Archive for Mathematical Logic manuscript No.
(will be inserted by the editor)

David Pierce

Model-theory of vector-spaces over unspecified fields

Received: date / Revised: date

Abstract Vector-spaces over unspecified fields can be axiomatized as one-sorted structures, namely, abelian groups with the relation of parallelism. Parallelism is binary linear dependence. When equipped with the n-ary relation of linear dependence for some positive integer n, a vector-space is existentially closed if and only if it is n-dimensional over an algebraically closed field. In the signature with an n-ary predicate for linear dependence for *each* positive integer n, the theory of infinite-dimensional vector-spaces over algebraically closed fields is the model-completion of the theory of vector-spaces.

Keywords vector-space · model-completeness · sort

Mathematics Subject Classification (2000) 03C10 · 03C60

0 Introduction

In Abraham Robinson's original definition [14, §2.2], a (first-order) theory T is **model-complete** if the theory $T \cup \operatorname{diag} \mathfrak{M}$ is complete whenever $\mathfrak{M} \models T$. Two equivalent formulations are as follows, in current notation, where $\mathfrak{M}$ and $\mathfrak{N}$ are models of T:

1. $\mathfrak{M} \subseteq \mathfrak{N} \implies \mathfrak{M} \preccurlyeq_1 \mathfrak{N}$ (**Robinson's Test**) [14, 2.3.1];
2. $\mathfrak{M} \subseteq \mathfrak{N} \implies \mathfrak{M} \preccurlyeq \mathfrak{N}$ [14, 2.4.1].

Model-completeness is of use in the discovery of complete theories, since a model-complete theory with a model that embeds in every other model is complete [14, 4.1.6]. Discovery of *model*-complete theories—complete or not—is also of interest in itself.

David Pierce
Mathematics Department, Middle East Technical University, Ankara, Turkey, 06531
E-mail: dpierce@metu.edu.tr

Robinson gives some examples of model-complete theories, including the theories of non-trivial torsion-free divisible abelian groups [14, §3.1], algebraically closed fields [14, §3.2], and real-closed fields [14, §3.3]; these examples are now standard [12, ch. 3]. Robinson notes also, in effect, that the theory of non-trivial vector-spaces *over a given field* is model-complete [14, §3.6]. However, although his notation would seem to allow it, he does not appear to consider the theory of vector-spaces *tout court* (over an unspecified field).

In the most usual formulation, a vector-space is a two-sorted structure. About early versions of model-theory, Angus Macintyre notes [11, §2.4, p. 198]:

> For no good reason, natural structures were forced into a one-sorted formulation (and by now it is a considerable nuisance that there is no basic model-theory text in a many-sorted setting).

Nonetheless, the possibility of treating vector-spaces as two-sorted structures has been acknowledged in one textbook [15, §5.4]; also, modules in general have been treated as two-sorted structures [8, ch. 9]. A natural signature for this treatment of vector-spaces includes:

1. the signature of abelian groups, for the vectors;
2. the signature of rings, for the scalars;
3. a symbol for multiplication of vectors by scalars (in the present paper, this symbol is $*$).

In this signature, let T be the theory of vector-spaces, and if $n \in \{1,2,3,\dots\} \cup \{\infty\}$, let T_n be the theory of vector-spaces of dimension n. Andrey Kuzichev [10] gives an explicit elimination of quantified vector-variables in T_n. Indeed, suppose C is a formula whose vector-variables are from the tuple $(x_1,\dots,x_m,y)$. (Kuzichev says $m \geqslant 1$, but this is not necessary if summations $\sum_{i=1}^{0} f_i$ are understood to be 0.) Let C' be the result of replacing each y in C with $\sum_{i=1}^{m} \alpha_i * x_i$, and let C^* be the result of replacing each atomic sub-formula $\lambda * y = \sum_{i=1}^{m} \mu_i * x_i$ of C with $\lambda = 0 \wedge \sum_{i=1}^{m} \mu_i * x_i = 0$. Then

$$T \vdash \exists y\, C \leftrightarrow \Bigl(\exists \boldsymbol{\alpha}\, C' \vee \bigl(C^* \wedge \exists y\, \forall \boldsymbol{\alpha}\; y \neq \sum_{i=1}^{m} \alpha_i * x_i\bigr)\Bigr).$$

In each of the theories T_n, we can express the formula $\exists y\, \forall \boldsymbol{\alpha}\; y \neq \sum_{i=1}^{m} \alpha_i * x_i$ without quantified vector-variables. Kuzichev gives the corollary that, if U is a complete theory of the scalar-field, then $T_n \cup U$ is complete.

