdot y_{n+1}$ be minimal in this ordering. $\square$

We henceforth assume that the transversal R above is a Schreier transversal S. In a free group, if $\ell(ab) = \ell(a) + \ell(b)$, we may write ab as

$$a_{\triangle}b;$$

but if $\ell(ab) < \ell(a) + \ell(b)$, we may write ab as

$$a_{\sqcup}b.$$

Lemma 4. *If $x \in X \sqcup X^{-1}$ and $\delta(s, x) \neq 1$, then*

$$\delta(s, x) = s_{\triangle}x_{\triangle}(\overline{sx})^{-1}.$$

Proof. Suppose $sx = s_\sqcup x$. Then $s = tx^{-1}$ for some t, which is also in S since this is a Schreier set. Then

$$\delta(s, x) = sx(\overline{sx})^{-1} = t(\overline{t})^{-1} = tt^{-1} = 1.$$

The other possibility to consider is $x(\overline{sx})^{-1} = x_\sqcup(\overline{sx})^{-1}$. Then $\overline{sx} \cdot x^{-1} = \overline{sx}_\sqcup x^{-1}$, so $\overline{sx} = tx$ for some t, which again must be in S. We now have

$$\delta(s, x) = sx(\overline{sx})^{-1} = sx(tx)^{-1} = st^{-1},$$

so this is in H. But then $Hs = Ht$, so $s = t$ and therefore $\delta(s, x) = 1$. $\square$

Lemma 5. *If $x, y \in X \sqcup X^{-1}$ and $\delta(s, x) = \delta(t, y) \neq 1$, then*

$$(s, x) = (t, y).$$

Proof. By Lemma 4, we have

$$s_\triangle x_\triangle(\overline{sx})^{-1} = t_\triangle y_\triangle(\overline{ty})^{-1}.$$

If $\ell(s) = \ell(t)$, then we must have $s = t$ and then $x = y$. In the other case, we may assume $\ell(s) < \ell(t)$, so that $t = sxu$ for some u, and in particular $sx \in S$. But then $\overline{sx} = sx$, so $\delta(s, x) = 1$. $\square$

Lemma 6. *Suppose $h = \delta(s_1, x_1)^{\varepsilon_1} \cdots \delta(s_n, x_n)^{\varepsilon_n}$, where none of the factors $\delta(s_i, x_i)^{\varepsilon_i}$ is 1 or is the inverse of the following. Then*

$$h = \cdots_\triangle {x_1}^{\varepsilon_1}{}_\triangle \cdots_\triangle {x_2}^{\varepsilon_2}{}_\triangle \cdots;$$

in particular, $h \neq 1$ unless $n = 0$.

Proof. By Lemma 4, we have

$$\delta(s, x) \cdot \delta(t, y) = (s_\triangle x_\triangle(\overline{sx})^{-1}) \cdot (t_\triangle y_\triangle(\overline{ty})^{-1}).$$

By (‡) in the proof of Theorem 1, it is enough to show that this product is $s_\triangle x_\triangle \cdots {}_\triangle y_\triangle(\overline{ty})^{-1}$, unless one of the factors, or the whole product, is 1. There are three possibilities to consider.

Suppose that t, as a reduced word in X, has an initial segment equal to $\overline{sx} \cdot x^{-1}$. Then this segment is in S, so

$$\overline{sx} \cdot x^{-1} = \overline{\overline{sx} \cdot x^{-1}}.$$

By (†) in the proof of Theorem 1, we now have $s = \overline{sx} \cdot x^{-1}$, so $\delta(s, x) = 1$.

Now suppose that $\overline{sx}$, as a reduced word in X, has an initial segment equal to ty. Then $ty \in S$, so $\overline{ty} = ty$, and hence $\delta(t, y) = 1$.

Suppose finally

$$x_{\triangle}(\overline{sx})^{-1} = (t_{\triangle} y)^{-1}.$$

Then $x = y^{-1}$ and $\overline{sx} = t$, so again by (‡)

$$\delta(t, y)^{-1} = \delta(\overline{sx}, x^{-1})^{-1} = \delta(s, x).$$ □

Theorem 7. *Every subgroup of a free group is free. Indeed, if*

$$Y = \{(s, x) \colon (s, x) \in S \times X \wedge \delta(s, x) \neq 1\},$$

then the free group on Y is isomorphic to H under the map

$$(s, x) \mapsto \delta(s, x).$$

Proof. By Theorem 1, H is generated by the image of Y under δ. By Lemma 5, δ is injective on Y. By (‡) and Lemma 5, we never have $\delta(s, x) = \delta(t, y)^{-1}$ for any (s, x) and (t, y) in Y. Then also by Lemma 6, the extension of δ to a homomorphism from the free group on Y to H is an embedding. □

Example 1. Let $F = \langle x, y \rangle$ and let H be the kernel of the homomorphism

$$x \mapsto 1, \qquad y \mapsto 0$$

from F onto $\mathbb{Z}/2\mathbb{Z}$. Then $[F : H] = 2$, so we may let $R = \{1, x\}$. Recalling the definition (∗) and filling out the table

R	X	H	R
r	u	$\delta(r, u)$	$\overline{ru}$
1	x	1	x
1	y	y	1
x	x	x^2	1
x	y	xyx^{-1}	x

we have that H is free on $\{y, x^2, xyx^{-1}\}$, that is,

$$\{y, x^2, y^{x^{-1}}\},$$

where we use the notation

$$a^b = b^{-1}ab.$$

Example 2. Now letting the homomorphism be

$$x \mapsto (1,0), \qquad\qquad y \mapsto (0,1)$$

to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, we have $[F : H] = 4$, and we may let

$$R = \{1, x, y, xy\},$$

so that, from the table

R	X	H	R
r	u	$\delta(r,u)$	$\overline{ru}$
1	x	1	x
1	y	1	y
x	x	x^2	1
x	y	1	xy
y	x	$yxy^{-1}x^{-1}$	xy
y	y	y^2	1
xy	x	$xyxy^{-1}$	y
xy	y	xy^2x^{-1}	x

H is freely generated by $\{x^2, yxy^{-1}x^{-1}, y^2, xyxy^{-1}, xy^2x^{-1}\}$, that is,

$$\{x^2, [y^{-1}, x^{-1}], y^2, xx^{y^{-1}}, (y^2)^{x^{-1}}\},$$

where we use the notation

$$[a,b] = a^{-1}b^{-1}ab.$$

Example 3. Now let the 'same' homomorphism be onto $\mathbb{Z} \oplus \mathbb{Z}$. Then $F' \leqslant H$ (where F' is the subgroup of F generated by the commutators $[a,b]$). We can let

$$R = \{x^i y^j : (i,j) \in \mathbb{Z} \oplus \mathbb{Z}\}.$$

Since

$$x^i y^j \cdot x = x^i y^j x y^{-j} x^{-i-1} \cdot x^{i+1} y^j,$$
$$x^i y^j \cdot y = 1 \cdot x^i y^{j+1},$$

we have that H is freely generated by the $x^i y^j x y^{-j} x^{-i-1}$, that is,

$$[(x^i y^j)^{-1}, x^{-1}].$$

In particular, $H \leqslant F'$, so

$$H = F'.$$

Theorem 8. *Suppose H is finitely generated. Then there is a subgroup K of F such that the subgroup $\langle H \cup K \rangle$ of F is the free product $H * K$, and this is of finite index in F.*

Proof. Let Y be as in Theorem 7. We want to extend $\delta(Y)$ to a set $\delta(Y) \sqcup W$ that freely generates a subgroup of F of finite index; then K can be $\langle W \rangle$.

Let T be the set of all initial segments of those s and $\overline{sx}$ such that $(s, x) \in Y$. Since H is assumed finitely generated, T is finite.

Since S is a Schreier set, it includes T; also then, T is itself a Schreier set. If $x \in X$, define

$$T(x) = \{s \colon s \in T \wedge \overline{sx} \in T\}.$$

Then

$$s \mapsto \overline{sx} \colon T(x) \to T.$$

This function is injective, since if $\overline{sx} = \overline{tx}$, then $Hsx = Htx$, so $Hs = Ht$. Hence the function $s \mapsto \overline{sx}$ extends to a permutation φ_x of T. Since F is free on X, we obtain a right action $g \mapsto \varphi_g$ of F on T.

Suppose $(s, x) \in T \times X$. If $sx \in T$, then $sx = \overline{sx} = s \cdot \varphi_x$. If $sx^{-1} \in T$, then

$$(sx^{-1}) \cdot \varphi_x = \overline{sx^{-1}x} = \overline{s} = s,$$

and therefore $s \cdot \varphi_{x^{-1}} = (sx^{-1}) \cdot \varphi_x \cdot \varphi_{x^{-1}} = sx^{-1}$. In short, if $sx^\varepsilon \in T$, where $\varepsilon = \pm 1$, then

$$s \cdot \varphi_{x^\varepsilon} = sx^\varepsilon.$$

Now say t is an element $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ of T. Then since T is a Schreier set, we have

$$1 \cdot \varphi_t = 1 \cdot \varphi_{x_1^{\varepsilon_1}} \cdots \varphi_{x_n^{\varepsilon_n}} = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} = t.$$

Now define

$$J = \{f \colon f \in F \wedge 1 \cdot \varphi_f = 1\},$$

the stabilizer of 1 in F. Then

$$F = JT,$$

since if $1 \cdot \varphi_f = t$, then $ft^{-1} \in J$ and $f = ft^{-1}t$. Moreover, if $f, g \in J$ and $s, t \in T$, and $fs = gt$, then

$$s = 1 \cdot \varphi_{fs} = 1 \cdot \varphi_{gt} = t,$$

so $s = t$ and then $f = g$. Therefore T is a complete set of right-coset representatives of J in F. Hence it is a Schreier transversal with respect to X and J. Moreover, if $1 \cdot \varphi_f = s$, then $1 \cdot \varphi_{fs^{-1}} = 1$, so $fs^{-1} \in J$ and therefore

$$Jf = Js.$$

That is, the representative of Jf in T is $1 \cdot \varphi_f$. Hence, by Theorem 1, J is generated by the $sx \cdot (1 \cdot \varphi_{sx})^{-1}$ such that $(s, x) \in T \times X$; it is freely generated by an appropriate subset, by Theorem 7. Finally,

$$1 \cdot \varphi_{sx} = 1 \cdot \varphi_s \cdot \varphi_x = s \cdot \varphi_x = \overline{sx}.$$

So J is freely generated by certain $sx \cdot (\overline{sx})^{-1}$, that is, $\delta(s, x)$. Those generators that are not in Y generate the desired subgroup K; indeed, we have then $J = H * K$, and the index of J in F is the size of T. □

As a partial converse, we have that, if F is finitely generated, and $H * K$ is a subgroup of F of finite index, then $H * K$ is finitely generated by the corollary to Theorem 1, and therefore H (and K) must be finitely generated. We also have:

Corollary 9. *If H is finitely generated* normal *subgroup of F, then either H is trivial or H has finite index in F.*

A group is **residually finite** if the intersection of all of its subgroups of finite index is trivial: that is, every nontrivial element lies outside some subgroup of finite index.

Theorem 10 (F.W. Levi)**.** *Free groups are residually finite.*

Proof. Let $f \in F \smallsetminus \langle 1 \rangle$. By Theorem 8, there is a subset X of F such that $\langle f \rangle * \langle X \rangle$ is a subgroup H of F of finite index. Then

$$f \notin \langle f^2, X, H' \rangle,$$

and this group has index 2 in H, since H/H' is (isomorphic to) the free *abelian* group generated by $\{f\} \cup X$. $\square$

A group G is **Hopfian** its its quotient by a nontrivial normal subgroup is never isomorphic to G. For Theorem 12 below, we need:

Lemma 11. *A finitely generated group has only finitely many subgroups of a given finite index.*

Proof. Supposing G is generated by a set of size m, we consider subgroups of index n. Let H be such. We can embed G in $\mathrm{Sym}(G/H)$ by sending g to $xH \mapsto gxH$. There is a bijection from G/H to $\{0, \dots, n-1\}$, that is, to n in the von-Neumann definition, that takes H to 0. This bijection induces an isomorphism from $\mathrm{Sym}(G/H)$ to $\mathrm{Sym}(n)$. Composing, we have an embedding of G in $\mathrm{Sym}(n)$; that is, we have an action of G on n. Moreover, the stabilizer of 0 under this action is just H; so H is recovered from the embedding.

The number of embeddings of G in $\mathrm{Sym}(n)$ is at most

$$(n!)^m.$$

Therefore this is an upper bound on the number of subgroups H of G. $\square$

The foregoing proof uses the Axiom of Choice. Indeed, let A be the set of subgroups of G of index n, and let B be the set of embeddings of G in $\mathrm{Sym}(n)$. Then the proof uses an embedding of A in B that requires, for each H in A, the choice of a bijection from G/H to n.

It might appear that we could argue as follows. Suppose it is known that A has an *ordinal* cardinality κ. (Such would be the case if A were already known to be finite.) For each H in A, the set of bijections from G/H to n has cardinality $n!$. The set $\kappa \times (n!)$ has an ordinal cardinality, which is either a finite number or κ. So the set can be well-ordered, and therefore it might appear that we can obtain an embedding of A in B as desired. However, the set that we want to well-order is the set C comprising all bijections from G/H to n for all H in A. Without the Axiom of Choice, we do not have a bijection from C to $\kappa \times (n!)$.

The solution is, given H in A, to let B_H comprise those elements of B under which the stabilizer of 0 is H. We have proved that each B_H is nonempty; so the function $H \mapsto B_H$ establishes our claim with use of the Axiom of Choice.

It is of interest that the proof of Theorem 7 *does* (apparently) require the Axiom of Choice (because Lemma 3 requires it). Because of this, for a given subgroup of a free group, it may be impossible to *exhibit* a set that freely generates the subgroup (just as it is impossible to exhibit, say, a well-ordering of the set of real numbers).

