Commensurability and Symmetry

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Abstract

Commensurability and symmetry have diverged from a common Greek origin. In the Latin of Boethius, commensurable numbers are numbers not prime to one another. With Billingsley's translation of Euclid, commensurable magnitudes, including numbers, have come to be what Euclid himself called symmetric: possessed of a common measure, which for numbers can be unity alone. Symmetry has always had a vaguer sense as well: a quality that contributes to, if it does not constitute, the beauty of an object. The symmetry of a mathematical structure is given by its automorphism group; the size, by its underlying set. We measure the set by counting it, and we may express the result by a particular cardinal number: in Cantor's definition, made precise by von Neumann, this number is a certain set that is equipollent with the original set. For measuring symmetry, strictly speaking, we have no corresponding activity, because we have no simple way to select a representative from each isomorphism class of groups. Nonetheless, we allude to such representatives, as when we use the definite article to refer to the infinite cyclic group, instead of an arbitrary infinite cyclic group. Equality sometimes means identity, sometimes isomorphism or congruence. In our sign of equality, invented by Recorde, two line segments are depicted that are not the same, but their lengths are the same. It is worthwhile to pay attention to the distinction between equality and sameness, precisely because recognizing the possibility of confusing them has often been a mathematical advance.

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1 Introduction

This is about the development of commensurability and symmetry, two distinct mathematical notions with a common linguistic origin. The adjective "commensurable" is the Anglicized form of the Latin *commensurabilis,* which is itself a loan-translation of the Greek σύμμετρος. The corresponding Greek abstract noun συμμετρία comes to us as "symmetry" via the Latin transliteration *symmetria.* Thus, though having different meanings today, "commensurability" and "symmetry" are cognate words, even doublets, in the sense of deriving from the same Greek source.

Taking up the slogan, "numbers measure size, groups measure symmetry," I consider *how* numbers can measure size, before considering the corresponding question for groups and symmetry.

This consideration of numbers raises the question of whether equal numbers are the *same* number. Equality of numbers may be taken to correspond to isomorphism of groups, and isomorphic groups are usually not the same group.

Born just after the extinction in 476 of the Western Roman Empire, Boethius coined the Latin adjective *commensurabilis* for either of two numbers that are *not* relatively prime. Robert Recorde used (and perhaps created) the English term "commensurable" with the same meaning in 1557. For Recorde then, commensurable numbers had a common measure that was a *number* of units, and not simply unity itself. Thirteen years later, in translating Euclid, Billingsley used the term "commensurable" with Euclid's meaning of σύμμετρος, namely, having *any* common measure, even unity in the case of numbers.

The abstract noun "symmetry" also came into English in the sixteenth century, but not with a technical mathematical sense. Like its Greek source, συμμετρία, it referred to an interrelation of parts, and to their *proportions,* as in architecture. The adjective "symmetric" seems to have taken two more centuries to come into use, as does the crystallographic or more generally geometric notion of symmetry *with respect to* a straight line, a point, or a plane.

As "commensurability" is in origin the Latin for the Greek word "symmetry," so "proportion" is the Latin for the Greek "analogy" (ἀναλογία). Summary conclusions of the present work might be taken as negative; in particular, the analogy or proportion

$$\text{numbers} : \text{size} :: \text{groups} : \text{symmetry}$$

is imperfect, and if beauty is symmetry, this is not exactly the symmetry defined in mathematics. However, negative conclusions can sometimes (if not always) be expressed in positive terms. Gödel's *In*completeness Theorem means that mathematics is *not* the cranking out of all of the logical consequences of a given set of axioms. Negative in form, this conclusion is positive in content: "mathematical thinking is, and must remain, essentially creative," as Post said in 1944 [46, p. 295], in a passage quoted by Soare in his 1987 recursion-theory text [49, p. x]. One may object that there *are* complete axiomatizations of some interesting theories, such as the first-order theory of the ordered field of real numbers; but then one still has to decide for oneself, and perhaps to convince others, that the theory is worth studying. This is liberating. Likewise must one decide for oneself what is beautiful.

2 Numbers

2.1 Symmetry and size

A slogan from the textbook *Groups and Symmetry* by M. A. Armstrong is,

> Numbers measure size, *groups measure symmetry.*

This is how Armstrong begins his Preface [10, p. vii]. As far as I can tell, the author never defines symmetry explicitly. The word does not appear in his index. The adjective form "symmetric" does appear, as the first element of the phrase "symmetric group," and this has one reference (to page 26).

Perhaps Armstrong's slogan is to be taken as an implicit definition of symmetry. Groups will get an explicit axiomatic definition in Armstrong's Chapter 2, "Axioms," pages 6–11. Symmetry then might be understood as whatever a group can be used to measure. Similarly, intelligence has been defined as whatever an IQ test measures.[1] But whether there is any

[1] This definition was apparently first given, derisively, in 1923 by Edwin Boring, who said, "Thus we see that there is no such thing as a test for pure intelligence. Intelligence is not demonstrable except in connection with some special ability. It would never have been thought of as a separate entity had it not seemed that very different mental abilities had something in common, a 'common factor'" [12]. I encountered the reference in Lilienfeld *& al.*, "Fifty psychological and psychiatric terms to avoid: a list of inaccurate, misleading, misused, ambiguous, and logically confused words and phrases" [33]. One term

value in it or not, at least it is clear how to administer an IQ test. How would a "symmetry test" be administered?

From early childhood, we know how to use a "size test." We can measure the size of a set by counting. To measure the size of a set *is* to count it, as to measure the heaviness of a body is to weigh it. However, we can also make a precise explanation of size, independently of counting. Two sets have the **same size,** or are **equipollent,** if there is a one-to-one correspondence between them. By one definition then, **the size** of a set is its equipollence class, namely the class of all sets that have the same size as the original set. A **number** is just the size of *some* set. This definition does not require counting.

Alternatively, if possible, we can select from each equipollence class a standard element, calling *this* the number of each element of the class. For example, the sets having five elements are precisely those sets that can be put in one-to-one correspondence with the words "one, two, three, four," and "five," by the process called counting. We can now think of the number five itself in two ways:

(1) as what all five-element sets have in common, or
(2) as the set of the five words listed above, or as some other standard set of five elements.

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that the authors recommend for avoidance is "Operational definition," "the best known example in psychology [being] Boring's (1923) definition of intelligence as whatever intelligence tests measure." Thanks to Nevit Dilmen for this reference.

2.2 Cantor's aggregates

In his *Contributions to the Founding of the Theory of Transfinite Numbers,* Cantor initially takes something like the first approach. First he defines sets, or what in translation from his German are called aggregates [14, §1, pp. 85]:

> By an **aggregate** (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects m of our intuition or our thought.[2] These objects are called the **elements** of M. In signs we express this thus:
>
> $$M = \{m\}.$$

I pause to note that the the sign "=" here denotes *sameness,* which Cantor (like many of us today) will confuse with equality. Euclid *distinguishes* between equality and sameness. An isosceles triangle has two equal sides, but of course they are not the same side.

2.3 Recorde's equality

We may say that the two equal sides of an isosceles triangle have the same *length.* The sign "=" of equality is an *icon* of just this situation, in the precise sense of Peirce [40, p. 104]:

> A sign is either an *icon,* an *index,* or a *symbol.* An *icon* is a sign which would possess the character which renders it

[2] I transcribe Jourdain's translation faithfully, down to his parenthetical inclusion of Cantor's German, although I do not know German myself. However, where Jourdain puts words between quotation marks, I put the words in boldface (if they are being defined) or in italics (if they are otherwise being emphasized). For the aggregate M, Jourdain uses an upright M, although its arbitrary element m is italic, as here.

significant, even though its object had no existence; such as a lead-pencil streak as representing a geometrical line.

Robert Recorde had just this idea, when he introduced the "equals" sign in 1557 on the verso of folio Ff.i. (in roman font, Ff.i.)[3] of *The Whetstone of Witte* [47]:[4]

Howbeit, for easie alteratiõ of equations. I will propounde a fewe exãples,

[3]Recorde's book is evidently a quarto. The sheets used in printing are numbered, and the four leaves that result from folding each sheet twice are numbered. On the recto of each of first three leaves is printed a letter for the number of the original sheet, followed by a Roman numeral for the number of the leaf. Thus what we should call pages 1, 3, 5, and 9 are designated respectively A.i, A.ii, A.iii, and B.i; the intervening sheets are unmarked. The 23-letter Latin alphabet is used: A, B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, T, V, X, Y, and Z. After this come sheets Aa, Bb, Cc, and so on to Rr. The front matter consists of sheet a for the title and The Epistle Dedicatorie, and sheet b for The Preface to the gentle Reader. Thus the book is made of 2+23+17 or 42 sheets, making 336 pages, except that there are oddities: the leaves R.i. and Dd.iii. are larger, with tables. Having the book only as a `pdf` image, I do not know how these larger leaves were made.