However, these complete theories $T_n \cup U$ are never model-complete, unless $n = 1$. They are not even inductive: the union of a chain of models need not be a model (no matter what U is, unless $n = 1$; see §2 below). The problem is the lack of a way to express linear independence with an existential formula. The picture changes if predicates for linear independence or dependence are added.

I arrive, myself, at the example of vector-spaces by noting first that, in the *Elements* [5], Euclid gives a geometric formulation of certain field-theoretic identities. For example, the identity $(x+y)(x-y)+y^2 = x^2$ is Euclid's Proposition II.5, but Euclid expresses it in terms of the squares and rectangles bounded by certain straight lines (*εὐθεῖαι γραμμαί*, what we might call *line-segments;* see Figure 0.1).

At the beginning of the *Geometry* [2], René Descartes shows how straight lines (*lignes droites*) can be multiplied to produce *lines* rather than rectangles, if one line is chosen as unit:

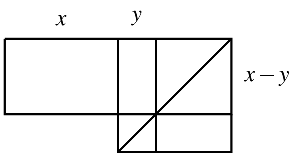

Fig. 0.1 Euclid's II.5

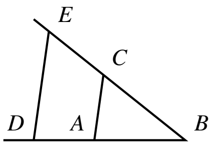

Fig. 0.2 Descartes's geometric multiplication

> *Soit par exemple AB l'vnité, & qu'il faille multiplier BD par BC, ie n'ay qu'a ioindre les poins A & C, puis tirer DE parallele a CA, & BE est le produit de cete Multiplication*[1] *[Fig. 0.2].*

By means of Descartes's observation, we can interpret the scalar-field in a vector-space of dimension at least two by means of the binary relation of *parallelism,* that is, binary linear dependence (§1). So vector-spaces of dimension at least two are axiomatizable, as one-sorted structures, in the signature of abelian groups with a binary predicate. Also, the axioms can be chosen to be $\forall\exists$.

When the predicate for parallelism is used, then the existentially closed vector-spaces are two-dimensional. (It does not matter whether a separate sort for the scalars is retained or not.) If, more generally, an n-ary predicate for linear dependence is used, then the existentially closed vector-spaces are n-dimensional over algebraically closed fields (§2). The class of such models being elementary, its theory is model-complete, although it does not admit full quantifier-elimination.

On the value of looking at basic structures, I quote again Descartes, this time from near the end of the fourth of his *Rules for the Direction of the Mind* [4]:

> *Quant à moi, conscient de ma faiblesse, j'ai décidé d'observer opiniâtrement, dans ma quête de connaissances, un ordre tel qu'en partant toujours des choses les plus simples et les plus faciles, je m'interdise de passer à d'autres, avant que dans les premières il ne m'apparaisse qu'il ne reste plus rien à désirer; c'est pourquoi j'ai poussé jusqu'à ce jour, aussi loin que j'ai pu, l'étude de cette mathématique universelle...*[2]

The result about existentially closed vector-spaces means, in part, that a vector-space of $n+1$ dimensions can be embedded in an n-dimensional space so that linear independence of n-element sets (more precisely, n-tuples) is preserved. A sim-

[1] 'For example, let *AB* be taken as unity, and let it be required to multiply *BD* by *BC*. I have only to join the points *A* and *C*, and draw *DE* parallel to *CA*; then *BE* is the product of *BD* and *BC*' [2, p. 5].