Theorem 12 (Mal'cev). *A finitely generated residually finite group is Hopfian. In particular, a finitely generated free group is Hopfian.*

Proof. Suppose G is finitely generated and residually finite. Say N is a normal subgroup of G, and

$$G \cong G/N.$$

The isomorphism sends a subgroup H of G to a subgroup of G/N, and the latter subgroup must have the form H^*/N for some subgroup H^* of G such that

$$N \leqslant H^* \leqslant G.$$

We have

$$[G : H] = [G : H^*].$$

If this index is required to be a certain finite number, then, by Lemma 11, there are only finitely many possibilities for H, and then the map $H \mapsto H^*$ is just a permutation of those possibilities. This shows N is a subgroup of every subgroup of G of finite index. Therefore N is trivial. □

Theorem 13 (Nielsen)**.** *If F is free of rank n, and X generates F and has size n, then X freely generates F.*

Proof. Suppose F is freely generated by Y. Extend a bijection from Y to X to an endomorphism φ of F. Then φ must be surjective (since X generates F), so

$$F/\ker\varphi \cong F.$$

By Theorem 12, $\ker\varphi$ must be trivial, so φ is an automorphism of F. In particular, since Y freely generates F, so does X. $\square$

2. Groups acting on trees

Considering 3 under the von-Neumann definition whereby it is the set $\{0, 1, 2\}$, let X consist of the finite sequences of elements of 3. That is, X contains, for each n in ω, the n-tuples

$$(x_0, \dots, x_{n-1}),$$

where $x_i \in 3$. If $n = 0$, the only such tuple is the empty set. In general, strictly, the n-tuple above is the set

$$\{(0, x_0), \dots, (n-1, x_{n-1})\}$$

(where now (s, t) stands for $\{\{s\}, \{s, t\}\}$; but we do not need this). Then X is (partially) ordered by inclusion, so that we have

$$(y_0, \dots, y_{m-1}) \leqslant (x_0, \dots, x_{n-1})$$

if and only if $m \leqslant n$ and $(y_0, \dots, y_{m-1}) = (x_0, \dots, x_{m-1})$. In particular, as a (partially) ordered set, X is a *tree,* the lower levels of which can be depicted as in Figure 2.1.

Now let $\mathfrak{S}$ be the group of automorphisms of X as a (partially) ordered set.[1] We shall define α and τ in $\mathfrak{S}$ so that $\langle \alpha, \tau \rangle$ is an infinite 3-group. The definition of τ is easy: it is given by

$$\tau(x_0, \dots, x_n) = (x_0 + 1, \dots, x_n),$$

where the addition is *modulo* 3. Then

$$\tau^3 = 1,$$

and τ permutes the (set of) the subtrees $\{x \colon (i) \leqslant x\}$.

[1] Here $\mathfrak{S}$ is a German letter S, obtained in $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX with `\mathfrak{S}`. From `\mathfrak{G}` one gets $\mathfrak{G}$. See Appendix B, p. 60.

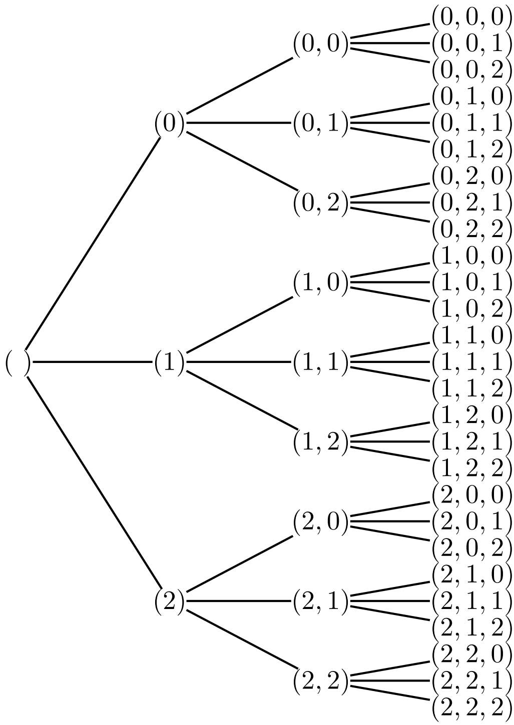

Figure 2.1. A ternary tree

To define α, we introduce the following notation. If x is an n-tuple $(x_0, \dots, x_{n-1})$ in X, and $i \in 3$, then by $i \cdot x$ we mean the $(n+1)$-tuple $(i, x_0, \dots, x_{n-1})$ in X. So τ permutes the (set of) subtrees $\{i \cdot x : x \in X\}$. If also $\beta \in \mathfrak{S}$, we define an element β_i of $\mathfrak{S}$ by requiring

$$\beta_i(i \cdot x) = i \cdot \beta(x),$$

but $\beta_i(x) = x$ if $x_i \neq i$. Now we define

$$\alpha = \tau_0 \circ {\tau_1}^2 \circ \alpha_2.$$

This is a recursive definition in the lengths of the arguments: $\alpha(\varnothing) = \varnothing$

of course, and then

$$\begin{aligned}\alpha(0\cdot x)&=0\cdot\tau(x),\\ \alpha(1\cdot x)&=1\cdot\tau^2(x),\\ \alpha(2\cdot x)&=2\cdot\alpha(x).\end{aligned}$$

By induction on those lengths,

$$\alpha^3=1.$$

Let

$$G=\langle\alpha,\tau\rangle.$$

Theorem 14 (Grigorchuk, Gupta–Sidki)**.** *G is an infinite 3-group.*

Proof. We first show G is infinite. To this end, let

$$H=\langle\alpha,\alpha^\tau,\alpha^{\tau^2}\rangle.$$

This is a normal subgroup of G. We shall show that H has a quotient (by a nontrivial subgroup) that is isomorphic to G.

For each i in 3, the group $\mathfrak{S}$ has a subgroup $\mathfrak{S}_i$, comprising those σ such that

$$\sigma(j\cdot x)=j\cdot x$$

if $j\in 3\smallsetminus\{i\}$. Then the subgroup $\langle\mathfrak{S}_0,\mathfrak{S}_1,\mathfrak{S}_2\rangle$ is an internal direct product,

$$\mathfrak{S}_0\times\mathfrak{S}_1\times\mathfrak{S}_2.$$

Also, G is a subgroup of this. A typical element of the direct product is (π_0,ρ_1,σ_2) for some π, ρ, and σ in $\mathfrak{S}$. We consider H under the projection

$$(\pi_0,\rho_1,\sigma_2)\mapsto\pi_0.$$

Conjugation in the direct product by τ is a kind of permutation of coordinates: we have

$$\begin{aligned}(\pi_0,\rho_1,\sigma_2)^\tau&=(\rho_0,\sigma_1,\pi_2),\\ (\pi_0,\rho_1,\sigma_2)^{\tau^2}&=(\sigma_0,\pi_1,\rho_2).\end{aligned}$$

Therefore

$$H = \langle (\tau_0, {\tau_1}^2, \alpha_2), ({\tau_0}^2, \alpha_1, \tau_2), (\alpha_0, \tau_1, {\tau_2}^2) \rangle,$$

so the image of H in $\mathfrak{S}_0$ is just

$$\langle \alpha_0, \tau_0 \rangle,$$

which is isomorphic to G. Since the kernel of the projection of H onto this is nontrivial, it follows that G is infinite.

Now we show that G is a 3-group, that is, every element has order a (finite) power of 3. Since $\langle H \cup \{\tau\} \rangle = G$, it is enough to establish the claim for elements of H. An arbitrary element β of this can be written as

$$\alpha^{\tau^{\varepsilon_0}} \circ \cdots \circ \alpha^{\tau^{\varepsilon_{n-1}}} \circ \tau^{\varepsilon_n}$$

for some ε_i in 3 and some minimal n in ω. Then n can be called the **length** of the element. In β^3, the factors that are powers of τ can be eliminated. In particular, if the ε_i are the same for all i in n, then $\beta^3 = 1$. If they are not all the same, consider the image of β under the projection onto $\mathfrak{S}_0$. This image will be a composition of $n+1$ powers of α_0 and τ_0; but at least two of these are powers of τ_0; these two can be combined, and so the length of the image is less than n. By induction then, the order of every element of H (and therefore of G) is a power of 3. □

3. The Burnside problem

Question 15 (Burnside, 1902). *Is there an infinite, finitely generated:*

1) torsion group?

2) group of bounded exponent?

Theorem 16 (Godod, 1964). *There is an infinite* 3*-generated* p*-group for every prime* p.

Theorem 17 (Aleshin, 1972). *There is an infinite* 2*-generated* p*-group for every prime* p.

The proof uses finite automata.

Theorem 18 (Grigorchuk, 1980). *There is an infinite* 3*-generated* 2*-group of automorphisms of the regular* 2*-tree.*

Theorem 19 (Gupta–Sidki). *There is an infinite* 2*-generated* p*-group of automorphisms of the regular* p*-tree for every odd primes* p.

Theorem 20 (Adian–Novikov, 1968). *There is an infinite* 2*-generated group of* odd *exponent* n*, whenever* $n \geqslant 4381$.

The question of whether there is an infinite 2-generated group of exponent 5 is open.

Theorem 21 (Ivanov). *There is an infinite* 2*-generated group of exponent* 2^{48}.

4. Limit groups

The reference now is

> Champetier and Guirardel, 'Limit groups as limits of free groups: compactifying the set of free groups', February 1, 2008.

We aim to show that limit groups are models of the universal theory of free groups.

Our signature $\mathscr{S}$ is $\{1, {}^{-1}, \cdot\}$, and n and m range over ω. When X is a set, then F_X is a free group on X. Also

$$F_n = F_{\{x_0,\dots,x_{n-1}\}}$$

(a free group on n generators).

Question 22 (Tarski, 1945). *Have we*

$$F_n \equiv F_m$$

when n and m are greater than 1?

Theorem 23 (Sela, Kharlampovich–Myasnikov, 2006). *Yes.*

The theory of nonabelian free groups interprets $\mathbb{Z}$ (as a group: it is the centralizer of a nontrivial element of a model), so the theory does not have finite Morley rank. It does not have infinite Morley rank either, nor U-rank. However, Sela showed that it is stable.

To solve Tarski's problem, we take an $\mathscr{S}$-sentence σ and show $F_n \models \sigma$ if and only if $F_m \models \sigma$. What can σ look like? The $\mathscr{S}$-terms are **words.** So σ has the form

$$\forall \boldsymbol{x}\, \exists \boldsymbol{y}\, \forall \boldsymbol{z}\ \cdots \bigvee_i \bigwedge_j \bigl(w_{ij}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\dots) = 1\bigr)^{\varepsilon_{ij}}$$

where $\varphi^{\varepsilon_{ij}}$ is either φ or $\neg\varphi$. The first step is to look at very simple σ:

$$\forall \boldsymbol{x} \bigwedge_j w_j(\boldsymbol{x}) = 1,$$

a *[positive] universal sentence;* here $w_j(\boldsymbol{x}) = 1$ is a **word equation.**

For the moment, let $\mathscr{S}$ be arbitrary, and let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathscr{S}$-structures. A **universal $\mathscr{S}$-sentence** is one of the form

$$\forall \boldsymbol{x}\ \psi(\boldsymbol{x}),$$

where ψ is quantifier-free. For the set of universal $\mathscr{S}$-sentences that are true in $\mathfrak{M}$, we may write[1]

$$\mathrm{Th}(\mathfrak{M})_\forall.$$

If $\mathfrak{M}$ embeds in $\mathfrak{N}$ (in particular, if $\mathfrak{M} \subseteq \mathfrak{N}$), then

$$\mathrm{Th}(\mathfrak{N})_\forall \subseteq \mathrm{Th}(\mathfrak{M})_\forall.$$

Fact 24. $\mathrm{Th}(F_n)_\forall = \mathrm{Th}(F_m)_\forall$.

Proof. F_n embeds in F_2, considered as $F_{\{x,y\}}$, under $x_k \mapsto x^{y^k}$. $\square$

Let G be a group, and let $\varphi(\boldsymbol{x})$ be the formula

$$\bigwedge_{i<m} w_i(\boldsymbol{x}) = 1.$$

We may write φ^G for the set of solutions of φ in G:

$$\varphi^G = \{\boldsymbol{a} \colon G \models \varphi(\boldsymbol{a})\}.$$

Let E be the finitely presented group

$$\langle \boldsymbol{x} |\, w_0, \dots, w_{m-1}\rangle,$$

which is a quotient F_n/N.

Fact 25. *There is a natural bijection between φ^G and* $\mathrm{Hom}(E, G)$.

[1] Our authors use $\mathrm{Univ}(\mathfrak{M})$.

Proof. If $\alpha\colon E \to G$, let

$$\boldsymbol{a} = (\alpha(x_0N), \dots, \alpha(x_{n-1}N)).$$

Then

$$w_i(\boldsymbol{a}) = \alpha(w_i(x_0N, \dots, x_{n-1}N)) = \alpha(w_i(\boldsymbol{x})N) = \alpha(N) = 1,$$

so $\boldsymbol{a} \in \varphi^G$. For the other way around, if $\boldsymbol{a} \in \varphi^G$, define β from F_n to G by

$$\beta(x_i) = a_i.$$

Since $w_i(\boldsymbol{a}) = 1$ for all i, the homomorphism β factors through N, giving an element α of $\mathrm{Hom}(E, G)$ such that $\alpha(x_iN) = a_i$. □

The foregoing is an instance of a general fact of universal algebra:

Example 4. If K is a commutative unital ring (such as a field), and $F_i \in K[X_0, \dots, X_{n-1}]$, and R is a K-algebra, then there is a bijection between

$$\{\boldsymbol{a}\colon \boldsymbol{a} \in R^n \wedge \bigwedge_{i<m} F_i(\boldsymbol{a}) = 0\}$$

and

$$\mathrm{Hom}_K(K[\boldsymbol{X}]/(F_0, \dots, F_{m-1}), R).$$

The group E has a distinguished sequence of generators, namely

$$(x_0N, \dots, x_{n-1}N),$$

which we may denote by $(s_0, \dots, s_{n-1})$. Therefore $(E, \boldsymbol{s})$ is called a *marked group.* In general, a **marked group** is the expansion of a group to a signature $\mathscr{S}(\boldsymbol{s})$, where $\boldsymbol{s}$ is a finite sequence of new constants, and the interpretations of these constants in the expanded group *generate* the group.

Example 5. In $\mathrm{Sym}(3)$, if $\sigma = (0\ 1)$ and $\tau = (0\ 1\ 2)$, then the two marked groups

$$(\mathrm{Sym}(3), \sigma, \tau), \qquad\qquad (\mathrm{Sym}(3), \tau, \sigma)$$

are non-isomorphic as such.

If $\alpha \in \mathrm{Hom}(E, F_m)$, then we have a diagram thus:

$$\begin{tikzcd}
E \arrow[r, two heads] & \alpha(E) \arrow[r, "\leqslant"] \arrow[d, leftrightarrow, "\cong"] & F_m \\
& F_k &
\end{tikzcd}$$

We therefore restrict our attention to epimorphisms. We consider all pairs (α, G), where α is a homomorphism on E, and G is its image. We may denote this pair by

$$E \overset{\alpha}{\twoheadrightarrow} G$$

(or just by $E \twoheadrightarrow G$ if the name of the epimorphism is unimportant). We define

$$(E \overset{\alpha_0}{\twoheadrightarrow} G_0) \cong (E \overset{\alpha_1}{\twoheadrightarrow} G_1)$$

if there is a group-isomorphism β from G_0 to G_1 such that the diagram

$$\begin{tikzcd}
E \arrow[r, two heads, "\alpha_0"] \arrow[dr, two heads, "\alpha_1"'] & G_0 \arrow[d, two heads, "\beta"] \\
& G_1
\end{tikzcd}$$

commutes. If $\boldsymbol{s}$ is a generating sequence of E, and $\alpha \colon E \twoheadrightarrow G$, then G is marked by $\alpha(\boldsymbol{s})$. The following are equivalent conditions on $E \overset{\alpha_0}{\twoheadrightarrow} G_0$ and $E \overset{\alpha_1}{\twoheadrightarrow} G_1$.