[4]I try to reproduce the blackletter of Recorde's book. The `yfont` package for LaTeX provides Gothic, Schwabacher, and Fraktur fonts, and the Gothic seems closest to what Recorde's printer uses. However, `yfont` Gothic uses as many of Gutenberg's ligatures as possible [36, p. 395]. Recorde's printer uses no obvious ligatures, except maybe between cee (c) and tee (t), albeit not with the loop of ct. I have tried to maintain Recorde's spellings, including the tilde in place of a following en (as in õ for on). The `yfont` package does not provide the italic letters that Recorde's printer uses: the use of Schwabacher for emphasis within Gothic text is said to be "historical practice" [36, p. 394], and so I follow this practice, as for example to set the word equations (as opposed to equations), which is italic in the original. Recorde's printer's numerals are not so heavy and stylized as in `yfont` Gothic. I try to follow the printer's use of periods, which come before and after most numerals, though not all.

> bicauſe the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouſe repetition of theſe woordes : is equalle to : I will ſette as I doe often in woorke uſe, a paire of paralleles, or Gemowe[5] lines of one lengthe, thus: $=$,[6] bicauſe noe. 2. thynges, can be moare equalle. And now marke theſe nombers.

Recorde gives several examples of equations, numbered in the left margin; with $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX, I reproduce them as follows:

$$14.x. + .15.u = 71.u. \tag{2.1}$$

$$20.x. - .18.u = .102.u. \tag{2.2}$$

$$26.z + 10x = 9.z - 10x + 213.u. \tag{2.3}$$

$$19.x + 192.u = 10x + 108u - 19x \tag{2.4}$$

$$18.x + 24.u. = 8.z. + 2.x. \tag{2.5}$$

$$34z - 12x = 40x + 480u - 9.z \tag{2.6}$$

Periods are thus used freely, but inconsistently. I approximate Recorde's peculiar indeterminates or "cossic signs" with Latin letters. One should understand u here as unity, and z as x^2. On the verso of folio S.i., Recorde tells how to express each of what we should call the powers of x, from the zeroth to the twenty-fourth:

[5] Recorde's "gemowe" is an obsolete word, found in the *Oxford English Dictionary* [38] under "gemew, gemow": it derives from the Old French plural *gemeaux,* whose singular is *gemel.* The modern French singular is *jumeau,* meaning "twin," although the form *gémeau* was created in 1546, on the basis of the Latin *gemellus,* to indicate the sign of the Zodiac called in English "Gemini" [17, 48]. The older singular *gemel* also came into English, where, in the plural form "gemels," it is a heraldic term meaning "bars, or rather barrulets, placed together as a couple." Thus two gemels would seem to be like Recorde's sign of equality. The Latin *gemellus* is the diminutive of *geminus.*

[6] Recorde's sign is much longer, more like $=\!=\!=\!=$.

$[u]$	Betokeneth nomber abſolute aſ if it had no ſigne.
$[x]$	Signifieth the roote of any nomber.
$[z]$	Repreſenteth a ſquare nomber.
$[c]$	Expreſſeth a Cubike nomber.
$[zz]$	Is the ſigne of a ſquare of ſquares, or Zenzizenzike.
$[sz]$	Standeth for a Surſolide.
$[zc]$	Doeth ſignifie a Zenzicubike, or a ſquare of Cubes.
$[bsz]$	Doeth betoken a ſeconde Surſolide.
	....
$[zzzc]$	Signifieth a ſquare of ſquares, of ſquared Cubes.

Recorde thus varies the word (“betokeneth,” “signifieth,” *&c.*) used to say that the meaning of a sign is being given. Along with the zeroth and first, each prime power of what we call x is for Recorde a different new symbol. The fifth power, the *sursolid,* is obtained from the second power by prefixing an elongated ess, like our integral sign $\int$. The higher prime powers, from seventh to 23rd, are the second to sixth sursolids respectively; their symbols are obtained from that of the first sursolid by prefixing the letters from b to f. The symbols for composite powers are the appropriate composites of the symbols for prime powers. Four pages later (on the verso of folio S.iiii.), The table of Coßike ſignes, and their peculier nombers (see Figure 2.1) gives what we should call the exponents for the first 14 signs, and it is explained that multiplying the signs corresponds to adding the exponents.

Thus we see one stage in the development of the form of the polynomial equation. The main point is that our sign of equality, as introduced by Recorde, is an icon of two equal, but distinct, straight lines. Equality in origin is not sameness, though today we use the sign of equality to indicate that

0.	1.	2.	3.	4.	5.	6.
$[u]$	$[x]$	$[z]$	$[c]$	$[zz]$	$[sz]$	$[zc]$
7.	8.	9.	10.	11.	12.	13.
$[bsz]$	$[zzz]$	$[cc]$	$[zsz]$	$[csz]$	$[zzc]$	$[dsz]$

Figure 2.1: "The table of Cossike signes"

two different expressions denote the same thing. This is what Cantor will do explicitly below.

2.4 Cantor's cardinal numbers

After considering what we call *unions* of sets, and *subsets* of particular sets, Cantor continues [14, §1, pp. 86]:

> Every aggregate M has a definite *power,* which we will also call its *cardinal number.*
>
> We will call by the name **power** or **cardinal number** of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given.
>
> We denote the result of this double act of abstraction, the cardinal number or power of M, by
>
> $$\overline{\overline{M}}.$$

So $\overline{\overline{M}}$ is a "general concept." This is as vague as "what all sets having the size of M have in common." However, Cantor has not yet defined having the same size. He immediately starts groping towards a second approach to number, where a number is a standard element of an equipollence class:

> Since every single element m, if we abstract from its nature, becomes a **unit,** $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M.
>
> We say that two aggregates M and N are **equivalent,** in signs
>
> $$M \sim N \text{ or } N \sim M,$$
>
> if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other.[7]

Cantor goes on to observe that "equivalence" (what we have called equipollence, or having the same size) is indeed what we now call an equivalence relation: it is symmetric (as above), reflexive, and transitive. Moreover,

> Of fundamental importance is the theorem that two aggregates M and N have the same cardinal number if, and only if, they are equivalent: thus,
>
> $$\text{from } M \sim N \text{ we get } \overline{\overline{M}} = \overline{\overline{N}},$$
>
> and
>
> $$\text{from } \overline{\overline{M}} = \overline{\overline{N}} \text{ we get } M \sim N.$$
>
> Thus the equivalence of aggregates forms the necessary and sufficient condition for the equality of their cardinal numbers.

Here is where sameness and equality are explicitly confused. In any case, Cantor derives his latter implication from the general equivalence

$$M \sim \overline{\overline{M}}$$

[7] Cantor's symbol for equipollence, at least in Jourdain's translation, is curvier than the $\sim$ of TEX.

and the transitivity of equivalence. The former implication might be said to follow similarly from the implication

$$\overline{\overline{M}} \sim \overline{\overline{N}} \implies \overline{\overline{M}} = \overline{\overline{N}};$$

but Cantor himself does not seem to suggest such an intermediate step. He argues:

> In fact, according to the above definition of power, the cardinal number $\overline{\overline{M}}$ remains unaltered if in the place of each of one or many or even all elements m of M other things are substituted. If, now, $M \sim N$, there is a law of co-ordination by means of which M and N are uniquely and reciprocally referred to one another; and by it to the element m of M corresponds the element n of N. Then we can imagine, in the place of every element m of M, the corresponding element n of N substituted, and, in this way, M transforms into N without alteration of cardinal number. Consequently
>
> $$\overline{\overline{M}} = \overline{\overline{N}}.$$

The validity of this argument can be questioned, just as some of Euclid's arguments are questioned.

2.5 Ambiguity of equality

In the fourth proposition of Book I of the *Elements* [19], Euclid proves what today we call the "Side-Angle-Side" condition for congruence of triangles. We may say that the proposition is not a theorem, but a postulate, as it is for example in the Weeks–Adkins textbook that I used in high school [53, p. 61]. Nonetheless, Euclid gives a proof; but here he "applies" one triangle to another, and this is not accounted for among his postulates. As Fitzpatrick says in a note to his own translation, "The application of one figure to another should be counted as an additional postulate" [23, p. 11].

I do not agree; but I believe I can understand Fitzpatrick's inclination. On the originally blank last page of my copy of the Weeks–Adkins geometry text, I find a list, in my own hand, of "Statements unmentioned but neccessary [*sic*]":

- If $A = B$ at one place and time, then $A = B$ at any place and time, provided A and B always represent the same things.
- Line AB is the same as line BA, provided each A and each B represent the same points.
- If two people are to discuss geometry, they must have a common language.

Such were my concerns in high school. Though our proofs in the geometry course were supposed to make everything explicit, I had evidently been troubled to realize that we were not achieving this goal.

I do not think my list of tacit conventions in our text was the direct result of a lecture by the teacher, though above the list I find something that I could have copied from the blackboard: a table showing the converse, inverse, and contrapositive of the statement "If A, then B." From the course I remember an exercise involving the "trisector" of a line segment or angle. I refused to perform the exercise, since the concept of trisection had not been formally defined. I was not the only student troubled by this exercise. The teacher ridiculed us, observing that it was obvious what trisection meant. She was right, though I was incensed at the time. Had not the whole purpose of the geometry course been to establish that "obviousness" was not a sufficient criterion for mathematical truth?

