Compactness and choice

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Contents

1 Compactness

The recent model-theory text of Tent and Ziegler [2] introduces the Compactness Theorem as follows:[1]

[1] The quotation is taken from what is called an early second edition of the text.

> Its name is motivated by the results in Section 4.2 which associate to each theory a certain compact topological space.
>
> We call a theory T *finitely satisfiable* if every finite subset of T is consistent.
>
> **Theorem 2.2.1** (Compactness Theorem). *Finitely satisfiable theories are consistent.*

The section referred to begins:

> We now endow the set of types of a given theory with a topology. The Compactness Theorem 2.2.1 then translates into the statement that this topology is compact, whence its name.
>
> Fix a theory T. An n-type is a maximal set of formulas $p(x_1, \ldots, x_n)$ consistent with T. We denote by $\mathrm{S}_n(T)$ the set of all n-types of T. We also write $\mathrm{S}(T)$ for $\mathrm{S}_1(T)$.[2] [...]
>
> **Remark.** The Stone duality theorem asserts that the map
>
> $$X \mapsto \{C \mid C \text{ clopen subset of } X\}$$
>
> yields an equivalence between the category of 0-dimensional compact spaces and the category of Boolean algebras. The inverse map assigns to every Boolean algebra $\mathcal{B}$ its Stone space $\mathrm{S}(\mathcal{B})$, the set of all ultrafilters (see Exercise 1.2.4) of $\mathcal{B}$. For more on Boolean algebras see [Givant and Halmos, *Introduction to Boolean algebras*].

Nothing is incorrect here. But one might be led to believe that the type spaces are *by definition* Stone spaces of Boolean algebras of logically equivalent formulas. Since Stone spaces are always compact, the type spaces are compact, and one might then conclude that the Compactness Theorem follows. But this would be a wrong conclusion, since this theorem fails in second-order logic, and yet Stone spaces of algebras of second-order formulas are still compact.

By the definition above, the type spaces are *dense subspaces* of certain Stone spaces. The Compactness Theorem is that these subspaces are compact. Since Stone spaces are Hausdorff, the type spaces must then be closed; therefore they are the whole Stone spaces. The point of this section to spell out the details of these observations.

[2] $\mathrm{S}_0(T)$ can be considered as the set of all complete extensions of T, up to equivalence. [Footnote in source.]

Fix some logic L that extends ordinary propositional logic: it could be a first-order logic, a second-order logic, or something else. There is a class $\mathbf{Mod}$ of structures interpreting L, and a set Sn of sentences of L. If $\sigma \in \mathrm{Sn}$, we can define

$$\mathbf{Mod}(\sigma) = \{\mathfrak{A} \in \mathbf{Mod} \colon \mathfrak{A} \models \sigma\}.$$

If $\mathfrak{A} \in \mathbf{Mod}$, we can define

$$\mathrm{Th}(\mathfrak{A}) = \{\sigma \in \mathrm{Sn} \colon \mathfrak{A} \models \sigma\}.$$

If $\Gamma \subseteq \mathrm{Sn}$ and $\mathcal{K} \subseteq \mathbf{Mod}$, we define

$$\mathbf{Mod}(\Gamma) = \bigcap_{\sigma\in\Gamma} \mathbf{Mod}[\sigma], \qquad \mathrm{Th}(\mathcal{K}) = \bigcap_{\mathfrak{A}\in\mathcal{K}} \mathrm{Th}(\mathfrak{A}).$$

The classes $\mathbf{Mod}(\Gamma)$ are **elementary classes** (though usually this term assumes a first-order logic); the classes $\mathrm{Th}(\mathcal{K})$ are **theories.** The functions $\Gamma \mapsto \mathbf{Mod}(\Gamma)$ and $\mathcal{K} \mapsto \mathrm{Th}(\mathcal{K})$ determine a *Galois correspondence* between the theories and the elementary classes.[3]

Moreover, since

$$\mathbf{Mod}(\sigma \vee \tau) = \mathbf{Mod}(\sigma) \cup \mathbf{Mod}(\tau),$$

the elementary classes are the closed *classes* of a *topology* on $\mathbf{Mod}$ with basis consisting of the closed classes $\mathbf{Mod}(\sigma)$.

To say that the logic L has a **compactness theorem** is to say that if $\Gamma \subseteq \mathrm{Sn}$ and every finite subset of Γ has a model, then Γ itself has a model. But this just means that if $\{\mathbf{Mod}(\sigma) \colon \sigma \in \Gamma\}$ has the Finite Intersection Property, then $\bigcap_{\sigma\in\Gamma} \mathbf{Mod}(\sigma) \neq \varnothing$: that is, $\mathbf{Mod}$ is compact as a topological space.