1. $(E \overset{\alpha_0}{\twoheadrightarrow} G_0) \cong (E \overset{\alpha_1}{\twoheadrightarrow} G_1)$.
2. $(G_0, \alpha_0(\boldsymbol{s})) \cong (G_1, \alpha_1(\boldsymbol{s}))$.
3. $\ker(\alpha_0) = \ker(\alpha_1)$.

The point is to study the set $\mathscr{G}(E)$ of all isomorphism-classes of $E \twoheadrightarrow G$. This set corresponds to the set of normal subgroups of E. We want to understand the 'closure' in $\mathscr{G}(E)$ of the set of epimorphisms $E \twoheadrightarrow F_m$, where $m \geqslant 2$.

A logical or model-theoretic way to do this is to understand $\mathscr{G}(E)$ as the set of all complete quantifier-free n-types of $\mathscr{S}$ that contain the equations

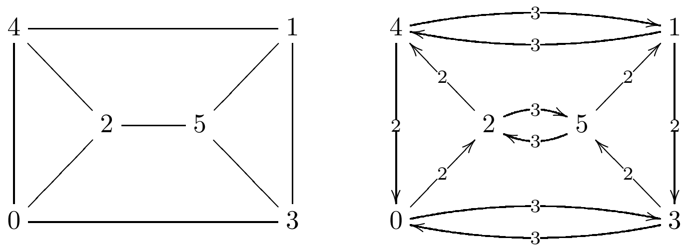

Figure 4.1. Cayley graph of $(\mathbb{Z}/6\mathbb{Z}, 2, 3)$

$w_i = 1$ along with all equations that hold universally in groups. Or we can replace the variables x_i with constants s_i; then $\mathscr{G}(E)$ consists of the quantifier-free theories in $\mathscr{S}(\boldsymbol{s})$ of quotients of E. See Appendix C, page 62. Otherwise, we can proceed as follows.

Step 1. On E there is a **word metric,** in which the distance between two elements is the length of the shortest path between them in the *Cayley graph* of $(E, \boldsymbol{s})$. For present purposes, the **Cayley graph** is a graph whose nodes are the group-elements, and where an edge is drawn between g and h if $g^{-1}h$ or $h^{-1}g$ is one of the given generators s_i.

There is also a more elaborate version of the Cayley graph: a *labelled, directed* graph, where again the nodes are the group-elements, and now an arrow is drawn from g to h, and is labelled as $g^{-1}h$, if this last group-element is one of the given generators.

Example 6. The two Cayley graphs of $(\mathbb{Z}/6\mathbb{Z}, 2, 3)$ are as in Figure 4.1.

Step 2. If (X, x, d) is a pointed metric space, the **pointed Gromov–Hausdorff metric** on $\mathscr{P}(X)$ is defined as follows. If r is a non-negative real number, two subsets C and D of X are called r**-close** if

$$C \cap \mathrm{B}(x, r) = D \cap \mathrm{B}(x, r),$$

where $\mathrm{B}(x, r)$ is the open ball about x of radius r; in this case we may write

$$C \sim_r D.$$

Then we can define $\mathrm{d}(C,D)$ as e^{-R}, where R is the supremum of the set of r such that $C \sim_r D$. Such a supremum will in fact be a maximum (unless it is ∞, that is, $C = D$). This metric is **ultrametric,** that is,

$$\mathrm{d}(C,E) \leqslant \max\bigl(\mathrm{d}(C,D), \mathrm{d}(D,E)\bigr).$$

Regardless of the metric, there is already a topology on $\mathscr{P}(X)$, the **Tychonoff topology:** This has a basis consisting of the sets $U_{A,B}$, where A and B are finite disjoint subsets of X, and for all subsets Y of X,

$$Y \in U_{A,B} \leftrightarrow A \subseteq Y \land B \cap Y = \varnothing.$$

The sets $U_{A,B}$ are both closed and open. The Tychonoff topology corresponds to the product topology on 2^X when 2—that is, $\{0,1\}$—is given the discrete topology.

Fact 26. *The Gromov–Hausdorff topology on $\mathscr{P}(X)$ is finer than, or the same as, the Tychonoff topology. If all balls around x are finite in the metric on X, then the two topologies are the same.*

Proof. Suppose A and B are arbitrary finite disjoint subsets of X. Then the set $\{\mathrm{d}(x,y)\colon y \in A \cup B\}$ has an upper bound. Let r be a strict upper bound. Then for every element C of $\mathscr{P}(X)$, every element of $\mathrm{B}(C,\mathrm{e}^{-r})$ contains just the points of $A \cup B$ that C does. Therefore

$$U_{A,B} = \bigcup_{A \subseteq C \subseteq X \smallsetminus B} \mathrm{B}(C,\mathrm{e}^{-r}).$$

Thus the Gromov–Hausdorff topology is at least as fine as the Tychonoff topology.

We have generally in the former topology

$$\begin{aligned}\mathrm{B}(C,\mathrm{e}^{-r}) &= \bigcap_{r<s}\{D\colon C \sim_s D\}\\ &= \bigcap_{r<s}\{D\colon\ \mathrm{B}(x,s) \cap C = \mathrm{B}(x,s) \cap D\}.\end{aligned}$$

Now suppose each open ball $\mathrm{B}(x,s)$ in X is finite. Then

$$\inf\{\mathrm{d}(x,y)\colon y \notin \mathrm{B}(x,r)\} > r.$$

Let s be the infimum (which is a minimum). Then

$$\mathrm{B}(C, \mathrm{e}^{-r}) = \{D\colon\ \mathrm{B}(x,s) \cap C = \mathrm{B}(x,s) \cap D\}.$$

But we may also let

$$A = \mathrm{B}(x,s) \cap C, \qquad\qquad B = \mathrm{B}(x,s) \smallsetminus C,$$

both finite sets. Then

$$\mathrm{B}(C, \mathrm{e}^{-r}) = U_{A,B}. \qquad \square$$

In particular, when X is the Cayley graph of $(E, \boldsymbol{s})$ with the word metric, and x is 1, then the Tychonoff and the Gromov–Hausdorff topologies coincide.

[This shows that the Gromov–Hausdorff topology on $\mathscr{G}(E)$ is independent of the choice of $\boldsymbol{s}$.

[If not all balls around x in X are finite, then the Gromov–Hausdorff topology on $\mathscr{P}(X)$ may be strictly finer than the Tychonoff topology. For example, if we give X the discrete metric, then the Gromov–Hausforff topology on $\mathscr{P}(X)$ is also discrete, so it is different from the Tychonoff topology unless X is itself finite.

[If X is countable, then every bijection with the set of positive integers induces a metric on X, hence a topology on $\mathscr{P}(X)$; and the topology is independent of the choice of bijection.]

Step 3. $\mathscr{G}(E)$, considered as the set of normal subgroups of E, is closed in $\mathscr{P}(E)$, hence compact.

Proof. Let $A \in \mathscr{P}(E) \smallsetminus \mathscr{G}(E)$.

Case 1. If $A = \varnothing$, then

$$A \in U_{\varnothing,\{1\}},$$

and no elements of this set are groups.

Case 2. If A is neither empty nor a subgroup of E, then it contains some a and b such that it does not contain ab^{-1}. Then

$$A \in U_{\{a,b\},\{ab^{-1}\}},$$

and no elements of this set are groups.

Case 3. If A is a subgroup of E, but not a normal subgroup, then it contains some a, but not a^g, for some g in E. Then

$$A \in U_{\{a\},\{a^g\}},$$

and no elements of this set are normal subgroups of E. □

Question 27. *How does $\mathscr{G}(E)$ change with E?*

Each element of E is $w(\boldsymbol{s})$ for some w in F_n. For each element g of E, and for each normal subgroup N of G, we have

$$g \in N \quad \text{if and only if} \quad G/N \models g = 1.$$

More generally, if A and B are disjoint finite subsets of E, let σ be the sentence

$$\bigwedge_{g \in A} g = 1 \wedge \bigwedge_{h \in B} h \neq 1.$$

Then

$$N \in U_{A,B} \quad \text{if and only if} \quad G/N \models \sigma.$$

Using the notation of Appendix C, we may write $U_{A,B}$ as

$$[\sigma].$$

A sequence $\big((G_m, \boldsymbol{s}_m) \colon m \in \omega\big)$ of marked groups whose isomorphism-classes are in $\mathscr{G}(E)$ converges to $(G, \boldsymbol{s})$ if and only if, for each word w in F_n, there is some M in ω such that, for all m in ω, if $m \geqslant M$, then

$$G_m \models w(\boldsymbol{s}_m) = 1 \quad \text{if and only if} \quad G \models w(\boldsymbol{s}) = 1.$$

Example 7.

1. $\lim_{m \to \infty} \langle \boldsymbol{x} | \, r_0, \dots, r_{m-1} \rangle = \langle \boldsymbol{x} | \, r_0, r_1, \dots \rangle$.

2. $\lim_{m \to \infty} (\mathbb{Z}/m\mathbb{Z}, 1) = (\mathbb{Z}, 1)$.

3. $\lim_{m \to \infty} (\mathbb{Z}, 1, m) = \big(\mathbb{Z}^2, (1,0), (0,1)\big)$.

If $\alpha\colon E' \twoheadrightarrow E$, this induces an embedding α^* of $\mathscr{G}(E)$ in $\mathscr{G}(E')$, where

$$\alpha^*(N) = \alpha^{-1}[N].$$

[The following is Lemma 2.2 of the paper; the proof there is left as an exercise. Examples like the foregoing are given *after* the lemma, but it seems better to put them before, as the proof of Lemma 2.2 can use convergent sequences.]

Fact 28. *Consider α as an epimorphism of marked groups.*

1. *α^* is a homœomorphism onto its image.*

2. *α^* is open if and only if* $\ker(\alpha)$ *is finitely generated as a normal subgroup.*

Proof. Since α is surjective, α^* is injective. The Cayley graph of $(E, \boldsymbol{s})$ results from identifying two nodes g and g_1 of $(E', \boldsymbol{s}')$ if $\alpha(g) = \alpha(g')$: therefore, for all g and h in E', there are g_1 and h_1 in E' such that

$$\alpha(g) = \alpha(g_1), \qquad \alpha(h) = \alpha(h_1), \qquad \mathrm{d}(\alpha(g), \alpha(h)) = \mathrm{d}(g_1, h_1).$$

Thus α^* is in fact an isometry.

If $\ker(\alpha)$ is infinitely generated as a normal subgroup, it is the limit of a strictly increasing sequence of normal subgroups; then none of these normal subgroups are in $\alpha^*[\mathscr{G}(E)]$, although $\ker \alpha$ itself is; so $\alpha^*[\mathscr{G}(E)]$ is not open.

Suppose now $\ker(\alpha)$ *is* finitely generated as a normal subgroup of E'. Let $(N_k\colon k \in \omega)$ be a sequence of elements of $\mathscr{G}(E')$ that converges to an element of $\alpha^*[\mathscr{G}(E)]$. Every normal subgroup N of E' is determined by $\alpha[N]$ and $\ker(\alpha) \cap N$, and if the latter is just $\ker(\alpha)$ itself, then $N \in \alpha^*[\mathscr{G}(E)]$. The sequence $(\ker(\alpha) \cap N_k\colon k \in \omega)$ converges to $\ker(\alpha)$. Therefore, if k is large enough, then N_k contains the generators of $\ker(\alpha)$, so it includes $\ker(\alpha)$ and is therefore in $\alpha^*[\mathscr{G}(E)]$. Thus every point of this set is an interior point, so the set is open. □

Apply this to $F_n \overset{\pi}{\twoheadrightarrow} E$: here π^* is open if (and only if) E is finitely presented. So we might as well replace $\mathscr{G}(E)$ with $\mathscr{G}(F_n)$.

A marked group is a **limit group** if it is in the closure of the set of all (marked) free groups in $\mathscr{G}(F_n)$, where F_n has the marking $(x_0, \dots, x_{n-1})$. A group is a limit group if it has a marking in which it is a limit group.

The following is Proposition 5.2 of the article [but we can use it to understand the examples below, which are based on those given in §2.6 of the paper].

Proposition 29. *If σ is a universal sentence of $\mathscr{S}(\boldsymbol{s})$, then the set of models of σ in $\mathscr{G}(F_n)$ is a closed subset; if existential, then open.*

Proof. Suppose σ is $\exists \boldsymbol{v}\ \psi(\boldsymbol{v})$, where ψ is quantifier-free, and $(G, \boldsymbol{s}) \models \sigma$. Then for some tuple $\boldsymbol{w}$ of words,

$$(G, \boldsymbol{s}) \models \psi(\boldsymbol{w}(\boldsymbol{s})),$$

and every model of this sentence is a model of σ. That is, in the notation of Appendix C as introduced above, we have

$$(G, \boldsymbol{s}) \in [\psi(\boldsymbol{w}(\boldsymbol{s}))],$$

and the open set $[\psi(\boldsymbol{w}(\boldsymbol{s}))]$ is included in the set of all models of σ in $\mathscr{G}(F_n)$. □

Example 8. The sets of (marked) abelian, k-nilpotent, and k-soluble groups are closed. The set of abelian groups is also open, since it consists of the groups satisfying

$$\bigwedge_{i,j} [s_i, s_j] = 1.$$

Similarly, the set of 2-nilpotent groups is open. Indeed, a group is 2-nilpotent if and only if its every element commutes with every element of the commutator subgroup. It is enough to check this property at generators; therefore the (marked) 2-nilpotent groups are those that satisfy

$$\bigwedge_{i,j,k} [[s_i, s_j], s_k] = 1.$$

Similarly, the k-nilpotent groups are those satisfying

$$\bigwedge_{i(0),\dots,i(k-1)} [[\dots[[s_{i(0)}, s_{i(1)}], s_{i(2)}], \dots], s_{i(k-1)}].$$

But the 2-soluble groups are those whose commutator subgroups are abelian, and it is not clear that the commutator subgroup is generated by the commutators *of the generators,* that is, by the $[s_i, s_j]$; so we do not have that the set of marked 2-soluble groups is open. In fact it is *not* open, provided one grants:

- there are free objects in the category of finitely generated 2-soluble groups;
- those free objects are infinitely presented groups.

Then there is a 2-soluble group G, namely $\langle s_0, \dots,_{n-1} \mid r_0, \dots \rangle$, such that:

- G is not equal to G_m, namely $\langle s_0, \dots,_{n-1} \mid r_0, \dots, r_{m-1} \rangle$, for any m;
- every 2-soluble group on n generators is a quotient of G.