It had; but I think our text itself had gone overboard with this idea. The book lists "Algebraic Properties of Equality and Inequality" [53, p. 41]. I see that I crossed out "Properties" and wrote "Theorems" above. The properties or theorems are of the form,

If $a = b$ and $c = d$, then $a + c = b + d$.

This is the "Addition Property of Equality," and there is also an "Addition Property of Inequality," which to my mind now is of different logical status, though this is not said:

If $a > b$ and $c > d$, then $a + c > b + d$.

Subtraction, multiplication, and division properties of equality and inequality are also given. As has been explained in the text, "The letters a, b, c, and d are symbols for positive numbers"; and before that,

> Statements of the form "a is equal to b" occur throughout algebra and geometry. The symbols a, b refer to elements of some set and the basic meaning of $a = b$ is that a and b are names for the same element . . . In our geometry, $AB = CD$ means that line segment AB and line segment CD have the same length, and $\angle X = \angle Y$ means that angle X and angle Y have the same measure. In each case the equality is a statement that the *same number* gives the measure of both geometric quantities involved.

If the "basic meaning" of equality is sameness, then the word "basic" is being used in its slang sense of "approximate," as in, "The proof is basically correct, but has some small errors." For Weeks and Adkins go on to tell us that in geometry, equality is not actually sameness, but sameness of some *property.* Thus, with geometric objects, it does need to be made explicit somehow that equality is preserved under addition. Recognizing this, Euclid gives what is counted now as his second "common notion":

If equals be added to equals, the wholes are equal.

But in the Weeks–Adkins "Addition Property of Equality," the letters stand for numbers, and equality of numbers *is* sameness. In this case, the "Addition Property" and the rest should go without saying. Indeed, my classmates and I were told this by a different teacher in the following year, in a precalculus class, when we started proving things from the axioms of $\mathbb{R}$ as an ordered field, and we asked the teacher why we were not proving the "Addition Property" as a theorem.

Since Euclid introduces no symbolism for the length of a line segment, as opposed to the segment itself, his notion of equality is unambiguous. It is congruence. This is made explicit in the common notion that is now numbered fifth, following Heiberg's bracketing of two earlier common notions in manuscripts:

> Things congruent to one another are equal to one another.

Heath uses "coincide" for "congruent" [22]; but Heiberg's Latin is, *quae inter se congruunt, aequalia sunt.* The Greek verb is *ἐφαρμόζω*, or *ἐπί* + *ἁρμόζω*, the root verb being the origin of our "harmony." To say that two line segments are equal is to say that one can be picked up and placed on the other so that they "harmonize," that is, coincide. In Euclid's Proposition I.4, it is assumed about given triangles ΑΒΓ and ΔΕΖ that ΑΒ, ΑΓ, and the included angle are respectively equal to ΔΕ, ΔΖ, and the included angle. By definition of equality, this means ΑΒ can be placed on ΔΕ so that they coincide, and then the angles will coincide, and then ΑΓ and ΔΖ will coincide, so that the remaining features of the triangle are respectively equal.

That is a proof. Or we can call it an "intuitive justification" for what is "really" a postulate. But Cantor's quoted argument for the implication

$$M \sim N \implies \overline{\overline{M}} = \overline{\overline{N}}$$

does not even rise to this level. I think the argument fails at the start for not observing more precisely that $\overline{\overline{M}}$ is unchanged if distinct elements of M are replaced with other *distinct* things. Despite the earlier description, $\overline{\overline{M}}$ cannot consist of "units" simply, without any way to distinguish between different units. Cantor does not provide such a way.

2.6 Euclid's numbers

Euclid does not have Cantor's problem in the *Elements.* The definitions at the head of Book VII are indeed vague [20]:

<u>Μονάς</u> *ἐστιν, καθ' ἣν*
ἕκαστον τῶν ὄντων ἓν λέγεται.
<u>Ἀριθμὸς</u> *δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.*
Unity is that according to which
each entity is said to be one thing.
And a **number** is a multitude of unities.

I translate Euclid's *μονάς* as "unity" here, although Heath uses "unit" [22]. In his "Mathematicall Preface" [16] to Billingsley's 1570 translation of the *Elements,* John Dee notes explicitly in the margin that he has *created* the word "unit" precisely to translate Euclid's *μονάς*. However, Billingsley uses "unity" in his own translation [18].[8] An abstract noun does seem called for, at least in the first instance above of *μονάς*. An alternative might be "oneness."[9] English also has the option of coining

[8]The relevant passages of Dee and Billingsley are quoted in the *OED* [38] in the articles "Unit" and "Unity" respectively.

[9]Euclid's *ἕν* "one" has neuter gender, but the feminine form of the adjective is *μία*, and both forms (along with the masculine *εἷς*) have the root SEM. However, it is not clear whether the M here relates these words to *μονάς* in the way that "one" is related to "oneness." Chantraine's *Dictionaire étymologique de la langue grecque* [15] gives no indication

the word "monad" for the Greek $\mu o \nu \acute{\alpha} \varsigma$, and English has in fact done this, as for rendering the philosophy of Leibniz, or in Jowett's translation of the words of Socrates in Plato's *Phaedo* [43, 105B–C, p. 245]:

> I mean that if any one asks you "what that is, of which the inherence makes the body hot," you will reply not heat (this is what I call the safe and stupid answer), but fire, a far superior answer . . . and instead of saying that oddness is the cause of odd numbers, you will say that the monad is the cause of them . . . [10]

In any case, it is possible that the definitions found in the *Elements* were not put there by Euclid. As the diagrams of Euclid's *propositions* indicate, the unities or units or monads that make up Euclid's numbers are not so abstract as to be devoid of distinctions. Each of Euclid's numbers can be conceived of as a bounded straight line, each of its units being a different part of the whole. The number itself is then the *set*

---

of a connection between μία and μονάς. On the other hand, neither does Chantraine suggest a connection between εἷς, μία, ἕν and the prefix συν- (originally ξυν-, and appearing as συμ- in συμμετρία), while the *American Heritage Dictionary* [37] alludes to a presumed connection. Here the entry **syn-** in the dictionary proper refers to **sem-¹** in the Appendix of Indo-European Roots. This may be an error, since in the Appendix itself, the modern "syn-" is found not under **sem-¹,** but under **ksun.** However, both **sem-¹** and **ksun** are referred to the same entry *sem-* in Pokorny's *Indogermanisches Etymologisches Wörterbuch.* Perhaps an editor of the *AHD* came to think Pokorny too bold in tracing συν- and ἕν unequivocally to a common root, but failed to make all changes needed to reflect this change of heart.

[10] The example of Jowett is quoted in the *OED.* The Loeb translation of Fowler [44, p. 363] has "the number one" for Socrates's μονάς and thus Jowett's "monad," but this may be misleading, inasmuch as a monad is not a number of things, but one thing: in short, one is not a number.

of these parts. Two *different* numbers can be equal: Euclid makes this clear in Proposition VII.8, where he lays down one number that is equal to another, though different from it. He does this for the convenience of diagramming the argument, since the equal numbers are going to be divided differently into parts.

At least one modern textbook seems to allow different numbers to be equal. Near the beginning of his *Fundamental Concepts of Algebra* [35, pp. 2, 3], Bruce Meserve writes:

> The numbers that primitive man first used in counting the elements of a set of objects are called *natural numbers* or *positive integers.* Technically, the positive integers are symbols. They may be written as /, //, ///, . . . ; i, ii, iii, . . . ; 1, 2, 3, . . . ; or in many other ways . . .
>
> Comparisons between cardinal numbers must agree with the corresponding comparisons between the sets of elements represented by the cardinal numbers. Accordingly, the cardinal numbers a, b associated with the sets A, B are equal (written $a = b$) and the sets are said to be *equivalent* if there exists a one-to-one correspondence between the elements of the two sets . . .

On page 1 of Meserve's book, a footnote has explained that "new terms will be italicized when they are defined or first identified." However, the word "equal" is not italicized in the passage above. It is not clear whether Meserve would write such equations as

$$/// = 3, \qquad\qquad 3 = \mathrm{iii}.$$

Still, ///, 3, and iii would seem to be different as symbols, and Meserve has said that numbers are symbols. Presently he seems almost pointedly to avoid treating equality as sameness [35, p. 4]:

> Given any two finite sets A, B with cardinal numbers a, b, we may compare the cardinal numbers using the subsets 1, 2, . . . , a and 1, 2, . . . , b of the set of positive integers. Let C be the set 1, 2, . . . , c of positive integers that are in both these subsets. If $c = a$ and $c \neq b$, then $a < b$. If $c = a$ and $c = b$, then $a = b$. If $c = b$ and $c \neq a$, then $b < a$. Thus we have proved that for any two finite sets A, B with cardinal numbers a, b exactly one of the relations $a < b$, $a = b$, $a > b$ must hold.

It is not clear why a third letter c is needed here after a and b; but its introduction is reminiscent of Euclid's introduction of a new number that is different from but equal to an earlier number. Meserve goes on to treat equality as a generic equivalence relation [35, pp. 7, 8]:

> Any relation having the three properties:
>
> reflexive, $a = a$,
> symmetric, $a = b$ implies $b = a$,
> transitive, $a = b$ and $b = c$ imply $a = c$,
>
> is called an *equivalence relation.* The equivalence of sets and therefore the equality of cardinal numbers as defined [above] can be proved to be an equivalence relation as follows . . .
>
> One can also prove under the usual definitions that "identity" ($\equiv$), "congruence" ($\cong$) of geometric figures, and "similarity" ($\sim$) of geometric figures are equivalence relations. Thus each of the symbols $=$, $\equiv$, $\cong$, $\sim$ represents "equals" in a well-defined mathematical sense. We now use the equivalence relation $=$ in a characterization of the positive integers by means of Peano's postulates . . .