A similar Galois correspondence arises in algebraic geometry. Suppose L/K is a field-extension, and X is an n-tuple of variables. If $f \in K[X]$, define

$$\mathrm{V}(f) = \{a \in L^n \colon f(a) = 0\}.$$

[3]Evry relation R between sets or classes A and B induces a Galois correspondence between certain subsets of A and of B. In one direction this one-to-one, order-reversing correspondence is $X \mapsto \bigcap_{x\in X}\{y \in B \colon x \mathrel{R} y\}$. The original Galois correspondence is induced by the relation $\{(a,\sigma) \in K \times \mathrm{Aut}(K) \colon \sigma(a) = a\}$, where K is a field.

If $a \in L^n$, define

$$\mathrm{I}(a) = \{f \in K[X] \colon f(a) = 0\}.$$

If $A \subseteq K[X]$ and $B \subseteq L^n$, define

$$\mathrm{V}(A) = \bigcap_{f \in A} \mathrm{V}(f), \qquad \mathrm{I}(B) = \bigcap_{a \in B} \mathrm{I}(a).$$

The sets $\mathrm{V}(A)$ are **algebraic sets** over K. The sets $\mathrm{I}(B)$ are radical ideals, but perhaps not every radical ideal of $K[X]$ is of this form, unless L is algebraically closed. In any case, there is a Galois correspondence between the algebraic sets and the radical ideals of the form $\mathrm{I}(B)$. Moreover, since

$$\mathrm{V}(fg) = \mathrm{V}(f) \cup \mathrm{V}(g),$$

the sets $\mathrm{V}(f)$ compose a basis of closed sets for a topology on L^n, the **Zariski topology,** in which the closed sets are just the algebraic sets. (Strictly, there is a Zariski topology for every subfield K of L.)

The radical ideals $\mathrm{I}(a)$ are prime ideals; but not necessarily every prime ideal is of this form, unless we have both that L is algebraically closed, *and* that the transcendence-degree of L/K is at least n.

Let us suppose this is so. If we identify points a and b of L^n if $\mathrm{I}(a) = \mathrm{I}(b)$, then the space becomes the **spectrum** of $K[X]$: the corresponding topological space whose underlying set consists of the prime ideals of $K[X]$. The spectrum need not be Hausdorff, since it is possible to have $\mathrm{I}(a) \subset \mathrm{I}(b)$, so that every closed set that contains $\mathrm{I}(a)$ contains $\mathrm{I}(b)$, but not conversely.[4] The spectrum is however compact, since L^n itself is compact. Indeed, suppose a collection $\{\mathrm{V}(f) \colon f \in A\}$ of basic closed subsets of L^n has the Finite Intersection Property. Since

$$\mathrm{V}(f) \cap \mathrm{V}(g) = \mathrm{V}(f, g),$$

the set A must generate a proper ideal of $K[X]$. This ideal then is included in a prime ideal $\mathrm{I}(a)$, so $a \in \bigcap_{f \in A} \mathrm{V}(f)$.

In the logical situation, we identify σ and τ if $\mathbf{Mod}(\sigma) = \mathbf{Mod}(\tau)$. Then Sn becomes a Boolean algebra in the usual way. Every subset $\mathrm{Th}(\mathcal{K})$ of

[4] Of course the symbol $\subset$ here is to $\subseteq$ as $<$ is to $\leqslant$. Two errors of TEX are that `\subset` gives $\subset$ and not $\subseteq$, and `\leq` and `\le` give $\leq$ and not $\leqslant$.

Sn can now be understood as a *filter* of this algebra; and every subset $\mathrm{Th}(\mathfrak{A})$, as an *ultrafilter.* (Note that $\mathrm{Th}(\varnothing)$ is the improper filter Sn.)

However, not every ultrafilter of Sn need be the theory of some structure. For, such an ultrafilter is just a subset $\mathscr{U}$ with two properties:

1. Every finite subset of $\mathscr{U}$ has a model.
2. If $\sigma \notin \mathscr{U}$, then $\neg\sigma \in \mathscr{U}$.

In the second-order logic for $(\mathbb{N}, 1, x \mapsto x+1)$ with an additional constant-symbol c, the Peano axioms, together with the sentences $c \neq 1$, $c \neq 2$, and so on, are included in a proper filter, and therefore an ultrafilter; but they have no model.