In particular, G_m is not 2-soluble; but G is the limit of the G_m.

Having k-torsion is an existential property, so the set of groups with this property is open, and the set of torsion-free groups is closed. In particular, limit groups are torsion-free.

The following is Proposition 5.3 of the paper.

Theorem 30. *If G is a finitely generated group, and H is a group such that*

$$\mathrm{Th}(H)_\forall \subseteq \mathrm{Th}(G)_\forall,$$

then for any tuple $\boldsymbol{s}$ that generates G, the marked group $(G, \boldsymbol{s})$ is a limit of marked subgroups of H.

Proof. Every open neighborhood of $(G, \boldsymbol{s})$ is $[\varphi(\boldsymbol{s})]$ for some quantifier-free formula φ of $\mathscr{S}$. Then

$$(G, \boldsymbol{s}) \models \varphi(\boldsymbol{s}), \qquad G \models \exists \boldsymbol{x}\ \varphi(\boldsymbol{x}), \qquad H \models \exists \boldsymbol{x}\ \varphi(\boldsymbol{x}),$$

so $H \models \varphi(\boldsymbol{c})$ for some $\boldsymbol{c}$. Thus $[\varphi(\boldsymbol{s})]$ contains $(\langle \boldsymbol{c} \rangle, \boldsymbol{c})$. □

This allows us to prove:

Theorem 31. *For all finitely generated groups G, the following are equivalent.*

1. *G is a limit group.*
2. $\mathrm{Th}(F_2)_\forall \subseteq \mathrm{Th}(G)_\forall$.
3. *For any finite marking $\boldsymbol{s}$ of G, the marked group $(G, \boldsymbol{s})$ is a limit group.*

Proof. Trivially, (3) → (1).

(1) → (2). Suppose $\sigma \in \mathrm{Th}(F_2)_\forall$ and $(G, \boldsymbol{s})$ is a limit group. Let

$$D = \{(H, \boldsymbol{s}_H) \colon H \models \sigma\}.$$

For any marking $\boldsymbol{s}'$ of the same length as $\boldsymbol{s}$, for any m, $(F_m, \boldsymbol{s}') \in D$ (since F_m and F_2 have the same universal theory). Since $(G, \boldsymbol{s})$ is in the closure of such marked free groups and D is closed, we have $(G, \boldsymbol{s}) \in D$. Therefore $\sigma \in \mathrm{Th}(G)_\forall$.

(2) → (3). This is a special case of the last theorem. □

The following is Proposition 2.20—called 'marked subgroups'—of the paper.

Theorem 32. *Suppose the marked groups $(G_m, \boldsymbol{s}_m)$ converge to $(G, \boldsymbol{s})$ in $\mathscr{G}(F_n)$, and H is a subgroup of G that is generated by $\boldsymbol{t}$, which is $\boldsymbol{v}(\boldsymbol{s})$ for some tuple $\boldsymbol{v}$ of words from F_n. Then the marked groups $(\langle \boldsymbol{v}(\boldsymbol{s}_m) \rangle, \boldsymbol{v}(\boldsymbol{s}_m))$ converge to $(H, \boldsymbol{t})$.*

Proof. Write $\boldsymbol{t}_m$ for $\boldsymbol{v}(\boldsymbol{s}_m)$. For all words w in as many letters as the length of $\boldsymbol{t}$, if m is large enough, then the following are equivalent:

$$\begin{gathered}(H,\boldsymbol{t}) \models w(\boldsymbol{t}) = 1,\\ (G,\boldsymbol{t}) \models w(\boldsymbol{t}) = 1,\\ (G,\boldsymbol{s}) \models w(\boldsymbol{v}(\boldsymbol{s})) = 1,\\ (G_m,\boldsymbol{s}_m) \models w(\boldsymbol{v}(\boldsymbol{s}_m) = 1,\\ (G_m,\boldsymbol{t}_m) \models w(\boldsymbol{t}_m) = 1,\\ (\langle \boldsymbol{t}_m\rangle,\boldsymbol{t}_m) \models w(\boldsymbol{t}_m) = 1.\end{gathered}$$ □

As a special case, we obtain Proposition 3.1(4) of the paper:

Theorem 33. *Every 2-generated subgroup of a limit group is either a free group or a free abelian group, that is, it is isomorphic to one of*

$$\langle 1\rangle, \qquad \mathbb{Z}, \qquad \mathbb{Z}\times\mathbb{Z}, \qquad F_2.$$

Proof. Let H be a subgroup $\langle a,b\rangle$ of a limit group. By the last theorem, (H,a,b) is the limit of a sequence of marked free groups $(F_{n(m)},a_m,b_m)$ (where $n(m) \leqslant 2$). Suppose H is not free of rank 2. Then

$$(H,a,b) \models w(a,b) = 1$$

for some nontrivial word w. Hence if m is large enough, we have

$$(F_{n(m)},a_m,b_m) \models w(a_m,b_m) = 1.$$

By Theorem 13 above, $F_{n(m)}$ must not be free of rank 2; so $n(m) = 1$, that is, $F_{n(m)}$ is isomorphic to $\mathbb{Z}$. In particular, $[a_m,b_m] = 1$, so $[a,b] = 1$, that is, H is abelian. We have already observed that limit groups are torsion-free; in particular, H is torsion-free. □

Corollary 34. *If G is a non-abelian limit group, then F_2 embeds in G.*

Now we have what is Theorem 5.1 of the paper:

Theorem 35. *Suppose G is a finitely generated nonabelian group. The following are equivalent.*

1. *G is a limit group.*

2. $\mathrm{Th}(G)_\forall = \mathrm{Th}(F_2)_\forall$.

Proof. We have (2) → (1) by Theorem 31 above. If G is a nonabelian limit group, then by the same theorem we have $\mathrm{Th}(F_2)_\forall \subseteq \mathrm{Th}(G)_\forall$, and we have the reverse inclusion by the last corollary. □

For ultraproducts and Łoś's Theorem, see Appendix D. Given an infinite sequence of groups, we define the normal subgroup N of the direct product as there.

Theorem 36. *If a sequence of marked groups $(G_m, \boldsymbol{s}_m)$ converges to $(G, \boldsymbol{s})$, then there is an embedding of G in the ultraproduct $\prod_{m\in\omega} G_m/N$, namely*

$$w(\boldsymbol{s}) \mapsto (w(\boldsymbol{s}_m) \colon m \in \omega) \cdot N$$

Proof. If the putative embedding is well-defined, it is a homomorphism. Therefore it *is* well-defined, since if $w(\boldsymbol{s}) = 1$, then $\{m \colon w(\boldsymbol{s}_m) = 1\}$ is cofinite and therefore large. Similarly it is an embedding, since if $w(\boldsymbol{s}) \neq 1$, then $\{m \colon w(\boldsymbol{s}_m) \neq 1\}$ is cofinite and therefore large. □

A **non-standard free group** is a nonprincipal ultraproduct of nonabelian free groups. Let us denote such an ultraproduct by

$$^*F.$$

By Łoś's Theorem,

$$\mathrm{Th}(^*F)_\forall = \mathrm{Th}(F_2)_\forall.$$

If $G \leqslant {}^*F$, then $\mathrm{Th}(^*F)_\forall \subseteq \mathrm{Th}(G)_\forall$; therefore G is a limit group. The previous theorem gives the converse:

Corollary 37. *Every limit group embeds in $^*F$.*

5. Limit groups 2

5.1. Topologies

Given a group E, we let

$$\mathscr{G}(E) = \{\text{quotient groups of } E\}.$$

We have topologized this set in two equal ways:

1. If $G \in \mathscr{G}(E)$, then
$$G = E/1^G.$$

Hence we can give $\mathscr{G}(E)$ the **Tychonoff topology** by embedding it in $\mathscr{P}(E)$ under
$$G \mapsto 1^G.$$

If A and B are disjoint finite subsets of E, then $\mathscr{G}(E)$ has the basic open set
$$U_{A,B}, \text{ which is } \{G\colon A \subseteq 1^G \land B \cap 1^G = \varnothing\}.$$

No presentation of E is required. (This is the advantage of the Tychonoff topology.)

Say $D \in \mathscr{G}(E)$. If $G \in \mathscr{G}(D)$, then

$$G = D/1^G = (E/1^D)/(N/1^D) \cong E/N,$$

where
$$N = \bigcup 1^G.$$

Thus $\mathscr{G}(D)$ embeds in $\mathscr{G}(E)$ under

$$G \mapsto E/\bigcup 1^G.$$

But the induced topology on $\mathscr{G}(D)$ is the same as its own Tychonoff topology.

2. Let

$$\mathscr{S} = \{1, {}^{-1}, \cdot\}.$$

Suppose E is presented as $\langle S|\, R\rangle$. We may suppose

- S is a set of new constants,
- R is a set of closed terms of $\mathscr{S}(S)$.

Let

$$T_{S,R}$$

be the quantifier-free theory of groups in $\mathscr{S}(S)$, with an additional axiom

$$w = 1$$

for each w in R. Then $E \models T_{S,R}$, but $T_{S,R}$ is generally not complete. Let

$$\mathrm{S}(T_{S,R})$$

be the space of quantifier-free completions of $T_{S,R}$. Then $\mathscr{G}(E)$ is in bijection with this space under

$$G \mapsto \mathrm{diag}_S(G),$$

where $\mathrm{diag}_S(G)$ is the quantifier-free theory of G in $\mathscr{S}(S)$. (It is practically the **Robinson diagram** of G.) Then $\mathscr{G}(E)$ inherits the **Stone topology:** if σ is a quantifier-free sentence of $\mathscr{S}(S)$, then $\mathscr{G}(E)$ has a basic open set

$$[\sigma], \text{ which is } \{G\colon G \models \sigma].$$

If $g \in E$, then $g = {w_g}^E$ for some closed term w_g of $\mathscr{S}(S)$. Then

$$U_{A,B} = [\sigma],$$

where

$$\sigma \text{ is } \bigwedge_{a\in A} w_a = 1 \wedge \bigwedge_{b\in B} w_b \neq 1.$$

So the Tychonoff and Stone topologies are the same. Elements of $\mathscr{G}(E)$ are **marked** by S: they are groups in $\mathscr{S}(S)$ that are generated by their interpretations of S.

We can now consider elements of $\mathscr{G}(E)$ as isomorphism-classes of groups marked by S. (This is the advantage of the Stone topology.)

The foregoing holds for any group E. However, if E has a metric d, then $\mathscr{G}(E)$ has the **Gromov–Hausdorff topology.** In this, for each positive real number r, an element G of $\mathscr{G}(E)$ has an open neighborhood comprising those H in $\mathscr{G}(E)$ such that, for all g in E,

$$g \in 1^G \vartriangle 1^H \to \mathrm{d}(g, h) \geqslant r.$$

The Gromov–Hausdorff topology on $\mathscr{G}(E)$ agrees with the Tychonoff topology if and only if every ball around 1 in (E, d) is finite. Such is the case when E is finitely generated, and d is the **word metric.** But even if E is not finitely generated, but is still countable, an appropriate metric can be defined, as for example by means of a bijection with $\mathbb{Z}$.

In any case, we shall just use the Tychonoff topology.

If E is countable, then the $\mathscr{G}(E)$ topology is *first countable* (points have countable neighborhood bases; in fact the topology is *second countable*—itself has a countable basis). So the following are equivalent conditions on a subset A and a point P of $\mathscr{G}(E)$:

1. $P \in \mathrm{c}\ell(A)$.
2. A sequence $(P_k : k \in \omega)$ of points of A converges to P.

Theorem 38. *Given a set Ω and a sequence $(A_k : k \in \omega)$ of elements of $\mathscr{P}(\Omega)$, we have*

$$\liminf_k A_k = \bigcup_{k\in\omega} \bigcap_{k\leqslant m} A_m \subseteq \bigcap_{k\in\omega} \bigcup_{k\leqslant m} A_m = \limsup_k A_k.$$

[The equations *are by definition.] The two limits are equal to the same set A if and only if the sequence converges to A in the Tychonoff topology on $\mathscr{P}(\Omega)$. In particular, the sequence converges to A in either of two cases:*

1. *$A_0 \subseteq A_1 \subseteq \dots$ and $\bigcup_{k\in\omega} A_k = A$.*
2. *$A_0 \supseteq A_1 \supseteq \dots$ and $\bigcap_{k\in\omega} A_k = A$.*

Proof. Say $\liminf_k A_k = A$. For every open neighborhood $U_{X,\varnothing}$ of A, we have $X \subseteq A$, and therefore (since X is finite) $A_m \in U_{X,\varnothing}$ if m is large enough.

Similarly, if $\limsup_k A_k = B$, then for every neighborhood $U_{\varnothing,Y}$ of B, we have $Y \cap B = \varnothing$, so $A_m \in U_{\varnothing,Y}$ if m is large enough.

Finally, in any case,

$$U_{X,Y} = U_{X,\varnothing} \cap U_{\varnothing,Y}.$$

Therefore, if $A = B$, then $(A_k : k \in \omega)$ converges to this.

But suppose now $c \in B \smallsetminus A$.

- If $c \in C$, then $C \in U_{\{c\},\varnothing}$, but for all k in ω, there is m such that $k \leqslant m$, and $c \notin A_m$, so $A_m \notin U_{\{c\},\varnothing}$.
- If $c \notin C$, then $C \in U_{\varnothing,\{c\}}$, but for all k in ω, there is m such that $k \leqslant m$, and $c \in A_m$, so $A_m \notin U_{\varnothing,\{c\}}$. □

We have examples of each of the special cases:

1. $\lim_{m\to\infty} \langle \omega | m \rangle = \langle \omega | \omega \rangle = \langle \; \rangle$, that is,

$$\lim_{m\to\infty} \langle s_0, s_1, \dots | s_0, \dots, s_{m-1} \rangle = \{0\}.$$

2. $\lim_{m\to\infty} \langle \omega | \omega \smallsetminus m \rangle = \langle \omega | \; \rangle = F_\omega$, that is,

$$\lim_{m\to\infty} \langle s_0, s_1, \dots | s_m, s_{m+1}, \dots \rangle = F_\omega.$$

There is no convergence here in the Gromov–Hausdorff topology induced by the word metric. We are in a more general situation with:

3. $\lim_{m\to\infty} (\mathbb{Z}/m\mathbb{Z}, 1 + m\mathbb{Z}) = (\mathbb{Z}, 1)$, because we can write

$$\mathbb{Z}/m\mathbb{Z} = \langle s | s^m \rangle, \qquad \mathbb{Z} = \langle s | \; \rangle$$

(this makes the claim meaningful), and

$$\begin{aligned} \langle \; \rangle = \{0\} &\subseteq \bigcap_{k\in\omega} \bigcup_{k\leqslant m} m\mathbb{Z} \\ &\subseteq \bigcap_{k\in\omega} \Big(\mathbb{Z} \smallsetminus \big(\{-k+1, -k+2, \dots, -1\} \cup \{1, 2, \dots, k-1\} \big) \Big) = \{0\}. \end{aligned}$$

The last is an example of:

Theorem 39. *Suppose* $\mathscr{A} \subseteq \mathscr{G}(E)$. *If* $E \in \mathrm{c}\ell(\mathscr{A})$, *then*

$$\bigcap_{G \in \mathscr{A}} 1^G = \langle\ \rangle.$$

Finitely presented groups that are limits can always be considered as being of the type of the last example. The following corresponds to Lemma 2.3 of Champetier and Guirardel [2]; with the Stone topology, it becomes a triviality.