It is not clear what Meserve means by identity symbolized by $\equiv$. His book's word index features identity only in the phrases "identity element under an operation," "identity relation," and "identity transformation." Under "identity relation," the corresponding pages are only 102 and 134, where it is established that an equation of polynomials is an identity if it holds for all values of the indeterminates; otherwise the equation is conditional. Meserve's index of symbols and notation features $\equiv$ only for congruence of integers with respect to a modulus. Gauss establishes this use of the symbol at the beginning of the *Disquisitiones Arithmeticae* [24, p. 1] and remarks in a footnote,

> We have adopted this symbol because of the analogy between equality and congruence. For the same reason Legendre . . . used the same sign for equality and congruence. To avoid ambiguity we have made a distinction.

Presumably the analogy between equality and congruence lies in their being what we now call equivalence relations.

Meserve *is* sensitive to one foundational issue. Unlike what many people, including Peano himself, seem to think, while induction establishes that only one operation of addition can be defined recursively by the rules $a + 1 = a^+$ and $a + b^+ = (a + b)^+$, induction does not *obviously* establish that such an operation exists at all. Meserve knows this, at least through Landau [30], whom he cites. See my own article, "Induction and Recursion" [41].

2.7 Von Neumann's ordinal numbers

We have now seen that Euclid's geometry provides a way to understand numbers as sets of distinct units, which is something that Cantor and some of his successors have failed to do. However, today we may prefer not to rely on geometry as a foundation of our mathematics. For example, geometry may not well accommodate a straight line consisting of uncountably many units. In this case, we can understand numbers as von Neumann does.

First we should note that, in addition to cardinal numbers, Cantor defines *ordinal numbers* [14, §7, pp. 111–2, & §12, p. 137]:

> Every ordered aggregate M has a definite **ordinal type,** or more shortly a **type,** which we will denote by
>
> $$\overline{M}.$$
>
> By this we understand the general concept which results from M if we only abstract from the nature of the elements m, and retain the order of precedence among them. Thus the ordinal type $\overline{M}$ is itself an ordered aggregate whose elements are units which have the same order of precedence amongst one another as the corresponding elements of M, from which they are derived by abstraction.
>
> ....
>
> Among simply ordered aggregates *well-ordered aggregates* deserve a special place; their ordinal types, which we call **ordinal numbers,** form the natural material for an exact definition of the higher transfinite cardinal numbers or powers,—a definition which is throughout conformable to that which was given us for the least transfinite cardinal number Aleph-zero by the system of all finite numbers ν (§6).

On the contrary, Cantor's definitions are not exact. Von Neumann points this out as follows [52].

> The aim of the present paper is to give unequivocal and concrete form to Cantor's notion of ordinal number.
>
> Ordinarily, following Cantor's procedure, we obtain this notion by "abstracting" a common property from certain classes of sets [14]. We wish to replace this somewhat vague procedure by one that rests upon unequivocal set operations. The procedure will be presented below in the language of

naive set theory, but, unlike Cantor's procedure, it remains valid even in a "formalistic" axiomatized set theory . . .

What we really wish to do is to take as the basis of our considerations the proposition: "Every ordinal is the type of the set of all ordinals that precede it." But, in order to avoid the vague notion "type," we express it in the form: "Every ordinal is the set of ordinals that precede it." This is not a proposition proved about ordinals; rather, it would be a definition of them if transfinite induction had already been established. According to it, we have

$$\begin{aligned}
&0 = \varnothing,\\
&1 = \{0\},\\
&2 = \{0, 1\},\\
&3 = \{0, 1, 2\},\\
&\ldots\ldots\ldots\ldots,\\
&\omega = \{0, 1, 2, \ldots\},\\
&\omega + 1 = \{0, 1, 2, \ldots, \omega\},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$ [11]

Thus, for von Neumann, the number five becomes a certain set of five elements, namely $\{0, 1, 2, 3, 4\}$.

Most mathematicians seem not to think of numbers as sets. When they need a set with five elements, they use $\{1, 2, 3, 4, 5\}$. When they need a set with n elements, they use $\{1, \ldots, n\}$. They may however prefer a simpler notation for this set. For example, during the development of groups in his *Algebra,* Lang writes in two different places [31, pp. 13, 30]:

[11] I have simplified von Neumann's equations by allowing numbers already defined to be used in later definitions. Von Neumann writes out all of the definitions here in terms of the empty set, which he denotes by O; and he denotes sets by $(\ldots)$ rather than $\{\ldots\}$.

> Let $J_n = \{1, \ldots, n\}$. Let S_n be the group of permutations of J_n. We define a **transposition** to be a permutation τ such that there exist two elements $r \neq s$ in J_n for which $\tau(r) = s$, $\tau(s) = r$, and $\tau(k) = k$ for all $k \neq r, s \ldots$
>
> ....
>
> Let S_n be the group of permutations of a set with n elements. This set may be taken to be the set of integers $J_n = \{1, \ldots, n\}$. Given any $\sigma \in S_n$, and any integer i, $1 \leqq i \leqq n$, we may form the orbit of i under the cyclic group generated by σ. Such an orbit is called a **cycle** for $\sigma \ldots$

This seems like a needless profusion of symbols.[12] If one uses von Neumann's definition, then n itself is an n-element set, and one has no need for notation like J_n. If one blanches at the thought of saying "Let S_n be the group of permutations of n," one may of course introduce notation like Lang's J_n; but why not define it to mean $\{0, \ldots, n-1\}$, namely von Neumann's n? Our theme is that numbers measure size; and the beginning of size in general is not 1 but 0. When we measure a line with a ruler, at one end of the line we place the point of the ruler that is marked 0. See Figure 2.2.

[12] In another sense, Lang displays parsimony with symbols, or at least with words, allowing the expression $J_n = \{1, \ldots, n\}$ to serve both for the clause "J_n be equal to $\{1, \ldots, n\}$" and for the noun phrase "J_n, which is equal to $\{1, \ldots, n\}$." The inequation $r \neq s$ stands for the noun phrase "r and s, which are unequal"; strictly speaking it is not even necessary to say that they are unequal, since they have already been described as "two." The equation $\tau(r) = s$ stands not for a noun, but for the declarative sentence "$\tau(r)$ is equal to s." I have known students to be confused by such sloppiness, and Halmos somewhere inveighs against it. Nonetheless, its prevalence does show that there is a difference between doing good mathematics and expressing mathematics well.

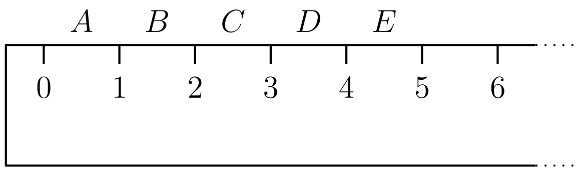

Figure 2.2: The measure of the set $\{A, B, C, D, E\}$ is 5

3 Symmetries

3.1 Symmetry

If groups measure symmetry, what does this mean? The object whose symmetry is being measured may not be simply a set. It is best considered (if only implicitly) as an object in a so-called *category.* From one object to another in a category, there may be *homomorphisms.* Some of these may be *invertible,* in which case they are *isomorphisms.* An invertible homomorphism from an object to itself is an *automorphism.* The automorphisms of an object compose a *group,* the group operation being functional composition. Then by the most general definition, two objects, possibly in two different categories, have the **same symmetry** if their automorphism groups are isomorphic to one another as objects in the category of groups.

The objects of a *concrete category* have "underlying sets," and the objects themselves are "sets with structure"; a homomorphism from one object to another is a function from the one underlying set to the other that "preserves" this structure. Then two objects of (possibly different) concrete categories have the **same size** if their underlying sets are isomorphic to one another in the category of sets.

Is there now perhaps some lack of parallelism, some *asymmetry,* in the slogan, "Numbers measure size, groups measure symmetry"? In the "categorical" definition of sameness of symmetry, groups are mentioned; in the “categorical” definition of sameness of size, not numbers but *sets* are mentioned. One might say that it is sets that measure size; more precisely, the underlying set of an object of a concrete category is the measure of the size of the object itself.

One might then ask whether extracting this underlying set is parallel to extracting the automorphism group of an arbitrary category. Symbolically, let an object A of a category have the automorphism group $\mathrm{Aut}(A)$; if the category is concrete, let A have the underlying set $\mathrm{Dom}(A)$, the “domain” of A. Objects A and B have the same size if

$$\mathrm{Dom}(A) \cong \mathrm{Dom}(B);$$

A and B have the same symmetry, if

$$\mathrm{Aut}(A) \cong \mathrm{Aut}(B).$$

The operation $X \mapsto \mathrm{Dom}(X)$ somewhat corresponds to Cantor’s operation $X \mapsto \overline{\overline{X}}$, but has the advantage of a clear meaning.