[2] 'Aware how slender my powers are, I have resolved in my search for knowledge of things to adhere unswervingly to a definite order, always starting with the simplest and easiest things and never going beyond them till there seems to be nothing further which is worth achieving where they are concerned. Up to now, therefore, I have devoted all my energies to this universal mathematics...' [3, p. 20].

ilar result arises in the model-theory of fields: Let $(a_0,\dots,a_n)$ be algebraically independent over a field K, and let (b,c) be a generic solution to $\sum_{i=0}^{n} a_i x^i = y$. (This generalizes [12, Example 8.1.2, p. 291]; see also [13, Example 2.1.8, p. 77].) Then $\text{tr-deg}(K(a_0,\dots,a_n)/K) = n+1$, while $\text{tr-deg}(K(a_0,\dots,a_n,b,c)/K(b,c)) = n$, although every n-tuple of elements of $K(a_0,\dots,a_n)$ that is algebraically independent over K is still independent over $K(b,c)$.

Recently and independently, Moshe Kamensky [9] has worked with (two-sorted) vector-spaces and has given theories of them that have elimination of quantifiers; this elimination is achieved by means of additional sorts, as for the Grassmannians of the finite powers of the scalar-field.

1 Interpretation of the field in the vector-space

In a chapter of his own textbook on the subject, Robin Hartshorne [6, ch. 3] presents Descartes's approach to geometry. It is perhaps a Cartesian spirit that allows Alfred Tarski [17, 18] to axiomatize Euclidean geometry by means of the ternary relation of between-ness and the quaternary relation of equidistance; these relations allow the underlying (real-closed) field to be recovered from the space.

In the present section, I specify an origin in space and work with the group-structure determined by it; then only a binary relation is needed to recover the underlying (arbitrary) field. This observation was useful to me in showing that a field equipped with a space and Lie-ring of derivations can be described as a one-sorted structure in two ways: as a field with certain operators, or as a Lie-ring with certain operators.

A **vector-space** can be defined as a triple $(V,K,*)$, where V is an abelian group (of **vectors**), K is a field (of **scalars**), and $*$ is a function $(x,\boldsymbol{v}) \mapsto x * \boldsymbol{v}$ from $K \times V$ to V so that:

1. the functions $\boldsymbol{v} \mapsto a * \boldsymbol{v}$ (where $a \in K$) are endomorphisms of V, and
2. the function $x \mapsto (\boldsymbol{v} \mapsto x * \boldsymbol{v})$ is a ring-homomorphism from K into $(\mathrm{End}(V), \circ)$.

Since K is a field, the homomorphism into $(\mathrm{End}(V), \circ)$ is an embedding, unless V is trivial; excluding this case, we may treat the homomorphism as an inclusion, so that multiplication in K is literally composition. Even in the trivial case, let us retain the symbol for composition to denote multiplication of scalars.

Model-theoretically then, a vector-space is a two-sorted structure, in the signature $\{+,-,\mathbf{0},\circ,0,1,*\}$; here the elements of $\{+,-\}$ do double duty, taking arguments from the sort of vectors *or* the sort of scalars. I shall use normal-face variables for scalars (as 0), and boldface for vectors (as $\mathbf{0}$) or, in more general contexts, for tuples of variables.

In every vector-space, the binary relation $\parallel$ on the sort of vectors is defined by the sentence

$$\boldsymbol{x} \parallel \boldsymbol{y} \leftrightarrow \exists u\, \exists v\, (u * \boldsymbol{x} + v * \boldsymbol{y} = \mathbf{0} \wedge (u \neq 0 \vee v \neq 0)). \tag{1.1}$$

(Here and throughout the paper, outer universal quantifiers are suppressed.) The sentence (1.1) has the form

$$\alpha \leftrightarrow \exists z_0 \cdots \exists z_{n-1}\, \beta, \tag{1.2}$$

where α and β are quantifier-free, and none of the z_i appears in α; such a sentence is equivalent to the conjunction of the $\forall\exists$-sentence $\exists z_0 \cdots \exists z_{n-1}\ (\alpha \to \beta)$ and the universal sentence $\beta \to \alpha$.