Theorem 40. *Suppose* $E = \langle S \mid R \rangle$, *where* R *is finite. Then* $\mathscr{G}(E)$ *is a neighborhood of* E *in* $\mathscr{G}(F_S)$. *Thus if* $\mathscr{A} \subseteq \mathscr{G}(F_S)$ *and* $E \in \mathrm{c}\ell(\mathscr{A})$, *then*

$$\bigcap_{G \in \mathscr{A} \cap \mathscr{G}(E)} 1^G = \langle\ \rangle.$$

Proof. E has the neighborhood $[\sigma]$ in $\mathscr{G}(F_S)$, and $[\sigma] \subseteq \mathscr{G}(E)$, where

$$\sigma \text{ is } \bigwedge_{w \in R} w = 1. \qquad \square$$

The converse of Theorem 39 fails: if G_m is $\langle \omega \mid \omega \smallsetminus \{m\} \rangle$ in $\mathscr{G}(F_\omega)$, then

$$\bigcap_{m \in \omega} 1^{G_m} = \langle\ \rangle, \qquad \lim_{m \to \infty} G_m = \langle \omega \mid \omega \rangle \neq F_\omega.$$

By definition, a **limit group** is the limit of a sequence of free groups in some $\mathscr{G}(F_n)$ (where $n \in \omega$). If E is a *finitely presented* limit group, then, by Theorem 40, it is the limit of a sequence of free groups in $\mathscr{G}(E)$. Thus, being a finitely presented limit group is invariant under isomorphism. Champetier and Guirardel [2] show this for all limit groups in §2.7, Corollary 2.18. We shall have it instead as a consequence of Theorems 48 and 57.

5.2. Ultrapowers

[Not presented; just a summary of Appendix D.]

A **(nonprincipal) ultrapower** of a group G is a quotient G^ω/N, understood as follows.

The group G^ω, or $\prod_\omega G$, is the group of sequences $(g_k : k \in \omega)$, where $g_k \in G$ in each case. We can write $(g_k : k \in \omega)$ as g. The **support** of g is the set

$$\{k : g_k \neq 1\};$$

we can write this as $\operatorname{supp}(g)$.

Lemma 41. *Among subgroups H of G such that*

$$g \in H \to \operatorname{supp}(g) \subset \omega,$$

there is a maximal instance, N, which is normal; and we may require

$$\sum_\omega G \leqslant N.$$

Proof. Let $\mathfrak{m}$ be a maximal ideal of $\mathscr{P}(\omega)$ as a Boolean ring:

$$A \in \mathfrak{m} \wedge B \in \mathfrak{m} \leftrightarrow \omega \smallsetminus (A \cup B) \notin \mathfrak{m}.$$

Elements of $\mathfrak{m}$ can be called **small;** of $\mathscr{P}(\omega) \smallsetminus \mathfrak{m}$, **large.** Then we can let N be the set of elements of G with small support:

$$N = \{g : g \in G^\omega \wedge \operatorname{supp}(g) \in \mathfrak{m}\}.$$

To meet the second condition, we require $\mathfrak{m}$ to be non-principal. $\square$

When N is as in the lemma, G^ω/N is a non-principal ultrapower of G, and we may denote it by

$$^*G.$$

The proof of the lemma suggests how to replace G with an arbitrary structure, such as a ring.

Theorem 42 (Łoś). *The homomorphism $g \mapsto (g, g, \dots) \cdot N$ from G to ${}^*G$ is an elementary embedding:*

$$G \prec {}^*G.$$

Likewise with arbitrary structures for G.

Piotr showed:

Theorem 43. *The limit groups are just the finitely generated subgroups of ${}^*F_2$.*

5.3. Residual properties

Say $\mathfrak{X}$ is an adjective of groups that is true of the trivial group. More precisely, $\mathfrak{X}$ is a class of groups that is closed under isomorphism and contains $\langle\ \rangle$. Some quotients E/N are $\mathfrak{X}$. The quotient

$$E/\bigcap_{E/N\in\mathfrak{X}} N$$

may not be $\mathfrak{X}$; but it is called **residually** $\mathfrak{X}$. Thus, E is residually $\mathfrak{X}$ if and only if

$$(*) \qquad \bigcap_{E/N\in\mathfrak{X}} N = \langle\ \rangle;$$

equivalently, for all g in $E \smallsetminus \{1\}$, there is an epimorphism φ from E to an $\mathfrak{X}$ group H such that

$$g \notin \ker(\varphi).$$

Note that $(*)$ can be written as

$$\bigcap_{G\in\mathfrak{X}\cap\mathscr{G}(E)} 1^G = \langle\ \rangle.$$

Therefore, by Theorem 39, we have:

Theorem 44. *A group is residually $\mathfrak{X}$ if it is a limit of $\mathfrak{X}$ quotients of itself.*

We shall see that the converse fails. Meanwhile, by Theorem 40, we have the following, which Champetier and Guirardel [2, §2.4 (e)] state in case $\mathfrak{X}$ is free.

Theorem 45. *Finitely presented groups that are limits of $\mathfrak{X}$ groups are residually $\mathfrak{X}$.*

In particular, finitely presented limit groups are residually free. We shall show (Theorem 57) that *all* limit groups are residually free.

Ali showed the following. How easily is it obtained from Theorem 44?

Theorem 46. *Free groups are residually finite.*

One has the following, for example, though it is obvious for free groups. (It is found in Derek J.S. Robinson [5, 2.2.5].)

Theorem 47. *The word problem of a finitely presented residually finite group is soluble.*

Proof. Say $G = \langle s_0, \dots, s_{n-1} | w_0, \dots, w_{p-1} \rangle$. Given w in F_n, writing G as F_n/N, we want to know whether $w \in N$.

1. We can effectively list the elements of N as $(g_k : k \in \omega)$. (This requires only that G be recursively presented.)

2. We can effectively list, as $(a_k : k \in \omega)$, the images of w under homomorphisms from G into (finite) groups $(\{0, \dots, m\}, \cdot)$, where $n \in \omega$.

Either $w \in N$, so $g_k = w$ for some k; or $w \notin N$, so $a_k \neq 1$ for some k. $\square$

The group E is **fully residually** $\mathfrak{X}$, or **ω-residually** $\mathfrak{X}$, if for all finite subsets A of $E \smallsetminus \{1\}$ there is an epimorphism φ from E to an $\mathfrak{X}$ group such that
$$A \cap \ker(\varphi) = \varnothing;$$
equivalently, there is G in $\mathscr{G}(E)$ such that
$$G \in \mathfrak{X}, \qquad A \cap 1^G = \varnothing,$$
that is, $\mathfrak{X} \cap U_{\varnothing,A}$ is nonempty in $\mathscr{G}(E)$. In this space, E has a neighborhood base consisting of such sets $U_{\varnothing,A}$. Now Theorem 44 leads to what is basically a restatement of the new definition:

Theorem 48. *A group G is fully residually $\mathfrak{X}$ if and only if it is a limit point of $\mathfrak{X} \cap \mathscr{G}(G)$.*

In particular, finitely generated fully residually free groups are limit groups. The converse is true for *finitely presented* limit groups, by Theorem 40. We shall show that the converse is true generally in Theorem 57.

Meanwhile, for which $\mathfrak{X}$ can a group be residually $\mathfrak{X}$, but not fully?

Theorem 49. *Residually $\mathfrak{X}$ groups are fully residually $\mathfrak{X}$, provided $\mathfrak{X}$ is closed under subgroups and (binary) products.*

Proof. Under the hypothesis, suppose G is residually $\mathfrak{X}$, and $G \smallsetminus \{1\}$ contains g_0 and g_1. Then G has normal subgroups N_i such that $g_i \notin N_i$. But $G/(N_0 \cap N_1)$ embeds in $G/N_0 \times G/N_1$. □

In particular, residually finite groups are fully residually finite. (So free groups are fully residually finite.) Champetier and Guirardel [2, §2.4(e)] show that residually finite groups are limits of finite groups, by using the Gromov–Hausdorff topology; but the claim is immediate from the definition, understood as Theorem 48.

Theorem 49 does not apply to residually free groups. Indeed, the group

$$F_2 \times \mathbb{Z}, \text{ which is } \langle x, y, z \mid [x, z], [y, z] \rangle,$$

is residually free, but not fully residually free: the three generators, along with $[x, y]$, cannot be mapped nontrivially into a free group. The result is in the paper [1] of Benjamin Baumslag based on his doctoral thesis of 1965 at London University. The result is a consequence of his theorem that a residually free group is fully residually free if and only if it is residually free and **commutative transitive:**

$$[x, y] = 1 \wedge [y, z] = 1 \wedge y \neq 1 \to [x, z] = 1.$$

Champetier and Guirardel repeat this as [2, Thm 2.12]. Benjamin Baumslag is the younger brother of Gilbert Baumslag, who was Ali's reference.

Again, finitely presented limit groups are fully residually free. To show that all limit groups are fully residually free (Theorem 57), we shall use an application of residualness to rings:

Theorem 50 (Hilbert's Nullstellensatz)**.** *A finitely generated integral domain that includes a field K is fully residually algebraic over K.*

Proof. Suppose $K[x_0, \dots, x_{n-1}]$ is an integral domain, and

$$\{f_0(\boldsymbol{x}), \dots, f_{p-1}(\boldsymbol{x})\} \subseteq K[\boldsymbol{x}] \smallsetminus \{0\}.$$

By Hilbert's Basis Theorem,

$$K[\boldsymbol{x}] \cong K[\boldsymbol{X}]/(g_0, \dots, g_{q-1})$$

for some g_k. The system

$$\bigwedge_{j<p} f_j(\boldsymbol{X}) \neq 0 \wedge \bigwedge_{k<q} g_k(\boldsymbol{X}) = 0$$

is solved by $\boldsymbol{x}$ in $K[\boldsymbol{x}]$ and hence in $K[\boldsymbol{x}]^{\mathrm{alg}}$; therefore it has a solution $\boldsymbol{t}$ in K^{alg}, since

$$K^{\mathrm{alg}} \preccurlyeq K[\boldsymbol{x}]^{\mathrm{alg}}$$

by the model-completeness of ACF. Then the rule $x_i \mapsto t_i$ determines a well-defined K-homomorphism φ from $K[\boldsymbol{x}]$ into K^{alg} such that

$$\varphi(f_j(\boldsymbol{x})) \neq 0. \qquad \square$$

Then we have the following **porism,** which is Champetier and Guirardel [2, Lemma 6.7] (attributed to Remeslennikov). As Proclus [4] writes,

> 'Porism' is a term applied to a certain kind of problem, such as those in the *Porisms* of Euclid. But it is used in its special sense when as a result of what is demonstrated some other theorem comes to light without our propounding it. Such a theorem is therefore called a 'porism',[1] as being a kind of incidental gain resulting from the scientific demonstration. [212]

[1] From πορίζω, 'furnish', 'provide' [translator's note].

> 'Porism' is a geometrical term and has two meanings. We call 'porism' a theorem whose establishment is an incidental result of the proof of another theorem, a lucky find as it were, or a bonus for the inquirer. Also called 'porisms' are problems whose solution requires discovery, not merely construction or simple theory. [301]

Porism 51. *A finitely generated sub-ring of* $^*\mathbb{Z}$ *is fully residually* $\mathbb{Z}$ *(that is, infinite cyclic).*

Proof. The sub-ring is $\mathbb{Z}[x_0, \dots, x_{n-1}]$, and

$$\mathbb{Z} \preccurlyeq {}^*\mathbb{Z}. \qquad \square$$

To use this for the full residual freeness of limit groups, we relate free groups to $\mathbb{Z}$ as follows.

5.4. A linear free group

If R is a ring, then the **special linear group** $\mathrm{SL}_2(R)$ comprises the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over R such that

$$ad - bc = 1.$$

The **projective special linear group** $\mathrm{PSL}_2(R)$ is $\mathrm{SL}_2(R)/\,\mathrm{C}(\mathrm{SL}_2(R))$.