If the slogan “Numbers measure size, groups measure symmetry” is to express a thorough-going analogy, we should understand a number to be nothing other than a pure set, that is, an object in the category of sets. The number of an object in a concrete category would then be the underlying set of the object. This usage of “number” would be compatible with Euclid’s usage, though not with ours, since equipollent sets are not necessarily equal.

Today, every equipollence class of sets contains an ordinal number and therefore a least ordinal number, which is the cardinal number of every set in the class. However, there is no useful way to designate, within every isomorphism class of automorphism groups, a particular element that shall serve as *the* group of every object whose automorphism group belongs to the class.[1] Thus it would be more accurate to say,

- numbers measure size, *isomorphism classes* of groups measure symmetry; or
- sets measure size, groups measure symmetry; or even
- sets have size, groups have symmetry.

3.2 Groups of symmetries

Lang hints at the understanding of groups as automorphism groups. Right after the abstract definition of a group as a monoid with inverses, he gives several examples, although they are abstract as well:

- If a group and a set are given, then the set of maps from the set into the group is itself a group.
- The set of permutations of a set is a group.
- The set of invertible linear maps of a vector space into itself is a group, as is the set of invertible $n \times n$ matrices over a field.

This is at [31, I, §2, p. 8]. The next "example" is:

> **The group of automorphisms.** We recommend that the reader now refer to §11, where the notion of a category is defined, and where several examples are given. For any object A in a category, its automorphisms form a group denoted by $\mathrm{Aut}(A)$. Permutations of a set and the linear automorphisms of a vector space are merely examples of this more general structure.

[1] If one works in Gödel's universe of *constructible* sets, then one does have a way to select a representative from each isomorphism-class of groups; but it is not a useful way, for present purposes.

We may understand $\mathrm{Aut}(A)$, or rather its isomorphism class, as the measure of the symmetry of A. Lang however does not speak of symmetry as such. Between the two instances quoted above where the notation J_n is used, Lang observes [31, I, §5, p. 28]:

> The symmetric group S_n operates transitively on $\{1, 2, \ldots, n\}$.

The term "symmetric group" here is not given any special typographical treatment, although it represents the first use of the term "symmetric" in the index (and the term "symmetry" is not in the index). Other terms are made bold when Lang defines them.

According to the index in his own *Algebra,* Hungerford uses the term "symmetry" once, to refer to any of the eight symmetries of the square, defined as an example [27, I.1, p. 26]. In his philosophical book *Mathematics: Form and Function,* Mac Lane defines a symmetry this way, as a rigid motion of a figure ("a collection of points") onto itself [34, I.6, pp. 17 & 19].

Armstrong uses the term "symmetry" in this way too, but also more abstractly. Again, he does not actually define the term: perhaps this would not be in keeping with his informal treatment. After his opening slogan, Armstrong says what he expects of his audience, which is basically that they have some experience of undergraduate mathematics:

> The first statement ["numbers measure size"] comes as no surprise; after all, that is what numbers "are for". The second ["groups measure symmetry"] will be exploited here in an attempt to introduce the vocabulary and some of the highlights of elementary group theory.
>
> A word about content and style seems appropriate. In this volume, the emphasis is on examples throughout, with

> a weighting towards the symmetry groups of solids and patterns. Almost all the topics have been chosen so as to show groups in their most natural role, acting on (or permuting) the members of a set, whether it be the diagonals of a cube, the edges of a tree, or even some collection of subgroups of the given group . . .
>
> . . . .
>
> As prerequisites I assume a first course in linear algebra (including matrix multiplication and the representation of linear maps between Euclidean spaces by matrices, though not the abstract theory of vector spaces) plus familiarity with the basic properties of the real and complex numbers. It would seem a pity to teach group theory without matrix groups available as a rich source of examples, especially since matrices are so heavily used in applications.

Armstrong goes on to use the word "symmetry" as if it were a word like "language": it denotes a concept, but also an instance of the concept. We use *language* to communicate; Turkish is *one language.* The definitions in the *Elements* discussed above use *μονάς* in this twofold way: it is the concept of unity, and it is anything that has unity. Thanks to John Dee, we can use the word "unit" for something with unity. Armstrong's twofold use of "symmetry" is seen, even at the beginning of his Chapter 1, "Symmetries of the Tetrahedron":

> How much symmetry has a tetrahedron? Consider a regular tetrahedron T and, for simplicity, think only of rotational symmetry. Figure 1.1 [Figure 3.1] shows two axes. One, labelled L, passes through a vertex of the tetrahedron and through the centroid of the opposite face; the other, labelled M, is determined by the midpoints of a pair of opposite edges. There are four axes like L and two rotations about each of these, through $2\pi/3$ and $4\pi/3$, which send the tetrahedron to itself. The sense of the rotations is as shown:

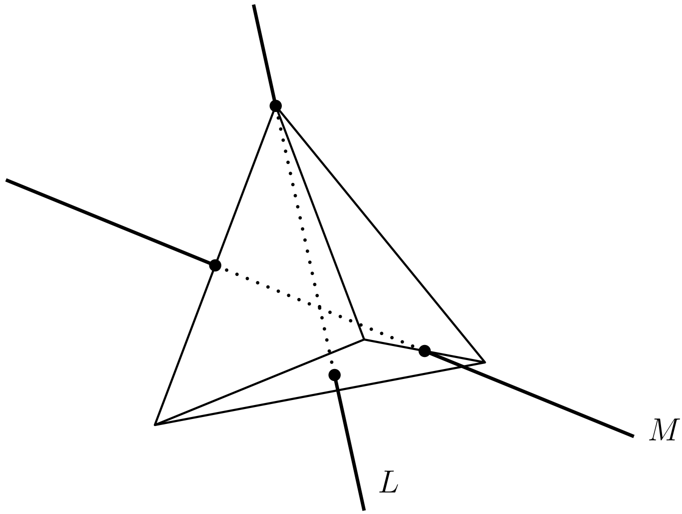

Figure 3.1: A recasting of Armstrong's Figure 1.1

> looking along the axis from the vertex in question the opposite face is rotated anticlockwise. Of course, rotating through $2\pi/3$ (or $4\pi/3$) in the opposite sense has the same effect on T as our rotation through $4\pi/3$ (respectively $2\pi/3$). As for axis M, all we can do is rotate through π, and there are three axes of this kind. So far we have $(4 \times 2) + 3 = 11$ symmetries. Throwing in the identity symmetry, which leaves T fixed and is equivalent to a full rotation through 2π about any of our axes, gives a total of twelve rotations.

Each of these twelve rotations is *a symmetry* of the tetrahedron. Presumably twelve of them together constitute a measure of *the symmetry* of the tetrahedron. However, Armstrong goes on to observe that this measure is not simply the number twelve:

> We seem to have answered our original question. There are precisely twelve rotations, counting the identity, which

> move the tetrahedron onto itself. But this is not the end of the story. A flat hexagonal plate with equal sides also has twelve rotational symmetries (Fig. 1.2), as does a right regular pyramid on a twelve sided base (Fig. 1.3).

The respective groups of rotational symmetries of the three objects have order twelve, but no two are isomorphic to one another, and therefore none embeds in another. Thus the collection of isomorphism-classes of symmetry groups is only partially ordered. This does happen to be true for the collection of equipollence-classes of sets as well, unless we assume the Axiom of Choice.

4 *Symmetria*

Symmetry then is a way of understanding a mathematical structure that is more subtle than simply counting the number of its underlying individuals. Why is it called symmetry? The Greek abstract noun *συμμετρία* is evidently the source of the English noun, and citations in the *Greek–English Lexicon* of Liddell and Scott [32] provide one way to research the meaning of the former.

4.1 Commensurability

The citations of the corresponding adjective *σύμμετρος -ον* do not include the first of the definitions at the head of Book X of Euclid's *Elements* [21]:

> <u>*Σύμμετρα*</u> *μεγέθη λέγεται τὰ τῷ αὐτῷ μέτρῳ μετρούμενα,*
> <u>*ἀσύμμετρα*</u> *δέ, ὧν μηδὲν ἐνδεχεται κοινὸν μέτρον γενέσθαι.*
> Magnitudes measured by the same measure
> are called **commensurable;**
> those that admit no common measure, **incommensurable.**

Evidently the English word "commensurable" could have been formed out of Latin components precisely to translate Euclid's *σύμμετρος*. In fact the history will turn out to be more complicated.

The *Lexicon* gives Euclid's *meaning* for the word *σύμμετρος*. It also quotes the *words* of Euclid given above; but it does so in their earlier expression by Aristotle, and with the feminine gender of *γραμμά* “line,” rather than the neuter gender of *μέγεθος* “magnitude” (the masculine and feminine of *σύμμετρος* are identical). The lexicon entry reads:

> *commensurate with, of like measure* or *size with* . . . : esp. of Time, *commensurate with, keeping even with* . . . 2. in Mathematics, *having a common measure,* *σύμμετροι αἵ τῷ αὐτῷ μέτρῳ μετρούμεναι* (sc. *γραμμαί*) Arist. *LI*968b6; freq. denied of the relation between the diagonal of a square and its side . . . *μήκει οὐ σύμμετροι τῇ ποδιαίᾳ* not lineally *commensurate* with the one-foot side, Pl. *Tht.* 147d, cf. 148b . . . II. *in measure with, proportionable, exactly suitable* . . .