In general, if two structures have the same theory, we may say the structures are **elementarily equivalent,** though again this term is usually reserved for first-order logic. We may denote the relation by $\equiv$. As in algebraic geometry, we may now consider $\mathbf{Mod}/{\equiv}$ instead of $\mathbf{Mod}$ itself. The points of $\mathbf{Mod}/{\equiv}$ can be considered as the theories of structures; that is, we assume

$$(\mathbf{Mod}/{\equiv}) = \{\mathrm{Th}(\mathfrak{A})\colon \mathfrak{A} \in \mathbf{Mod}\}.$$

Then $\mathbf{Mod}/{\equiv}$ is a subspace of the Stone space $\mathrm{S}(\mathrm{Sn})$ of ultrafilters of Sn. We have seen that it may be a proper subspace.

However, it is a *dense* subspace. For, the basic closed subsets of $\mathrm{S}(\mathrm{Sn})$ are the subsets $[\sigma]$, where $\sigma \in \mathrm{Sn}$ and

$$[\sigma] = \{U \in \mathrm{S}(\mathrm{Sn})\colon \sigma \in U\}.$$

(Here U stands for ultrafilter. Again, it is not necessarily the complete theory of some structure. Therefore $\sigma \in U$ should not be written as $U \vdash \sigma$.) Since

$$[\neg\sigma] = \mathrm{S}(\mathrm{Sn}) \smallsetminus [\sigma],$$

the basic closed sets are also basic open sets. If $U \in [\sigma]$, then $\sigma \neq \bot$, so σ has a model $\mathfrak{A}$, and then $\mathrm{Th}(\mathfrak{A}) \in [\sigma]$. Thus $\mathbf{Mod}/{\equiv}$ is dense in $\mathrm{S}(\mathrm{Sn})$.

The Stone space of a Boolean algebra can be identified with the spectrum of the corresponding Boolean ring. This is because prime ideals of a

Boolean ring are maximal and are the duals of ultrafilters: If $\mathfrak{p}$ is a prime ideal, then $\{\neg x \colon x \in \mathfrak{p}\}$ is an ultrafilter. This ultrafilter is also the complement of $\mathfrak{p}$.

In particular, Stone spaces are compact. They are also Hausdorff, so that compact subspaces are closed. Therefore the following are equivalent:

1. L has a compactness theorem,
2. $\mathbf{Mod}/{\equiv}$ is compact,
3. $\mathbf{Mod}/{\equiv}$ is a closed subspace of $\mathrm{S}(\mathrm{Sn})$,
4. $\mathbf{Mod}/{\equiv}$ is all of $\mathrm{S}(\mathrm{Sn})$.

In case L is a first-order logic, we can give direct proofs of (2) and (3). (Poizat gives them both.) We use **ultraproducts** in each case, and **Łoś's Theorem.** Specifically, for every indexed family $(\mathfrak{A}_i \colon i \in \Omega)$ of structures in $\mathbf{Mod}$, for every ultrafilter $\mathscr{U}$ on Ω, there is a structure $\mathfrak{A}$ such that, for all σ in Sn,

$$\mathfrak{A} \models \sigma \iff \{i \in \Omega \colon \mathfrak{A}_i \models \sigma\} \in \mathscr{U}. \tag{$*$}$$

In fact, $\mathfrak{A}$ can be taken as the ultraproduct denoted by

$$\prod_{i \in \Omega} \mathfrak{A}_i/\mathscr{U};$$

and when one proves ($*$), one will allow σ to have constants $(a_i \colon i \in \Omega)$ from $\prod_{i\in\Omega} A_i$, interpreted in each $\mathfrak{A}_i$ as a_i. This does not really give a more general result, since we can now go back and assume those constants were part of the language from the beginning.

Proof of compactness. Write $[\sigma]$ for $\{T \in \mathbf{Mod}/{\equiv} \colon \sigma \in T\}$. Suppose the collection $\{[\sigma] \colon \sigma \in B\}$ has the Finite Intersection Property. Then it generates a proper filter of subsets of $\mathbf{Mod}/{\equiv}$, so it is included in an ultrafilter $\mathscr{U}$ on $\mathbf{Mod}/{\equiv}$. If $T \in \mathbf{Mod}/{\equiv}$, then T has a model $\mathfrak{A}_T$. Let $\mathfrak{A}$ be the ultraproduct

$$\prod_{T \in \mathbf{Mod}/\equiv} \mathfrak{A}_T/\mathscr{U}.$$

Suppose $\sigma \in B$, so that $[\sigma] \in \mathscr{U}$. We have

$$\{T \in \mathbf{Mod}/{\equiv} \colon \mathfrak{A}_T \models \sigma\} = \{T \in \mathbf{Mod}/{\equiv} \colon \sigma \in T\} = [\sigma].$$

By Łoś's Theorem, $\mathfrak{A} \models \sigma$, so $\mathrm{Th}(\mathfrak{A}) \in [\sigma]$. So $\mathbf{Mod}/{\equiv}$ is compact. $\square$

We can streamline the proof by using that an arbitrary topological space is compact if and only if every ultrafilter on the underlying set includes the filter of neighborhoods of a point.[5] Then we can just start the proof with $\mathscr{U}$.