In Robinson [5], Proposition 3.2.10 is that, when K is a field and $n > 1$, then $\mathrm{SL}_n(K)$ is generated by the *transvections:* matrices $I + a \cdot E^i_j$, where every entry of E^i_j is 0, but entry (i, j) is 1. Left multiplication of a matrix X by $I + E^i_j$ effects the addition of a times row j of X to row i. By such operations, elements of $\mathrm{SL}_n(K)$ can be reduced to I. With this idea, we have:

Theorem 52. $\mathrm{SL}_2(\mathbb{Z})$ *is generated by*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Proof. Call these E_0 and E_1. Then $X \mapsto E_0 \cdot X$ on $\mathrm{SL}_2(\mathbb{Z})$ is adding the bottom row of X to the top row; and $X \mapsto E_1 \cdot X$ is adding top to bottom. We use these to perform the Euclidean algorithm on the first column of elements of $\mathrm{SL}_2(\mathbb{Z})$. Indeed, say

$$\begin{pmatrix} a_0 & * \\ a_1 & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

so $\gcd(a_0, a_1) = 1$. We obtain $(a_0, \dots, a_{n+1})$ for some n in ω, and $(k_0, \dots, k_{n-1})$, where, if $i < n$, then

$$a_i = a_{i+1} \cdot k_i + a_{i+2}, \qquad |a_{i+2}| \leqslant \frac{|a_{i+1}|}{2};$$

also

$$|a_n| = \gcd(a_0, a_1) = 1, \qquad a_{n+1} = 0.$$

Then

$$\begin{aligned}
\begin{pmatrix} a_0 & * \\ a_1 & * \end{pmatrix} &= {E_0}^{k_0} \cdot \begin{pmatrix} a_2 & * \\ a_1 & * \end{pmatrix} \\
&= {E_0}^{k_0} \cdot {E_1}^{k_1} \cdot \begin{pmatrix} a_2 & * \\ a_3 & * \end{pmatrix} \\
&= {E_0}^{k_0} \cdot {E_1}^{k_1} \cdot {E_0}^{k_2} \cdot \begin{pmatrix} a_4 & * \\ a_3 & * \end{pmatrix} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= {E_0}^{k_0} \cdot {E_1}^{k_1} \cdot {E_0}^{k_2} \cdots {E_1}^{k_{n-1}} \cdot \begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} \\
&= {E_0}^{k_0} \cdot {E_1}^{k_1} \cdot {E_0}^{k_2} \cdots {E_1}^{k_{n-1}} \cdot {E_0}^{\pm m} \cdot \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}
\end{aligned}$$

if n is even; but also

$$\begin{aligned}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} &= E_1\cdot\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}\\ &= E_1\cdot {E_0}^{-2}\cdot\begin{pmatrix}1 & -2\\ 1 & -1\end{pmatrix}\\ &= E_1\cdot {E_0}^{-2}\cdot E_1\cdot\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}\\ &= E_1\cdot {E_0}^{-2}\cdot E_1\cdot {E_2}^{-2}.\end{aligned}$$

If n is odd, then

$$\begin{aligned}\begin{pmatrix}a_0 & *\\ a_1 & *\end{pmatrix} &= {E_0}^{k_0}\cdot {E_1}^{k_1}\cdot {E_0}^{k_2}\cdots {E_0}^{k_{n-1}}\cdot\begin{pmatrix}0 & \mp 1\\ \pm 1 & m\end{pmatrix}\\ &= {E_0}^{k_0}\cdot {E_1}^{k_1}\cdot {E_0}^{k_2}\cdots {E_0}^{k_{n-1}}\cdot {E_1}^{\mp m}\cdot\begin{pmatrix}0 & \mp 1\\ \pm 1 & 0\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix}0 & \mp 1\\ \pm 1 & 0\end{pmatrix} &= {E_0}^{\mp 1}\cdot\begin{pmatrix}1 & \mp 1\\ \pm 1 & 0\end{pmatrix}\\ &= {E_0}^{\mp 1}\cdot E_1^{\pm 1}\cdot\begin{pmatrix}1 & \mp 1\\ 0 & 1\end{pmatrix}\\ &= {E_0}^{\mp 1}\cdot {E_1}^{\pm 1}\cdot {E_0}^{\mp 1}.\end{aligned}$$

□

Corollary 53. $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\langle -I\rangle$.

From the proof of the last theorem, we have the following:

Porism 54. *The quotient map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ induces a homomorphism φ from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, and $\ker(\varphi)$ is generated by*

$$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix},\qquad \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix},\qquad -I.$$

Proof. Modify the Euclidean algorithm. Given $\begin{pmatrix}a_0 & *\\ a_1 & *\end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$, we find $(a_0,\dots,a_{n+1})$ and $(k_0,\dots,k_n)$ such that

$$a_i = a_{i+1}\cdot 2k_i + a_{i+2},\qquad |a_{i+2}|\leqslant |a_{i+1}|,$$

and also (a_n, a_{n+1}) is one of

$$(\pm 1, 0), \qquad\qquad (1, 1).$$

Thus, using only $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, we can reduce any element of $\mathrm{SL}_2(\mathbb{Z})$ to one of

$$\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & m \end{pmatrix}, \qquad \begin{pmatrix} 1 & m \\ 1 & m+1 \end{pmatrix}.$$

The last two are not in $\ker(\varphi)$; if the first is, then m is even, so the matrix reduces to $\pm I$. □

The following is given as an example in Robinson [5, §2.1].

Lemma 55. *The subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *generated by*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is free.

Proof. Identify the matrices with the Möbius transformations

$$z \mapsto z + 2, \qquad\qquad z \mapsto \frac{z}{2z+1}.$$

Calling these α and β, and letting $1/z = \gamma(z)$, we have

$$\beta \circ \gamma(z) = \frac{1}{2+z} = \gamma \circ \alpha(z),$$
$$\beta = \gamma \circ \alpha \circ \gamma^{-1} = \gamma \circ \alpha \circ \gamma.$$

Then for all k in $\mathbb{Z} \smallsetminus \{0\}$:

- α^k sends the unit disk D into $\mathbb{C} \smallsetminus D$;
- β^k sends 0 to 0, and $\mathbb{C} \smallsetminus D$ into $D \smallsetminus \{0\}$;
- $\beta^k(1) = 1/(2k+1)$;

- for all words $w(x, y)$ that are nontrivial in x,

$$w(\alpha, \beta)(1) \neq 1. \qquad \square$$

Theorem 56. *If now φ is the induced homomorphism from $\mathrm{PSL}_2(\mathbb{Z})$ to $\mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z})$, then*

$$\ker \varphi \cong F_2,$$

that is, we have an exact sequence

$$1 \to F_2 \to \mathrm{PSL}_2(\mathbb{Z}) \xrightarrow{\varphi} \mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \to 1.$$

Champetier and Guirardel [2] say it is 'well known' that, if p is an odd prime, then the kernel of the natural homomorphism from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is a nonabelian free group; but I do not know how to prove this.

5.5. Full residual freeness of limit groups

The following theorem is Champetier and Guirardel [2, Proposition 6.6] and is attributed to Remeslennikov (in the form of the proof). According to Yves de Cornulier in *Mathematical Reviews,*

> The class of limit groups is known to coincide with the long-studied class of finitely generated fully residually free groups. The authors provide (Corollary 6.5 and Proposition 6.6) the first proof of this result not relying on the finite presentability of limit groups.

Again, for finitely presented groups, we have the converse of the following by Theorem 48 (basically the *definition* of being fully residually free); and we have the theorem itself for finitely presented groups.

Theorem 57. *Limit groups are fully residually free.*

Proof. We show that a finitely generated subgroup G of ${}^*F_2$ is residually free. Let φ be as in Theorem 56. Then we also have an exact sequence

$$1 \to {}^*F_2 \to \mathrm{PSL}_2({}^*\mathbb{Z}) \xrightarrow{{}^*\varphi} \mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \to 1.$$

(Here we use $^*\operatorname{PSL}_2(R) \cong \operatorname{PSL}_2(^*R)$, and also $^*R \cong R$ when R is finite.) We may assume $G \leqslant \ker(^*\varphi)$. Since G is finitely generated, we have then

$$G \leqslant \operatorname{PSL}_2(\mathbb{Z}[x_0, \dots, x_{n-1}])$$

for some x_i in $^*\mathbb{Z}$. Say $\{g_0, \dots, g_{p-1}\} \subseteq G \smallsetminus \{1\}$. Each g_j is a proper coset $M_j \cdot \langle -I \rangle$ of $\langle -I \rangle$, and M_j has the form

$$\begin{pmatrix} 2a_k(\boldsymbol{x}) + 1 & 2b_k(\boldsymbol{x}) \\ 2c_k(\boldsymbol{x}) & 2d_k(\boldsymbol{x}) + 1 \end{pmatrix}$$

for some a_k etc. in $\mathbb{Z}[\boldsymbol{X}]$, because g_j is a nontrivial element of $\ker(^*\varphi)$. Finally, $\mathbb{Z}[\boldsymbol{x}] = \mathbb{Z}[\boldsymbol{X}]/(f_0, \dots, f_{q-1})$ for some f_k. As in the proof of Porism 51, since $\mathbb{Z} \prec {^*\mathbb{Z}}$, we can find $\boldsymbol{t}$ in $\mathbb{Z}$ such that

- each $f_k(\boldsymbol{t}) = 0$, so $x_i \mapsto t_i$ is a well-defined homomorphism from $\mathbb{Z}[\boldsymbol{x}]$ into $\mathbb{Z}$,
- the induced homomorphism ψ from $\operatorname{PSL}_2(\mathbb{Z}[\boldsymbol{x}])$ to $\operatorname{PSL}_2(\mathbb{Z})$ takes the g_j to nontrivial elements.

We now have a commutative diagram as in Figure 5.1 (with top and bottom rows exact). In particular, there is a homomorphism from E to F_2 that is nontrivial at the g_j. □

$$\begin{tikzcd}
1 \arrow[r] & {^*F_2} \arrow[r] & \operatorname{PSL}_2(^*\mathbb{Z}) \arrow[r, "^*\varphi"] & \operatorname{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \arrow[r] & 1 \\
& E \arrow[u, hook] \arrow[r] \arrow[d] & \operatorname{PSL}_2(\mathbb{Z}[\boldsymbol{x}]) \arrow[u, hook] \arrow[r] \arrow[d, "\psi"] & \operatorname{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \arrow[u, hook, two heads] \arrow[d, hook, two heads] & \\
1 \arrow[r] & F_2 \arrow[r] & \operatorname{PSL}_2(\mathbb{Z}) \arrow[r, "\varphi"'] & \operatorname{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \arrow[r] & 1
\end{tikzcd}$$

Figure 5.1. Commutative diagram for full residual freeness of limit groups

Now, since a limit group E is fully residually free, it is a limit of free groups in $\mathscr{G}(E)$ by Theorem 48. Therefore being a limit group is invariant under isomorphism.

5.6. Finitude of maximal limit quotients

This section corresponds to Champetier and Guirardel [2, §6.4], which is said to be inspired by Zoé Chatzidakis [3].

Every group E has a unique maximal residually free quotient, namely $E/\tilde{N}$, where

$$\tilde{N} = \bigcap\{N \colon E/N \text{ is free}\}.$$

We shall show that a finitely generated group has finitely many maximal limit quotients—equivalently, *fully* residually free quotients.

Lemma 58. *Suppose G is a finitely generated group, and $(N_k \colon k \in \omega)$ is an increasing chain of normal subgroups of G such that each quotient G/N_k is residually free. Then the chain is eventually constant.*

Proof. Say G is generated by $\boldsymbol{s}$. Let V_k be the set of all $\rho(\boldsymbol{s})$, where ρ is a representation of G/N_k in $\mathrm{SL}_2(\mathbb{C})$; that is,

$$V_k = \bigcap_{w(\boldsymbol{s}) \in N_k} \{\boldsymbol{M} \colon \boldsymbol{M} \in \mathrm{SL}_2(\mathbb{C})^n \wedge w(\boldsymbol{M}) = 1\}.$$

Then $(V_k \colon k \in \omega)$ is a decreasing chain of algebraic varieties; so it is eventually constant. (The Hilbert Basis Theorem ensures that the corresponding increasing chain of ideals of polynomials that are zero on the varieties is eventually constant; then the Nullstellensatz, Theorem 50 above, ensures that the chain of varieties must be eventually constant.) However,

$$N_k \subset N_{k+1} \to V_k \supset V_{k+1},$$

by the residual freeness of G/N_k. Indeed, suppose $w(\boldsymbol{s}) \in N_{k+1} \smallsetminus N_k$. There is a homomorphism φ from G/N_k to F_2 such that $\varphi(w(\boldsymbol{s})) \neq 1$. But there is also an embedding ψ of F_2 in $\mathrm{SL}_2(\mathbb{C})$. Then

$$\psi(\varphi(w(\boldsymbol{s}))) \neq 1,$$
$$w(\psi(\varphi(s_0), \dots, \varphi(s_{n-1}))) \neq 1,$$

or in short $w(\psi(\varphi(\boldsymbol{s}))) \neq 1$, and so $\psi \circ \varphi(\boldsymbol{s}) \in V_k \smallsetminus V_{k+1}$. □

Lemma 59. *Every finitely generated residually free group is the maximal residually free quotient of a finitely presented group.*

Proof. Suppose $G = \langle s_0, \dots, s_n | \, w_0, w_1, \dots \rangle$. There is an increasing chain $(N_k \colon k \in \omega)$ of normal subgroups of F_n, where F_n/N_k is the maximal residually free quotient of $\langle \boldsymbol{s} | \, w_0, \dots, w_{k-1} \rangle$. By Lemma 58, the sequence stabilizes at some N_k, and then N_k contains each w_ℓ, so F_n/N_k is a quotient of G, even the maximal residually free quotient of G. If G is already residually free, then G must be F_n/N_k, so it is the maximal residually free quotient of $\langle \boldsymbol{s} | \, w_0, \dots, w_{k-1} \rangle$. □

Lemma 60. *Every residually free group G in $\mathscr{G}(F_n)$ has an open neighborhood whose every residually free element is in $\mathscr{G}(G)$.*

Proof. By Lemma 59, G is the maximal residually free quotient of some finitely presented group E. Say

$$E = \langle \boldsymbol{s} | \, w_0, \dots, w_{k-1} \rangle.$$

Then $\mathscr{G}(E)$ is the open set $[\bigwedge_{i<k} w_i = 1]$ of $\mathscr{G}(F_n)$. This open set contains G, and every residually free element, being a residually free quotient of E, must be a quotient of G. □

Theorem 61. *Every finitely generated group has finitely many limit quotients.*

Proof. Let E be a finitely generated group. The set $\mathscr{L}$ of limit quotients of E is a closed subset of $\mathscr{G}(E)$ (since limits of limit groups are limit groups). Therefore $\mathscr{L}$ is compact. By Lemma 60, for every D in $\mathscr{L}$, the set $\mathscr{G}(D) \cap \mathscr{L}$ is an open neighborhood of D in $\mathscr{L}$. Finitely many such neighborhoods over $\mathscr{L}$. The maximal limit quotients of E are among the corresponding groups D. □

6. Limit groups 3

Here are some results from [2, §2.7]. Given a finitely generated group E, we work in the topological space $\mathscr{G}(E)$ of group quotients of E. The **saturation** of a subset of $\mathscr{G}(E)$ is the set of all groups in $\mathscr{G}(E)$ that are isomorphic to some group in the subset. The following is the first part of [2, Lem. 2.17]; note that the argument does not require a metric.

Theorem 62. *The saturation of an open set is open.*

Proof. Let G and G_1 be groups marked by a finite set S or $\{s_0, \dots, s_{n-1}\}$. Suppose θ is an isomorphism from G_1 to G. Then for some terms w_i and v_i of $\mathscr{S}(S)$, we have

$$\theta({s_i}^{G_1}) = {w_i}^G, \qquad\qquad \theta^{-1}({s_i}^G) = {v_i}^{G_1},$$

so that

$${s_i}^G = \theta({v_i}^{G_1}) = v_i(\boldsymbol{w})^G,$$

that is,

$$G \models \bigwedge_{i<n} s_i = v_i(\boldsymbol{w}). \tag{$*$}$$

Moreover, for all terms u of $\mathscr{S}(S)$,

$$G_1 \models u = 1 \iff G \models u(\boldsymbol{w}) = 1. \tag{$\dagger$}$$

Conversely, the latter condition ($\dagger$) ensures that there is a well-defined monomorphism ${s_i}^{G_1} \mapsto {w_i}^G$ from G_1 to G; if we have also ($*$) for some v_i, then this monomorphism is surjective.