Here “Arist. *LI*” is *De Lineis Insecabilibus,* an obscure work attributed to Aristotle, but not with certainty, as Joachim says in his Introductory Note [2]. His comments serve as a reminder of the difficulty of making sense of ancient mathematics: it needs the knowledge, skills, and experience of both the classicist and the mathematician:

> The treatise *Περὶ ἀτόμων γραμμῶν*, as it is printed in Bekker’s Text of Aristotle, is to a large extent unintelligible. But . . . Otto Apelt, profiting by Hayduck’s labours and by a fresh collation of the manuscripts, published a more satisfactory text . . .
>
> In the following paraphrase, I have endeavoured to make a full use of the work of Hayduck and Apelt, with a view to reproducing the subtle and somewhat intricate thought of the author, whoever he might have been . . . there are grounds for ascribing [the treatise] to Theophrastus: whilst, for all we can tell, it may have been . . . by Strato, or possibly some one otherwise unknown. But the work . . . is interesting . . . Its value for the student of the History of Mathematics is no doubt considerable: but my own ignorance of this subject makes me hesitate to express an opinion.

In Bekker's edition, *De Lineis Insecabilibus* is five pages [1, pp. 968–72], The quotation in the *LSJ* lexicon is drawn from the following account of a specious argument:

> Again, the being of 'indivisible lines' (it is maintained) follows from the Mathematicians' own statements. For if we accept their definition of 'commensurate' lines as those which are measured by the same unit of measurement, and if we suppose that all commensurate lines actually are being measured, there will be some actual length, by which all of them will be measured. And this length must be indivisible. For if it is divisible, its parts—since they are commensurate with the whole—will involve some unit of measurement measuring both them and their whole. And thus the original unit of measurement would turn out to be twice one of its parts, viz. twice its half. But since this is impossible, there must be an indivisible unit of measurement.

The argument may be the following, which is more or less what Joachim suggests in his notes:

1. Every line is commensurable, in the sense of having a common measure with some other line.
2. Thus all lines are commensurable with one another.
3. In particular, all lines have a common measure.
4. A common measure of all lines must be indivisible.
5. Therefore there is an indivisible line.

Perhaps the first step is even simpler: every line is commensurable in the sense of being *mensurable,* that is, measurable. Perhaps also the second step is lacking. In any case, the second step does not follow from the first, and the third step follows from neither the second nor the first. In the notation of modern symbolic logic, the first three proposed steps above are

$$\forall x\ \exists y\ \exists z\ (z \mid x \wedge z \mid y),$$
$$\forall x\ \forall y\ \exists z\ (z \mid x \wedge z \mid y),$$
$$\exists z\ \forall x\ \forall y\ (z \mid x \wedge z \mid y).$$

The confusion of the argument may be reflected in the superficial similarity of sentences having different logical form, such as "These two angles are acute" and "These two angles are equal." The first abbreviates "These two angles are *each* acute"; the second, "These two angles are equal *to one another.*" Perhaps having recognized the potential ambiguity, Euclid often (though not always) uses the qualification, "to one another," when it fits. (See the example of *Elements* V.9 in §4.4 below.)

Again at the head of Book X, Euclid does provide a way to to call an individual magnitude commensurable, once some line of reference has been fixed. The reference line, along with any other straight line, the square on which is commensurable with the square on the reference line, is to be called *ῥητός*, as is each of these squares. In fact each of the straight lines is *ῥητή*, feminine, while the square is *ῥητόν*, neuter. Heath translates the adjective as "rational." Etymologically speaking, the rational is what is capable of speech; *ῥητός* refers originally to something spoken, as in our "rhetoric." In the present context, the irrational is ἄλογος, something without speech or reason or, in Latin, *ratio.*

Aristotle's (or pseudo-Aristotle's) own refutation of the argument above is at $969^{\mathrm{b}}6$, though perhaps it is not very illuminating. Joachim renders it thus:

> As to what they say about 'commensurate lines'—that all lines, because commensurate, are measured by one and the

> same actual unit of measurement—this is sheer sophistry; nor is it in the least in accordance with the mathematical assumption as to commensurability. For the mathematicians do not make the assumption in this form, nor is it of any use to them.
>
> Moreover, it is actually inconsistent to postulate both that every line becomes commensurate, and that there is a common measure of all commensurate lines.

Joachim describes his work as a paraphrase, but he seems here to follow Bekker's Greek reasonably:

> *τὸ δ' ἐπὶ τῶν συμμέτρων γραμμῶν, ὡς ὅτι αἱ πᾶσαι τῷ αὐτῷ τινὶ καὶ ἑνὶ μετροῦνται, κομιδῇ σοφιστικὸν καὶ ἥκιστα κατὰ τὴν ὑπόθεσιν τὴν ἐν τοῖς μαθήμασιν· οὔτε γὰρ ὑποτίθενται οὕτως, οὔτε χρήσιμον αὐτοῖς ἐστίν. ἅμα δὲ καὶ ἐναντίον πᾶσαν μὲν γραμμὴν σύμμετρον γίνεσθαι, πασῶν δὲ τῶν συμμέτρων κοινὸν μέτρον εἶναι ἀξιοῦν.*

In particular, the clause "every line becomes commensurate" is indeed singular in the Greek. However, we might try reading the whole last sentence to mean that, even if any two lines are commensurate, it does not follow that all lines have a common measure. At any rate, this would seem to be true. We might understand magnitudes of a given kind (lines, areas, solids) to compose an ordered commutative semigroup in which a less magnitude can always be subtracted from a greater. Then two magnitudes will be **commensurate** if the Euclidean algorithm can be applied effectively to produce a common measure. What we call the positive rational numbers compose such a structure, and any two of them are commensurate, but there is no least positive rational number.

The second oldest quotation in the *Oxford English Dictionary* [38] for "commensurable" is from Billingsley's version of the *Elements,* already mentioned above. The citation is:

1570 BILLINGSLEY *Euclid* x Def. i. 229 All numbers are commensurable one to another.

The quotation is actually on the verso of folio 228—facing the recto of 229—of Billingsley's book [18], and it is part of a commentary (possibly by John Dee) on the first definition in Book X, the definition itself having been translated,

> Magnitudes commensurable are such, which one and the selfe same measure doth measure.

As examples of *σύμμετρος*, the Index of Greek Terms in Thomas's *Selections Illustrating the History of Greek Mathematics* [50, 51] cites instances of what we should call commensurability or its negation:

1) Plato's *Theaetetus,* on Theodorus's theorem that the square roots of nonsquare numbers of square feet from two to seventeen are incommensurable with the foot;
2) Euclid's formal definition of commensurability, as above; and
3) Archimedes's theorem that commensurable magnitudes (*τὰ σύμμετρα μεγέθεα*) balance at distances inversely proportional to their weights. (By the Method of Exhaustion, the same is true for incommensurable magnitudes.)

In Heath's *History of Greek Mathematics* [25, 26], the Index of Greek Words does not show *συμμετρία* or *σύμμετρος* at all. Neither does Heath's English index show "symmetry" or "commensurability"; but the way to look up in Heath the topics listed from Thomas's index is through the word "irrational."

4.2 Nicomachus

According to the *Oxford English Dictionary,* "commensurable" derives from the Latin word *commensurabilis,* which Boethius coined or at least used; the English word may also be derived from Oresme's 14th-century French version of Boethius's word. The *Larousse dictionnaire d'étymologie* recognizes Oresme's 1361 derivation of the French *commensurable* from the 6th-century Latin of "Boèce" [17, p. 168].

Boethius's *Arithmetic* is considered [13, p. 212] an abridgment of Nicomachus's *Introductio,* and it was "the source of all arithmetic taught in the schools for a thousand years" [28, p. 201]. D'Ooge's edition of Nicomachus does not provide the Greek, except implicitly through an index of Greek terms. There is one instance of συμμετρία and one of σύμμετρος. The instance of the former is translated as follows [39, I.14.3, p. 208]:

> if when all the factors of a number are examined and added together in one sum, it proves upon investigation that the number's own factors exceed the number itself, this is called a superabundant number, for it oversteps the symmetry which exists between the perfect and its own parts.

Here "symmetry" seems to be a synonym for equality. In modern notation, a number n is superabundant (ὑπερτελὴς), perfect (τέλειος), or deficient (ἐλλιψής), according as

$$\sum_{d|n} d > 2n, \qquad \sum_{d|n} d = 2n, \qquad \sum_{d|n} d < 2n.$$

The number 28 is perfect because

$$\begin{gathered}\{d\colon d \mid 28\} = \{1, 2, 4, 7, 14, 28\},\\ 28 = 14 + 7 + 4 + 2 + 1,\end{gathered}$$

and this situation is one of "symmetry." By contrast, 12 is superabundant since $6 + 4 + 3 + 2 + 1 = 16 > 12$.