Moreover, using this criterion for compactness, we have a neat proof that the Stone space $\mathrm{S}(B)$ of an arbitrary Boolean algebra B is compact: For, if $\mathscr{U}$ is an ultrafilter on $\mathrm{S}(B)$, then it converges to the point

$$\{x \in B \colon [x] \in \mathscr{U}\},$$

where $[x] = \{F \in \mathrm{S}(B) \colon x \in F\}$. To see this, first note that the given 'point' is indeed a filter F on B, because the map $x \mapsto [x]$ from B to $\mathscr{P}(\mathrm{S}(B))$ is a Boolean algebra homomorphism; F is then an ultrafilter on B, because the homomorphism is injective. Finally, if $F \in [x]$, this just means $x \in F$, so $[x] \in \mathscr{U}$.

In this last proof, we can replace $\mathrm{S}(B)$ with an arbitrary subset Ω of it. We obtain that an ultrafilter $\mathscr{U}$ on Ω converges to the point

$$\{x \in B \colon [x] \cap \Omega \in \mathscr{U}\}$$

of $\mathrm{S}(B)$. The proof goes through as before, except that one needs to note also

$$[x] \cap \Omega \notin \mathscr{U} \iff [\neg x] \cap \Omega \in \mathscr{U}.$$

Now consider the case where B is Sn, and Ω is $\mathbf{Mod}/{\equiv}$. The limit of $\mathscr{U}$ is precisely the theory of the ultraproduct $\mathfrak{A}$ that we found above.

Indeed, Łoś's Theorem can be understood as being the statement that the limit of $\mathscr{U}$ is indeed the theory of this structure. In the original statement of the theorem, the indices are arbitrary; but we could treat the index of $\mathfrak{A}_i$ as $\mathrm{Th}(\mathfrak{A}_i)$ itself. We may have wanted $\mathfrak{A}_i$ and $\mathfrak{A}_j$ to be the same structure, or just to have the same theory, even though $i \neq j$; but we can deal with this by expanding the language.

In short, seen in the right light, the Compactness Theorem of first-order logic and Łoś's Theorem are the same, except that the latter theorem actually gives you the desired model.

As noted, we can also show directly that $\mathbf{Mod}/{\equiv}$ is closed:

[5] The topological reference that I happen to have on hand is Willard [3].

Proof of closedness. Let $U \in \mathrm{S}(\mathrm{Sn})$. Every element σ of U has a model $\mathfrak{A}_\sigma$. Also, the subsets $\{\tau\colon \tau \leqslant \sigma\}$ of U generate a proper filter, since

$$\{\tau\colon \tau \leqslant \sigma\} \cap \{\rho\colon \rho \leqslant \sigma\} = \{\tau\colon \tau \leqslant \sigma \wedge \rho\}.$$

(Remember that $\tau \leqslant \sigma$ means every model of τ is a model of σ; we can write this also as $\tau \vdash \sigma$.) Let $\mathscr{U}$ be an ultrafilter on U that includes this filter. Then the ultraproduct $\prod_{\sigma \in U} \mathfrak{A}_\sigma / \mathscr{U}$ is a model of U, since

$$\{\tau\colon \mathfrak{A}_\tau \models \sigma\} \supseteq \{\tau\colon \tau \leqslant \sigma\}. \qquad \square$$

2 Choice and Determinacy

In the first section, the Axiom of Choice was assumed. The purpose of this section is to suggest that this Axiom is not 'obviously' or 'intuitively' correct, since it contradicts another set-theoretic axiom that might be considered 'obviously' or 'intuitively' correct. That axiom is the Axiom of Determinacy, according to which, in certain *games* of infinite length, one of the players always has a winning strategy.