Now let φ be a quantifier-free formula of $\mathscr{S}$. If $G_1 \in [\varphi(\boldsymbol{s})]$ and $G_1 \cong G$, then

$$G \in [\varphi(\boldsymbol{w}) \wedge \bigwedge_{i<n} s_i = v_i(\boldsymbol{w})].$$

Conversely, if this holds for some choice of w_i and v_i, and we define G_1 as $\langle S | R\rangle$, where

$$u \in R \iff G \models u(\boldsymbol{w}) = 1,$$

then in particular $G_1 \in [\varphi(\boldsymbol{s})]$ and $G_1 \cong G$. Thus the saturation of $[\varphi(\boldsymbol{s})]$ is the union of the collection of all of the open sets $[\varphi(\boldsymbol{w}) \wedge \bigwedge_{i<n} s_i = v_i(\boldsymbol{w})]$. □

Theorem 63. *The closure of a saturated set is saturated.*

Proof. Let F ge saturated. The saturation of the complement of the closure of F is open, so if it contains a point of the closure of F, then it contains a point G of F itself—but then G is isomorphic to a group in the complement of F, which is absurd. □

Corollary 64. *The set of limit groups in $\mathscr{G}(E)$ is saturated.*

Proof. It is the closure of the set of free groups in $\mathscr{G}(E)$, and this set is saturated. □

In particular, we have now a simpler proof that being a limit group is invariant under isomorphism.

If $G \in \mathscr{G}(E)$, we may denote by

$$[G]_E$$

the set of groups in $\mathscr{G}(E)$ that are isomorphic to G. That is, $[G]_E$ is the saturation of $\{G\}$ in $\mathscr{G}(E)$.

Corollary 65. *If $1 \leqslant k \leqslant n$, then the closure of $[\mathbb{Z}^k]_{\mathscr{G}(F_n)}$ is*

$$[\mathbb{Z}^k]_{\mathscr{G}(F_n)} \cup \cdots \cup [\mathbb{Z}^n]_{\mathscr{G}(F_n)}.$$

Proof. If $p \leqslant n$, then $\mathbb{Z}^p \cong \langle S | R\rangle$, where $S = \{s_0, \dots, s_{n-1}\}$ and

$$R = \{[s_i, s_j] \colon i < j < p\} \cup \{s_m \colon p \leqslant m < n\}.$$

If also $1 \leqslant k \leqslant p$, then $\langle S | R\rangle$ is the limit of the groups

$$\langle S | R \cup \{{s_0}^r {s_\ell}^{-1} \colon k \leqslant \ell < p\}\rangle,$$

which are isomorphic to $\mathbb{Z}^k$. This shows that the closure of $[\mathbb{Z}^k]_{\mathscr{G}(F_n)}$ includes $[\mathbb{Z}^p]_{\mathscr{G}(F_n)}$.

Conversely, the set $[\mathbb{Z}^k]_{\mathscr{G}(F_n)} \cup \cdots \cup [\mathbb{Z}^n]_{\mathscr{G}(F_n)}$ is closed, since it is the intersection of the sets $[\sigma]$, where σ is one of

$$[w, v] = 1, \qquad w^{r+1} = 1 \to w = 1, \qquad \bigvee_{i<n} s_i \neq w_i(v_0, \dots, v_{k-1}). \qquad \square$$

Now we pass to [2, Ch. 3].

Theorem 66. *Every limit group in $\mathscr{G}(F_n)$ is of exactly one of three kinds:*

1) *trivial, or*
2) *a limit of free groups of rank 1, that is, a nontrivial free abelian group, or*
3) *a limit of free groups of rank 2.*

Proof. Every limit group in $\mathscr{G}(F_n)$ is in the closure of some $[F_\ell]_{\mathscr{G}(F_n)}$. Therefore, assuming $2 \leqslant \ell \leqslant n$, we have to show F_ℓ is a limit of elements of $[F_2]_{\mathscr{G}(F_n)}$. By induction, it is enough to show that, if $2 \leqslant k < n$, then F_{k+1} is a limit of elements of $[F_k]_{\mathscr{G}(F_n)}$. But F_{k+1} is a limit of the groups

$$\langle s_0, \dots, s_k \,|\, {s_0}^r \cdots {s_{k-1}}^r {s_k}^{-1} \rangle. \qquad \square$$

By Theorem 32, subgroups of limit groups are limit groups; also, by Theorem 33, a 2-generated subgroup of a limit group is either F_2 or a free abelian group of rank 2 or less.

Since we know by Theorem 57 that limit groups are fully residually free, we can take some examples of non-free limit groups from B. Baumslag [1]. The following is based on Lemma 7 (p. 412) of that paper.

Lemma 67. *Let F be a limit group with elements f_0, …, f_{k-1}, and u such that $[f_i, u] \neq 1$ in each case. Then*

$$u^{n_0} f_0 u^{n_1} f_1 \cdots f_{k-1} u^{n_k} \neq 1$$

if the n_i are integers large enough (in absolute value) when $1 \leqslant i < k$, and also when i is 0 or k, unless in this case $n_i = 0$.

Proof. Since F is fully residually free, we may replace it with a free quotient in which the $[f_i, u]$ remain nontrivial. We may also assume that u is *cyclically reduced,* that is, as a word, it does not begin with $g^{\pm 1}$ and end with $g^{\mp 1}$ for some generator g of F.

We shall show that, if n is large enough, then $u^n f u^n$ and $u^n f u^{-n}$, as reduced words, begin with u, and they end with $u^{\pm 1}$ respectively. Consider first the case of $u^n f u^n$, where n is positive. If n is large enough, then

$$u^n f = u_{\triangle} u^{n-1} f$$

in the notation introduced on page 8; that is, there is no cancellation between u and $u^{n-1}f$. Moreover, $u^{n-1}f \neq 1$. If r is large enough and positive, then $u^{n-1}fu^r = u^{n-1}fu^{r-1}{}_{\triangle}u$. Again, $u^{n-1}fu^{r-1} \neq 1$. Indeed we have

$$u^{n-1} f u^{r-1} = vgw, \tag{‡}$$

where v is an initial, and w a final, proper segment of u, and g is a segment of f, or possibly one or two of them are trivial, but not all three. If vgw is not simply v or w, then

$$u^n f u^r = u_{\triangle} u^{n-1} f u^{r-1}{}_{\triangle} u;$$

but in any case,

$$u^{n+1} f u^{r+1} = u_{\triangle} u^n f u^r{}_{\triangle} u. \tag{§}$$

If we go through the same steps with r negative, instead of (‡), we get

$$u^{n-1} f u^{r-1} = vgw^{-1},$$

where now w is a proper initial segment of u or is trivial; but we still get (§).

For each i less than k, there is now a positive integer m_i such that

$$\begin{aligned} u^{\pm(m_i+1)} f u^{\pm(m_i+1)} &= u^{\pm 1}{}_{\triangle} u^{\pm m_i} f u^{\pm m_i}{}_{\triangle} u^{\pm 1}, \\ u^{\pm(m_i+1)} f u^{\mp(m_i+1)} &= u^{\pm 1}{}_{\triangle} u^{\pm m_i} f u^{\mp m_i}{}_{\triangle} u^{\mp 1}. \end{aligned}$$

We can now require

$$|n_0| > m_0, \qquad |n_i| > m_{i-1} + m_i, \qquad |n_k| > m_k$$

where $1 \leqslant i < k$. $\square$

Theorem 68. *Let F be a limit group with an element u whose centralizer is $\langle u\rangle$, and let A be the free abelian group on $\{x_0, \dots, x_{n-1}\}$. Then the group*

$$F *_{u=x_0} A$$

(namely, $F * A/N$, where N is the normal closure of $\{u{x_0}^{-1}\}$) is a limit group.

Proof. Define a homomorphism θ from $F * A$ to F by requiring

- $\theta(f) = f$ for all f in F,
- $\theta(x_0) = u$,
- $\theta(x_{i+1}) = u^{p_i}$

for some p_i to be determined. We want θ to be nontrivial on some finite subset of $F * A \smallsetminus N$, where N is the normal closure of $\{x_0 u^{-1}\}$. We may assume that the elements of this finite subset have the form

$$a_0 f_0 a_1 \cdots a_{n-1} f_{n-1} a_n$$

for some f_j in $F \smallsetminus \langle u\rangle$ and a_j in $A \smallsetminus \langle x_0\rangle$; but possibly a_0 or a_n is trivial. We can now pick the p_i so that, by the lemma, each $\theta(a_j)$ is a power of u with exponent large enough that θ is as desired. □

A. Schedule

	chapter	date	speaker
1	Free groups	October 7	Ali Nesin
	”	October 14	”
2	Trees	”	”
3	Burnside problem	October 21	Oleg Belegradek
4	Limit groups	”	Piotr Kowalski
	”	October 28	”
	”	November 4	”
5	Limit groups 2	November 18	David Pierce
	”	December 2	”
	”	December 9	”
	”	December 16	”
6	Limit groups 3	December 23	Ayşe Berkman
	”	December 30	”

There was no meeting on November 11 (because of the Feast of the Sacrifice). Other talks, not recorded here:

- Cédric Milliet, ‘Definable envelopes around abelian, nilpotent or soluble subgroups in a group with simple theory’, October 28 and November 4
- Oleg Belegradek, ‘Coset-minimal groups’, November 18
- Salih Durhan, ‘Hahn fields’, December 2
- Özlem Beyarslan, ‘Algebraic closure in pseudofinite fields’, December 16 and 23

B. German letters

German letters are as follows; they are obtained by the `\mathfrak` command in $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX (specifically, in the `amssymb` package, which apparently loads the `amsfonts` package).

$$\begin{array}{ccccccccc}
\mathfrak{A} & \mathfrak{B} & \mathfrak{C} & \mathfrak{D} & \mathfrak{E} & \mathfrak{F} & \mathfrak{G} & \mathfrak{H} & \mathfrak{I} \\
\mathfrak{J} & \mathfrak{K} & \mathfrak{L} & \mathfrak{M} & \mathfrak{N} & \mathfrak{O} & \mathfrak{P} & \mathfrak{Q} & \mathfrak{R} \\
\mathfrak{S} & \mathfrak{T} & \mathfrak{U} & \mathfrak{V} & \mathfrak{W} & \mathfrak{X} & \mathfrak{Y} & \mathfrak{Z} & \\
\\
\mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} & \mathfrak{e} & \mathfrak{f} & \mathfrak{g} & \mathfrak{h} & \mathfrak{i} \\
\mathfrak{j} & \mathfrak{k} & \mathfrak{l} & \mathfrak{m} & \mathfrak{n} & \mathfrak{o} & \mathfrak{p} & \mathfrak{q} & \mathfrak{r} \\
\mathfrak{s} & \mathfrak{t} & \mathfrak{u} & \mathfrak{v} & \mathfrak{w} & \mathfrak{x} & \mathfrak{y} & \mathfrak{z} &
\end{array}$$

(The combined letters like ß are apparently not available in math mode.) A way to write German letters by hand is shown in Figure B.1, which is taken from a old textbook: Roe-Merill S. Heffner, *Brief German Grammar* (Boston: D.C. Heath and Company, 1931). According to this source, 'The great German poets, philosophers, and scientists usually wrote in German script.'

Figure B.1. The German alphabet by hand

C. Stone spaces

The topologies on the spaces $\mathscr{G}(E)$ introduced on page 25 can be understood as instances of a general construction, outlined here.

For some positive n in ω, we let $\boldsymbol{x}$ be the n-tuple $(x_0, \dots, x_{n-1})$ of letters, and we let F_n be the free group on $\boldsymbol{x}$. We also let $\boldsymbol{w}$ be an m-tuple $(w_0, \dots, w_{m-1})$ of elements of F_n, that is, reduced words on $\boldsymbol{x}$. Then we obtain the finitely presented group

$$\langle \boldsymbol{x} | \, \boldsymbol{w} \rangle,$$

which we call E. We can understand E as a structure in the signature $\mathscr{S}(\boldsymbol{s})$, where s_i is interpreted as the image of x_i in E. In this signature, let T_E be the theory of groups in which $w_i(\boldsymbol{s}) = 1$: this is the theory of groups G for which there is a homomorphism from E to G.

Let B_E consist of the quantifier-free sentences of $\mathscr{S}(\boldsymbol{s})$ *modulo* equivalence in T_E. Then B_E is a Boolean algebra with respect to $\vee$ and $\wedge$; it is an example of a **Lindenbaum–Tarski algebra.** The top element $\top$ of this algebra is (the equivalence-class of) $s_0 = s_0$; the bottom element, $\bot$, is $s_0 \neq s_0$. We can also understand B_E as a ring by defining

$$\sigma + \tau = \neg(\sigma \leftrightarrow \tau), \qquad \sigma \cdot \tau = \sigma \wedge \tau, \qquad 1 = \top, \qquad 0 = \bot.$$

Then B_E is a called a **Boolean ring** because

$$(*) \qquad \sigma \cdot \sigma = \sigma.$$

Any ring meeting this condition is commutative and of characteristic 2; but these features are already obvious for B_E.

A **filter** of B_E is a proper nonempty subset F of B_E such that

1) if σ and τ are in F, then $\sigma \wedge \tau$ is in F;

2) if $\sigma \in F$, then $\sigma \vee \tau \in F$.

Then a filter can be understood as a theory (in our case a quantifier-free theory) that includes the quantifier-free part of T_E. More algebraically, a filter of B_E is a subset $\{\neg\sigma\colon \sigma \in I\}$ for some ideal of B_E, when this is considered as a ring. In a word, filters are **dual** to ideals. We may refer to B_E as an **improper filter** of itself; then, for emphasis, a filter is a **proper filter.**

An **ultrafilter** is a maximal filter; equivalently, it is dual to a maximal ideal. Now, if $\mathfrak{m}$ is a maximal ideal of B_E, then the quotient $B_E/\mathfrak{m}$ is a field. This is a field in which every element satisfies $(*)$; so the field has only two elements, and therefore $\mathfrak{m}$ has index 2 as a subgroup of B_E. In particular, $\mathfrak{m}$ has only two cosets in B_E: itself, and $1 + \mathfrak{m}$. But

$$1 + \sigma = \neg\sigma.$$

Therefore an ultrafilter of B_E is a filter F meeting the additional condition

3) $\sigma \in F$ if and only if $\neg\sigma \notin F$.

In particular, an ultrafilter of B_E can be understood as a complete quantifier-free theory T that includes the quantifier-free part of T_E. If $G \models T$, let A be the subgroup of G generated by (the interpretation of) $\boldsymbol{s}$. Then T can be understood as the **Robinson diagram** of A (although strictly this diagram is the complete quantifier-free theory of A in the signature $\mathscr{S}(A)$, not just $\mathscr{S}(\boldsymbol{s})$). Also, A is isomorphic to a quotient E/N of E; and then N is uniquely determined by T.