The one indexed instance of *σύμμετρος* in Nicomachus [39, II.3.2, p. 232] could likewise be replaced with "equal." first Nicomachus sets up the general situation:

> Every multiple will stand at the head of as many superparticular ratios corresponding in name with itself as it itself chances to be removed from unity, and no more nor less under any circumstances.

What this means is that, for any number k, if for some n we take the nth power k^n, starting from there we obtain a continued proportion

$$k^n : k^{n-1}\ell : k^{n-2}\ell^2 : \cdots : k\ell^{n-1} : \ell^n,$$

where $\ell = k+1$. In the proportion, there are n terms after the first, and the ratio of each of these terms to the preceding is that of ℓ to k; this ratio is superparticular because the excess of ℓ over k (namely unity) is a part of k (that is, it measures k). The way n appears in two senses is apparently considered "symmetric." Nicomachus himself explains with an example, and here, apparently, the adjective *σύμμετρος* is used:

> The doubles, then, will produce sesquialters, the first one, the second two, the third three, the fourth four, the fifth five, the sixth six, and neither more nor less, but by every necessity when the superparticulars that are generated attain the proper number, that is, when their number <u>agrees with</u> the multiples that have generated them, at that point by a divine device, as it were, there is found the number which terminates them all because it naturally is not divisible by that factor whereby the progression of the superparticular ratios went on.

An illustration is provided as in Figure 4.1, where each column shows a continued proportion as above.

1	2	4	8	16	32	64
	3	6	12	24	48	96
		9	18	36	72	144
			27	54	108	216
				81	162	324
					243	486
						729

Figure 4.1: Superparticular ratios in Nicomachus

It does not appear that Nicomachus uses *συμμετρία* as a technical term.

4.3 Boethius and Recorde

Boethius, however, in *De Institutione Arithmetica* [11, I.18, p. 39, l. 14], does use "commensurable" as a technical term for numbers that are *not* prime to one another. In his example, by applying what we know as the Euclidean Algorithm, he shows that VIIII and XXVIIII are prime to one another (*contra se primos*); but XXI and VIIII have the common measure III, and therefore Boethius calls them *commensurabiles.*

Robert Recorde carried the usage of Boethius into English. He provides the *oldest* quotation for "commensurable" in the *Oxford English Dictionary*:

> **1557** Recorde *Whetst.* Bj, .20. and .36. be commensurable, seyng .4. is a common diuisor for theim bothe.

This from Recorde's *Whetstone of Witte* [47], cited earlier as the origin of our sign of equality. The book is formally a dialogue between the Scholar and the Master. It starts with an account of numbers that seems based on Euclid, though Recorde first mentions Euclid only to have the Scholar say,[1]

> Yet one thyng more I must demaunde of you, why Euclide, and the other learned men, refuse to accompte fractions emongest nombers.

The Master responds as follows, alluding to the definition of number quoted above from the *Elements*:

> Bicause all nombers doe consiste of a multitude of unities : and every proper fraction is lesse then an unitie, and therefore can not fractions exactly be called nombers : but maie bee called rather fractions of nombers.

Presently the Master introduces the term *commensurable* to mean *not relatively prime,* that is, *having a common measure other than unity*; this is the meaning of Boethius. Billingsley will use the term differently, thirteen years later, to mean *having any common measure at all,* as noted above; however, the *OED* takes no note of the difference. Recorde writes as follows; the *OED* quotation is here.

> Scholar . . . What saie you now of nõbers relatiue?
>
> Master. Some tymes their relation hath regarde to their partes, namely, whether these. 2. that bee so compared, haue any common parte, that will diuide theim bothe. For if thei haue so, then are thei called nombers commensurable. As. 12. and. 21. bee nombers commensurable: for. 3. will diuide eche of theim.
>
> Likewaies. 20. and. 36. be commensurable, seyng 4. is a commõ diuisor for them bothe. But if thei haue no suche common diuisor, then are thei called incommensurable. As 18 and 25. For 25 can bee diuided by no nomber more than by. 5. And. 18. can not be diuided by it.
>
> In like maner. 36. and. 49. are incommensurable: For 49. hath no diuisor but. 7. And 7. can not diuide. 36.
>
> Scholar. Doe you meane then, that incommensurable nombers, haue no cõparison nor proportion together?

[1] My quotations extend from the verso of A.ii. to B.i. (which is the folio number cited in the *OED*).

> Master. Naie, nothyng lesse. For any. 2. nombers maie haue comparison et[2] proportion together, although thei be incommensurable. As. 3. and. 4. are incommensurable, and yet are thei in a proportion together: as shall appeare anon.

Thus a number prime to another still has a ratio to the other; or in Recorde's terms, incommensurable numbers are still in proportion. One might here want to guard against the confusion that might have been seen in *De Lineis Insecabilibus* above: just because any two numbers are in proportion, it does not follow that they are in the *same* proportion as any other two numbers!

It might be convenient to have, as Recorde does, a single term for a pair of numbers that are not prime to one another; but it would seem that "commensurable" has not been used as such a term, at least not since Billingsley's rendition of Euclid.

4.4 Plato

In the Liddell–Scott *Lexicon,* the word *συμμετρία* is given two general meanings:

> *commensurability,* opp. *ἀσυμμετρία* . . . **II.** *symmetry, due proportion,* one of the characteristics of beauty and goodness . . .

We have considered the first meaning. The second seems not to be specifically mathematical. A key citation is to Plato's *Philebus* [45, 64D–65A]:

> SOCRATES. And it is quite easy to see the cause (*αἰτία*) which makes any mixture (*μῖξις*[3]) whatsoever either of the

[2] The original shows an obscure symbol here. It does not seem to be an ampersand, but could be the "Tironian et."

[3] The word also means "sexual intercourse."

highest value or none at all.

PROTARCHUS. What do you mean?

SOC. Why, everybody knows that.

PRO. Knows what?

SOC. That any compound (*σύγκρασις*), however made, which lacks measure and proportion (*μέτρου καὶ τῆς συμμέτρου φύσεως μὴ τυχοῦσα*),[4] must necessarily destroy its components, and first of all itself; for it is in truth no compound (*κρᾶσις*), but an uncompounded jumble (*ἄκρατος*[5] *συμπεφορημένη*), and is always a misfortune to those who possess it.

PRO. Perfectly true.

SOC. So now the power of the good has taken refuge in the nature of the beautiful; for measure and proportion (*μετριότης καὶ συμμετρία*) are everywhere identified with beauty and virtue.

PRO. Certainly.

SOC. We said that truth also was mingled with them in the compound.

PRO. Certainly.

SOC. Then if we cannot catch the good with the aid of one idea, let us run it down with three—beauty, proportion, and truth, and let us say that these, considered as one, may more properly than all other components of the mixture be regarded as the cause, and that through the goodness of these the mixture itself has been made good.

PRO. Quite right.

Thus Fowler in the Loeb edition translates *συμμετρία* as "proportion." Jowett uses "symmetry" [43, pp. 637–8]

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[4]More literally, I think, "which does not happen to have measure and a commensurate nature."

[5]Literally "unmixed," hence also "pure, perfect."

Is there any connection to mathematics here? Presumably Plato knows the technical meaning of *συμμετρία* as commensurability. Thus the words that he puts in the mouth of Socrates suggest an architectural theory whereby the sides of rectangles used in beautiful buildings ought to be in the ratios of small whole numbers, just as musical harmonies are played on strings whose lengths are in such ratios (assuming uniform density and tension).

It has been argued in modern times that the Greeks in fact used a different design principle, based on what we call the golden ratio, but Euclid calls extreme and mean ratio (*ἄκρος καὶ μέσος λόγος*) in Book VI of the *Elements*: two magnitudes A and B are in this ratio, A being the greater, if they satisfy the proportion

$$A + B : A :: A : B, \tag{4.1}$$

where the one extreme, $A + B$, is the sum of the other extreme, B, and the mean, A. In this case, A and B are incommensurable. One proof of this theorem is that the Euclidean algorithm, applied to A and B, does not terminate, since by "separation" of the ratios in (4.1) as in Book V of the *Elements,*

$$B : A :: A - B : B.$$

Knorr argues [29, ch. II] that the first discovered instance of incommensurability was that of the diagonal and side of a square; even to *define* the extreme and mean ratio takes too much mathematical sophistication. However, using the theory of incommensurability alluded to in Plato's dialogue the *Theaetetus* [42, 147D–E, p. 25], Theodorus could well have derived the incommensurability of two magnitudes in extreme and mean ratio—in our terms, the ratio of $\sqrt{5} + 1$ to 2—from that of the legs of the right triangle with sides that are, in our terms, 2, $\sqrt{5}$, and 3 [29, ch. VI]. In particular, Plato would likely have known that the extreme and mean ratio is, in our terms, "irrational." He might then have questioned its use in architecture, if it had been in use.

In any case, since we have seen that *συμμετρία* may be translated as "proportion," let us note that the word for a mathematical proportion is, for Euclid at least (as in Book V of the *Elements*), *ἀναλογία*, while to be proportional is to be *ἀνάλογος*, that is, "according to a [common] ratio." In particular, a proportion such as (4.1) is not an *equation* of ratios, but a "sameness" or *identification* of ratios. Knorr (for example) overlooks the distinction when he writes [29, p. 15],

> (c) A 'ratio' (λόγος) is a comparison of homogeneous quantities (i.e., numbers or magnitudes) in respect of size. A 'proportion' (ἀναλογία) is an equality of two ratios. Four magnitudes are 'in proportion' (ἀνάλογον) when the first and second have the same ratio to each other that the third and fourth have to each other . . .