We consider games with two players. Hodges [1] calls these players $\forall$ and $\exists$, after Abelard and Eloise; but I propose to call them simply 0 and 1, for notational purposes. A **game** that 0 and 1 can play is determined by a partition $A_0 \amalg A_1$ of the set ${}^{\omega}2$ of binary sequences on ω. A particular **play** of the game can be analyzed as a sequence of **rounds,** indexed by ω. In round m, player 0 chooses an element a_{2m} of 2; this is the **move** of 0 in this round. Then player 1 moves by choosing an element a_{2m+1} of 2. The play itself is then the sequence $(a_n : n \in \omega)$ or a, which is an element of ${}^{\omega}2$. The play is **won** by that player e such that $a \in A_e$; and then player $1 - e$ has **lost.**

Each player e may use a **strategy,** namely a function f_e from $\bigcup_{m \in \omega} {}^{m+e}2$ to 2. (So f_0 assigns an element of 2 to each finite binary sequence; f_1 does this to every *nonempty* finite binary sequence.) If both f_0 and f_1 are chosen, then a play is determined, namely the sequence $(a_n : n \in \omega)$ given by

$$a_{2m} = f_0(a_1, a_3, \dots, a_{2m-1}), \qquad a_{2m+1} = f_1(a_0, a_2, \dots, a_{2m}),$$

or simply by

$$a_{2m+e} = f_e(a_{1-e}, a_{3-e}, \dots, a_{2m-1+e}).$$

That is, f_e determines the move of player e from the previous moves by the *other* player. The player's own previous moves need not be formally considered, since they themselves were already determined by the player's strategy and the other player's previous moves.

Suppose player $1-e$ has chosen strategy f_{1-e}. For every b in ${}^{\omega}2$, player e might choose a strategy f_e that is constant on each set ${}^{m+e}2$, having the value b_m there. The resulting play will be a, where

$$a_{2m+1-e} = f_{1-e}(b_0, b_1, \dots, b_{m-e}), \qquad\qquad a_{2m+e} = b_m.$$

This shows that, for every choice of f_{1-e}, there are continuum-many plays that can result if player $1-e$ uses this strategy.

If, using a strategy f_e, player e wins all plays of a game, then f_e is a **winning** strategy for that game. The game is **determined** if one of the players has a winning strategy. The **Axiom of Determinacy** is that in every game, one of the players has a winning strategy: in other words, for every choice of the A_e, one of the following sentences of infinitary logic is true:

$$\exists x_0\ \forall x_1\ \exists x_2\ \cdots\ (x_0, x_1, x_2, \dots) \in A_0,$$
$$\forall x_0\ \exists x_1\ \forall x_2\ \cdots\ (x_0, x_1, x_2, \dots) \in A_1.$$

However, this Axiom is false under the assumption of the Axiom of Choice, or more precisely under the assumption that the Continuum can be well-ordered, so that there is a least ordinal, called 2^{ω}, whose cardinality is that of ${}^{\omega}2$.

Indeed, every ordinal is $\alpha + n$ for some unique limit ordinal α and finite ordinal n. Then $\alpha + n$ is even or odd, according as n is even or odd. Assuming the Axiom of Choice, we can list all possible strategies as $(f^{\alpha} : \alpha < 2^{\omega})$, where f^{α} will be a strategy for e if and only if $\alpha + e$ is even.

We shall now define a list $(a^{\alpha} : \alpha < 2^{\omega})$ of possible plays (that is, elements of ${}^{\omega}2$) so that,

- for all α, if $\alpha + e$ is even, then e can use strategy f^α for the play a^α; that is, for all m in ω,

$$a^\alpha_{2m+e} = f^\alpha(a^\alpha_{1-e}, a^\alpha_{3-e}, \dots, a^\alpha_{2m-1+e});$$

- $a^\alpha \neq a^\beta$ for all distinct α and β such that $\alpha + \beta$ is odd.

We do this recursively. If $(a^\beta \colon \beta < \alpha)$ has been defined, and $\alpha < 2^\omega$, then since there are continuum-many plays in which the strategy f^α is used, one of them, to be called a^α, is not among those a^β such that $\beta < \alpha$ and $\beta + \alpha$ is odd.

Since, if $\alpha + e$ is even, player e can use strategy f^α for the play a^α, this means player $1-e$ has *some* strategy that, with f^α, determines a^α. That is, player $1 - e$ can win against strategy f^α, provided $a^\alpha \in A_{1-e}$. We now choose the partition of ${}^\omega 2$ so that

$$\{a^\alpha \colon \alpha \text{ even}\} \subseteq A_1, \qquad \{a^\alpha \colon \alpha \text{ odd}\} \subseteq A_0.$$

Then neither player has a winning strategy for the game: the game is not determined.

References

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[3] Stephen Willard, *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970. MR MR0264581 (41 #9173)