The set of ultrafilters of B_E can be denoted by

$$\mathrm{S}(B_E).$$

This is the **Stone space** of B_E, considered as a Boolean algebra; and it is the **Tarski space** of B_E, considered in particular as a Lindenbaum-Tarski algebra.[1] Again, for every p in $\mathrm{S}(B_E)$, there is a corresponding normal subgroup of E, namely the set of $w(\boldsymbol{s})$ such that the sentence $w(\boldsymbol{s}) = 1$ is in p. Then p can be recovered from this normal subgroup.

[1] I have a memory of Angus Macintyre's using the term *Tarski space* for the space of completions of a theory (the particular theory in question was that of algebraically closed fields).

From B_E to the power-set $\mathscr{P}(\mathrm{S}(B_E))$ of the Tarski space, there is a function $\sigma \mapsto [\sigma]$, given by

$$[\sigma] = \{p \colon \sigma \in p\}.$$

If σ and τ are distinct elements of B_E, then one of $\sigma \wedge \neg\tau$ and $\tau \wedge \neg\sigma$ is not $\perp$. Say $\sigma \wedge \neg\tau$ is not $\perp$; then (by the Axiom of Choice) it is a member of some p in the Tarski space, and therefore

$$\sigma \in p, \qquad\qquad \tau \notin p.$$

Thus the function $\sigma \mapsto [\sigma]$ is an embedding of sets. It is an embedding of *Boolean algebras* since

$$[\sigma \vee \tau] = [\sigma] \cup [\tau], \qquad\qquad [\sigma \wedge \tau] = [\sigma] \cap [\tau],$$

and also

$$[\neg\sigma] = \mathrm{S}(B_E) \smallsetminus [\sigma], \qquad [\perp] = \varnothing, \qquad [\top] = \mathrm{S}(B_E).$$

In particular, the collection of sets $[\sigma]$ is closed under (finite) union and intersection, as well as complementation, so it is a basis of closed sets and of open sets of the (same) topology on $\mathrm{S}(B_E)$.

The topology on $\mathrm{S}(B_E)$ is compact. Indeed, suppose $\{[\sigma] \colon \sigma \in A\}$ is a set of basic closed sets whose every finite subset has nonempty intersection. This just means that the subset A of B_E generates a proper filter. By the Axiom of Choice, this filter is included in an ultrafilter p, and then

$$p \in \bigcap_{\sigma \in A} [\sigma].$$

In an alternative approach to compactness, we can understand $\mathrm{S}(B_E)$ as a subset of $\mathscr{P}(B_E)$. The latter is in bijection with 2^{B_E}, namely the set of functions from B_E into 2 (that is, $\{0, 1\}$). Suppose 2 has the discrete topology, which is compact since 2 is finite. Then 2^{B_E} can be given the product topology, which by Tychonoff's Theorem is compact. Then $\mathscr{P}(B_E)$ has the compact topology induced by the bijection. The topology on $\mathrm{S}(B_E)$ that we defined above is just the subspace topology, and $\mathrm{S}(B_E)$ is a closed subset of $\mathscr{P}(B_E)$, so it is compact.

We can do all of the foregoing with B_E replaced by the algebra of arbitrary or quantifier-free formulas in a given set of variables *modulo* equivalence in a given theory.

We can also metrize the topology on $\mathrm{S}(B_E)$. In this metric, the distance $\mathrm{d}\,pq$ between distinct points p and q is e^{-M}, where M is the greatest m such that, for all reduced words w belonging to F_n of length m or less, either both or neither of p and q belongs to $[w(\boldsymbol{s}) = 1]$. Note then

$$\mathrm{d}\,pr \leqslant \max\bigl(\mathrm{d}\,pq, \mathrm{d}\,qr\bigr).$$

The closed ball of radius e^{-M} about p is the intersection of the sets $[w(\boldsymbol{s}) = 1]$, where the equation $w(\boldsymbol{s}) = 1$ belongs to p and the length of w is M or less. So the closed ball is indeed closed in the Stone topology. In fact, since the number of such words w is finite, the ball is one of the basic closed sets. Conversely, suppose U is an open neighborhood of p in the Stone topology. Then there is some quantifier-free sentence σ such that

$$\sigma \in p, \qquad\qquad [\sigma] \subseteq U.$$

We may assume further that σ is a conjunction

$$(w'_0(\boldsymbol{s}) = 1)^{\varepsilon_0} \wedge \cdots \wedge (w'_{\ell-1}(\boldsymbol{s}) = 1)^{\varepsilon_{k-1}}$$

of word equations and inequations in $\boldsymbol{s}$. Let ℓ be the greatest length of one of the w'_i. Then U includes the closed ball of radius $\mathrm{e}^{-\ell}$ about p, which is also the open ball of radius $\mathrm{e}^{1-\ell}$ about p. So the topology on $\mathrm{S}(B_E)$ is indeed induced by the metric.

Each p in $\mathrm{S}(B_E)$ determines a normal subgroup N_p of E, namely the set of all elements $w(\boldsymbol{s})$ of E such that $p \in [w(\boldsymbol{s}) = 1]$. Then

$$\mathrm{d}\,pq = \mathrm{d}\,N_p N_q,$$

where the distance between normal subgroups of E—that is, between elements of $\mathscr{G}(E)$—is the Gromov–Hausdorff distance as on page 28.

D. Ultraproducts of groups

Suppose $\mathcal{G}$ is an infinite sequence $(G_k : k \in \omega)$ of groups. The **direct product** of $\mathcal{G}$ is the set of sequences $(g_k : k \in \omega)$, where $g_k \in G_k$ in each case. This product can be denoted by either of

$$\prod_{k\in\omega} G_k, \qquad\qquad \prod \mathcal{G};$$

it is a group in the obvious way. We may write $(g_k : k \in \omega)$ also as g. Then the **support** of g is the subset $\{k : g_k \neq 1\}$ of ω; we may denote this by

$$\operatorname{supp}(g).$$

The **direct sum** of $\mathcal{G}$ is the normal subgroup of G comprising those elements that have finite support; it can be denoted by either of

$$\sum_{k\in\omega} G_k, \qquad\qquad \sum \mathcal{G}.$$

By Appendix C, a maximal ideal of $\mathscr{P}(\omega)$ is a subset $\mathfrak{m}$ such that, for all subsets A and B of ω,

$$(*) \qquad A \in \mathfrak{m} \wedge B \in \mathfrak{m} \leftrightarrow A \cup B \in \mathfrak{m},$$

$$(\dagger) \qquad A \in \mathfrak{m} \leftrightarrow \omega \smallsetminus A \notin \mathfrak{m}.$$

We may then think of the elements of $\mathfrak{m}$ as the **small** subsets of ω; and their complements in ω, as **large.** Possibly there is an element ℓ of ω such that a subset is large if and only if it contains ℓ; in this case, $\mathfrak{m}$ is the principal ideal $(\omega \smallsetminus \{\ell\})$. In any case, we define $N_{\mathfrak{m}}$ as the set of elements of $\prod \mathcal{G}$ with small support.

Theorem 69. *Assume no G_k is trivial. The following are equivalent conditions on a subgroup of $\prod \mathcal{G}$:*

1. *It is maximal among the subgroups H of $\prod \mathcal{G}$ such that no element of H has full support.*

2. *It is $N_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ of $\mathscr{P}(\omega)$.*

The following are equivalent conditions on a maximal ideal $\mathfrak{m}$ of $\mathscr{P}(\omega)$:

1. *$\mathfrak{m}$ is nonprincipal.*

2. *$\sum \mathcal{G} \leqslant N_{\mathfrak{m}}$.*

Proof. For the first part, given $\mathfrak{m}$, suppose $g \in \prod \mathcal{G} \smallsetminus N_{\mathfrak{m}}$. Then $\operatorname{supp}(g)$ is large, so $N_{\mathfrak{m}}$ has an element h such that $\operatorname{supp}(h) = \omega \smallsetminus \operatorname{supp}(g)$. Then $gh \in \langle N_{\mathfrak{m}} \cup \{g\}\rangle$ and has full support. This establishes maximality of $N_{\mathfrak{m}}$.

Conversely, suppose N is maximal among the subgroups H of $\prod \mathcal{G}$ such that no element of H has full support. Let

$$\mathfrak{m} = \{\operatorname{supp}(g) \colon g \in N\}.$$

Then $\mathfrak{m}$ is a nonempty proper subset of $\mathscr{P}(\omega)$. If $g \in N$, and $A \subseteq \operatorname{supp}(g)$, then $h \in N$, where

$$h_k = \begin{cases} g_k, & \text{if } k \in A, \\ 1, & \text{if } k \in \omega \smallsetminus A, \end{cases}$$

so that $\operatorname{supp}(h) = A$. Thus $\mathfrak{m}$ contains all subsets of its elements. In particular, suppose also $g' \in N$, and $A = \operatorname{supp}(g) \cap \operatorname{supp}(g')$. With h as above, we have $gh^{-1}g' \in N$, and its support is $\operatorname{supp}(g) \cup \operatorname{supp}(g')$. So $\mathfrak{m}$ contains the unions of pairs of members. Therefore $\mathfrak{m}$ is a proper ideal of $\mathscr{P}(\omega)$. By maximality of N, the ideal must be maximal. □

Henceforth let $N = N_{\mathfrak{m}}$ for some nonprincipal ideal $\mathfrak{m}$ of $\mathscr{P}(\omega)$. The quotient $\prod \mathcal{G}/N$ is the **(nonprincipal) ultraproduct** of $\mathcal{G}$ with respect to $\mathfrak{m}$—or with respect to the dual ultrafilter of $\mathfrak{m}$. We have $gN = hN$ if and only if $\{k \colon g_k \neq h_k\}$ is small, that is, $\{k \colon g_k = h_k\}$ is large. These conditions make sense if the G_k are structures in an arbitrary signature; so they allow ultraproducts to be defined in any signature. If the G_k were fields, so that $\prod \mathcal{G}$ was a commutative ring (in fact a von Neumann regular ring), then N would be a maximal *ideal* of $\prod \mathcal{G}$. In our case, N just has the maximality given in the theorem above.

Theorem 70 (Łoś). *For all sentences σ of $\mathscr{S}$,*

$$\prod \mathcal{G}/N \models \sigma \text{ if and only if } \{k\colon G_k \models \sigma\} \text{ is large.}$$

Proof. If $n \in \omega$, the same element of $(\prod \mathcal{G})^n$ can be denoted by $\boldsymbol{g}$ and $(g^0, \dots, g^{n-1})$, where again $g^i = (g^i_k \colon k \in \omega)$. Then also $\boldsymbol{g}_k$ stands for $(g^0_k, \dots, g^{n-1}_k)$. Finally, $\boldsymbol{g}N$ means $(g^0N, \dots, g^{n-1}N)$ in $(\prod \mathcal{G}/N)^n$. We show by recursion that for all formulas φ of $\mathscr{S}$,

$$\prod \mathcal{G}/N \models \varphi(\boldsymbol{g}N) \text{ if and only if } \{k\colon G_k \models \varphi(\boldsymbol{g}_k)\} \text{ is large.}$$

1. The claim is true when φ is an atomic formula, that is, an equation, by definition of equality in $\prod \mathcal{G}/N$.

2. If the claim is true when φ is ψ, then it is true when φ is $\neg\psi$, simply because a subset of ω is large if and only if its complement is not, by (†).

3. If the claim is true when φ is ψ or χ, then the claim is true when φ is $\psi \vee \chi$, by $(*)$.

4. Suppose the claim is true when φ is $\psi(\boldsymbol{x}, y)$. The following are equivalent:

- $\prod \mathcal{G}/N \models \exists y\ \psi(\boldsymbol{g}N, y)$;
- $\prod \mathcal{G}/N \models \psi(\boldsymbol{g}N, hN)$ for some h in $\prod \mathcal{G}$;
- $\{k\colon G_k \models \psi(\boldsymbol{g}_k, h_k)\}$ is large for some h in $\prod \mathcal{G}$.

Now, for all h in $\prod \mathcal{G}$, we have

$$\{k\colon G_k \models \psi(\boldsymbol{g}_k, h_k)\} \subseteq \{k\colon G_k \models \psi(\boldsymbol{g}_k, f_k) \text{ for some } f_k \text{ in } G_k\};$$

and the inclusion is an equation for *some* choice of h. Therefore the following are equivalent.

- $\prod \mathcal{G}/N \models \exists y\ \psi(\boldsymbol{g}N, y)$;
- $\{k\colon G_k \models \psi(\boldsymbol{g}_k, f_k) \text{ for some } f_k \text{ in } G_k\}$ is large;

- $\{k\colon G_k \models \exists y\ \psi(\boldsymbol{g}_k, y)\}$ is large.

Therefore the claim is true when φ is $\exists y\ \psi(\boldsymbol{x}, y)$. $\square$

E. A summary

This was prepared on December 10. Here all groups are to be understood as finitely generated (though this may not always be required).

The set of quotients of a group E is given the 'Tychonoff topology': the weakest in which, for every g in E, the set of quotients E/N such that N contains g is both open and closed.

By definition, a limit group is a limit of free groups in the space of quotients of some group.

That space is a (closed) subspace of the space of quotients of some free group.

Therefore a limit group is just a limit of free quotients of some free group.

It is not immediate that being a limit group is invariant under isomorphism.

By definition, to be fully residually free is to be a limit of one's own free quotients.

Thus fully residually free groups are limit groups.

The space of quotients of a finitely presented group is an open neighborhood of the group (in any space of quotients that contains the group).

Therefore finitely presented limit groups are fully residually free.

It is supposedly true that every limit group is finitely presented, but we do not show this.

A limit group embeds in every ultraproduct of the free groups whose limit it is.

Since every free group embeds in the free group on two generators, every limit group embeds in an ultrapower of this free group.

With some work, we show that every limit group is fully residually free. The ideas used in the proof are:

1. The free group on two generators can be understood as a linear group over the integers.
2. This allows us to interpret a limit group E in an ultrapower of the integers.
3. We need only a finitely generated sub-ring of this ultrapower (since limit groups are finitely generated).
4. Then just as in the model-theoretic proof of Hilbert's Nullstellensatz, we get a homomorphism from E into a free group that is non-trivial on a predefined finite set of nontrivial elements.

So now we have that being a limit group and being a residually free group are the same thing.

In particular, every limit group is a limit of its own free quotients.

Therefore being a limit group is indeed invariant under isomorphism.

Every group has a unique maximal residually free quotient.

We are going to show that every group has finitely many maximal *fully* residually free quotients—that is, limit quotients.

To this end, we have shown that no group has an infinite descending chain of residually free quotients.

Again, this is connected to the Nullstellensatz, because a descending chain of quotients corresponds to a certain descending chain of varieties.

Because of this (as we shall show), every residually free group is the maximal residually free quotient of a finitely presented group.

Therefore, in the space of limit quotients of a group, every element has an open neighborhood consisting just of its own quotients.

But the space of limit quotients of a group is compact.

The finiteness of the set of maximal limit quotients will then follow.

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