We observed earlier that Euclid's equality is congruence, which can be detected by superposition. Equality is a possible property of two magnitudes. The presence of a *proportion* among *four* magnitudes is more subtle to detect. The magnitudes have ratios in pairs, but these ratios themselves are not magnitudes, and they cannot be placed alongside or atop one another. One does have such results as Proposition 9 of Book V of the *Elements*:

> Τὰ *πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα ἀλλήλοις ἐστιν· καὶ πρὸς ἃ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἐκεῖνα ἴσα ἐστίν.*
>
> Those having to the same the same ratio are equal to one another; also, those to which the same has the same ratio, they are equal:

$$A : C :: B : C \implies A = B,$$
$$C : A :: C : B \implies A = B.$$

This can be used to establish the equality of figures, such as pyramids, that are not congruent to one another, even part by part.

It is valuable to recognize the distinction between equality and sameness, if only because it can help prevent an error in interpreting Euclid's vague definition of proportions of numbers in Book VII of the *Elements.* The error has led modern mathematicians to think that the definition leads *Euclid* to error. The modern error is to think that, according to Euclid, we can establish a proportion

$$A : B :: C : D \tag{4.2}$$

of numbers simply by observing that for *some* numbers E and F and multipliers k and ℓ,

$$A = kE, \qquad B = \ell E, \qquad C = kF, \qquad D = \ell F.$$

Here the pair (k, ℓ) is not uniquely determined by the "ratio" (whatever that means) of A to B or of C to D. Since we are trying to establish *sameness* of those two ratios, and sameness *obviously* has the property that we call transitivity, while the proposed test for proportionality does not by itself establish transitivity, the test must not be Euclid's. We must first require E to be the *greatest* common measure of A and B; and F, of C and D. In other words, the proportion (4.2) means the Euclidean algorithm has the same steps, whether applied to A and B or C and D. I spell this out in another essay (in preparation).

4.5 Aristotle

In the *Metaphysics* [5, XIII.iii.10, $1078^{a}35$], Aristotle makes a general statement about *συμμετρία* that is more or less in agreement with Plato's *Philebus*:

> *τοῦ δὲ καλοῦ μέγιστα εἴδη τάξις καὶ συμμετρία καὶ τὸ ὡρισμένον, ἃ μάλιστα δεικνύουσιν αἱ μαθηματικαὶ ἐπιστῆμαι.*
>
> The main species of beauty are orderly arrangement, proportion, and definiteness; and these are especially manifested by the mathematical sciences.

It is not clear here whether mathematics *is* symmetric, or only concerns symmetrical (and orderly, well-defined) things. Aristotle's comment is preceeded by:

> And since goodness is distinct from beauty (for it is always in actions that goodness is present, whereas beauty is also in immovable things), they are in error who assert that the mathematical sciences tell us nothing about beauty or goodness . . .

The passage does not suggest what symmetry is. Earlier in the *Metaphysics* [6, IV.ii.18, $1004^{b}11$], Aristotle says:

> *ἐπεὶ ὥσπερ ἔστι καὶ ἀριθμοῦ ᾗ ἀριθμὸς ἴδια πάθη, οἷον περιττότης ἀρτιότης, συμμετρία ἰσότης, ὑπεροχὴ ἔλλειψις, καὶ ταῦτα καὶ καθ' αὑτοὺς καὶ πρὸς ἀλλήλους ὑπάρχει τοῖς ἀριθμοῖς . . . οὕτω καὶ τῷ ὄντι ᾗ ὂν ἔστι τινὰ ἴδια, καὶ ταῦτ' ἐστὶ περὶ ὧν τοῦ φιλοσόφου ἐπισκέψασθαι τὸ ἀληθές.*
>
> For just as number *qua* number has its peculiar modifications, *e.g.* oddness and evenness, commensurability and equality, excess and defect, and these things are inherent in numbers both considered independently and in relation to other numbers . . . so Being *qua* Being has certain peculiar modifications, and it is about these that it is the philosopher's function to discover the truth.

Thus properties of numbers are given as examples, and they come in correlative pairs:

περιττότης	*ἀρτιότης*	oddness	evenness
συμμετρία	*ἰσότης*	symmetry	equality
ὑπεροχὴ	*ἔλλειψις*	excess	defect

Every number is even or odd, but not both. Excess and defect could be a number's superabundance and deficiency of factors, as discussed by Nicomachus. This leaves out perfection, unless this is implied by equality; but in that case, what is symmetry? Possibly for Aristotle every *pair* of numbers is either equal or, if not equal, then at least symmetric in the sense of having a common measure (be this unity or a number of units).

Aristotle does recognize the possibility of "asymmetric" or incommensurable pairs of mathematical objects [5, XI.iii.7 (1061$^{\mathrm{a}}$28)]:

> And just as the mathematician makes a study of abstractions (for in his investigations he first abstracts everything that is sensible, such as weight and lightness, hardness and its contrary, and also heat and cold and all other sensible contrarieties, leaving only quantity and continuity—sometimes in one, sometimes in two and sometimes in three dimensions—and their affections *qua* quantitative and continuous, and does not study them with respect to any other thing; and in some cases investigates the relative positions of things and the properties of these, and in others their commensurability or incommensurability [*τὰς συμμετρίας καὶ ἀσυμμετρίας*], and in others their ratios; yet nevertheless we hold that there is one and the same science of all these things, viz. geometry), so it is the same with regard to Being.

Symmetry or commensurability in a more practical context arises in the *Nichomachean Ethics* [8, V.5, 1133$^{\mathrm{b}}$16, pp. 100–1]:

> *τὸ δὴ νόμισμα ὥσπερ μέτρον σύμμετρα ποιῆσαν ἰσάζει· οὔτε γὰρ ἂν μὴ οὔσης ἀλλαγῆς κοινωνία ἦν, οὔτ' ἀλλαγὴ ἰσότητος μὴ οὔσης, οὔτ' ἰσότης μὴ οὔσης συμμετρίας. τῇ μὲν οὖν ἀληθείᾳ ἀδύνατον τὰ τοσοῦτον διαφέροντα σύμμετρα γενέσθαι, πρὸς δὲ τὴν χρείαν ἐνδέχεται ἱκανῶς. ἓν δή τι δεῖ εἶναι, τοῦτο δ' ἐξ ὑποθέσεως· διὸ νόμισμα καλεῖται· τοῦτο γὰρ τάντα ποιεῖ σύμμετρα· μετρεῖται γὰρ πάντα νομίσματι.*

Crisp translates thus [9, p. 91]:

> So money makes things commensurable as a measure does, and equates them; for without exchange there would be no association between people, without equality no exchange, and without commensurability no equality. It is impossible that things differing to such a degree should become truly commensurable, but in relation to demand they can become commensurable enough. So there must be some one standard, and it must be on an agreed basis—which is why money is called *nomisma.* Money makes all things commensurable, since everything is measured by money.

The earlier Ross translation [3, p. 1101–2] of the first part is,

> Money, then, acting as a measure, makes goods commensurate and equates them; for neither would there have been association if there were not exchange, nor exchange if there were not equality, nor equality if there were not commensurability.

The following might be more literal:

> Money equalizes, as measure makes commensurable. For, there being no exchange, there would be no association;—no exchange, there being no equality; no equality, there being no commensurability.

In particular, it seems to me that "measure" can be understood as the subject of "make commensurable," while "money" is only the subject of "equalize." Evidently equating or equalizing is not making things the *same.* One might translate the verb *ἰσαζω* here also as "balance." Money makes it possible to balance dissimilar goods, though as Aristotle says, the balance is never perfect.

Symmetry in the sense of balance is mentioned in the *Physics* [4, VII.III, 246$^{\mathrm{b}}$3]:

> *ἔτι δὲ καί φαμεν ἁπάσας εἶναι τὰς ἀρετὰς ἐν τῷ πρός τι πὼς ἔχειν. τὰς μὲν γὰρ τοῦ σώματος, οἷον ὑγίειαν καὶ εὐεξίαν, ἐν κράσει καὶ συμμετρίᾳ θερμῶν καὶ ψυχρῶν τίθεμεν, ἢ αὑτῶν πρὸς αὑτὰ τῶν ἐντὸς ἢ πρὸς τὸ περιέχον.*

Apostle [7, pp. 139–40] renders this:

> Further, we also speak of virtues as coming under things which are such that they are somehow related to something. For we take the virtues of the body, such as health and good physical condition, to be mixtures and right proportions of hot and cold, in relation either to one another or to the surroundings.

Apostol's "right proportion"—what I would understand as balance—is just Aristotle's *συμμετρία.*

If a holy temple or a human face exhibits what we call bilateral symmetry, it is balanced. This would seem to be the connection between the ancient *symmetria* and modern mathematical symmetry. The connection is tenuous, as we should expect, since there can be no strict rule, no practical formula, for determining unambiguously what beautiful or balanced or symmetrical in life.

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