Let VS_2 be the theory of vector-spaces that have been *expanded* to the signature $\{+,-,\mathbf{0},\circ,0,1,*,\parallel\}$ so as to satisfy (1.1). (The subscript 2 alludes to the number of arguments that $\parallel$ takes.) Models of the theory VS_2 can be *extended* so as to satisfy also

$$\exists \boldsymbol{x}\ \exists \boldsymbol{y}\ \boldsymbol{x} \nparallel \boldsymbol{y}. \tag{1.3}$$

Let $\mathrm{VS}_2^{\mathrm{m}}$ be the theory of the resulting structures: the vector-spaces, with parallelism, of dimension at least two. (The superscript 'm' refers to a *m*inimum dimension.) This theory is inductive.

Reduction of a one-sorted structure is the discarding of some named operations or relations; we may understand reduction of a many-sorted structure to involve also the discarding of some sorts. In this sense, every model $(V,K,*,\parallel)$ of $\mathrm{VS}_2^{\mathrm{m}}$ has a one-sorted reduct $(V,\parallel)$ in the signature $\{+,-,\mathbf{0},\parallel\}$. This reduct is an abelian group with a certain binary relation; we might just call the reduct an **abelian group with parallelism.** Let $\mathrm{VS}_2^{\mathrm{r}}$ be the theory of these structures. The theories $\mathrm{VS}_2^{\mathrm{m}}$ and $\mathrm{VS}_2^{\mathrm{r}}$ will be related in Theorem 1.1.

In general, the models of a theory T in a signature $\mathscr{L}$ are the objects of a category whose morphisms are either embeddings or elementary embeddings. Let the two possibilities for the category be $\mathrm{Mod}^{\subseteq}(T)$ and $\mathrm{Mod}^{\preccurlyeq}(T)$ respectively. If $\mathscr{L} \subseteq \mathscr{L}'$, and T' is a theory of $\mathscr{L}'$, and reducts to $\mathscr{L}$ of models of T' are models of T, then we may speak of reduction from T' to T: it is a functor R from $\mathrm{Mod}^{\preccurlyeq}(T')$ to $\mathrm{Mod}^{\preccurlyeq}(T)$, and from $\mathrm{Mod}^{\subseteq}(T')$ to $\mathrm{Mod}^{\subseteq}(T)$ (this is a trivial case of Lemma 1.1 below).

Possibly, along with the reduction-functor R from $\mathrm{Mod}^{\preccurlyeq}(T')$ to $\mathrm{Mod}^{\preccurlyeq}(T)$, there is an **expansion-**functor E from $\mathrm{Mod}^{\preccurlyeq}(T)$ to $\mathrm{Mod}^{\preccurlyeq}(T')$ such that $R \circ E$ is the identity, and for each model $\mathfrak{A}$ of T', there is a 'reasonable' isomorphism $\sigma_{ER}^{\mathfrak{A}}$ from $\mathfrak{A}$ to $E(R(\mathfrak{A}))$—reasonable in the sense that, for all models $\mathfrak{A}$ and $\mathfrak{A}^*$ of T', for every elementary embedding h of $\mathfrak{A}$ in $\mathfrak{A}^*$, the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "h"] \arrow[d, "\sigma_{EF}^{\mathfrak{A}}"'] & \mathfrak{A}^* \arrow[d, "\sigma_{ER}^{\mathfrak{A}^*}"] \\
E(R(\mathfrak{A})) \arrow[r, "E(R(h))"'] & E(R(\mathfrak{A}^*))
\end{tikzcd}$$

In a word then, R is an **equivalence** of categories; we might say also that R is a **weakly conservative** reduction. For example, R is weakly conservative in case T' is VS_2, and $\mathscr{L}$ is $\{+,-,\mathbf{0},\circ,0,1,*\}$ (so that T is just the theory of vector-spaces). However, in this case, E is not functorial on $\mathrm{Mod}^{\subseteq}(T)$. For example, $(\mathbb{C},\mathbb{R},*) \subseteq (\mathbb{C},\mathbb{C},*)$, but $(\mathbb{C},\mathbb{R},*,\parallel) \nsubseteq (\mathbb{C},\mathbb{C},*,\parallel)$. This is a point to be developed in §2. Briefly, E is not functorial here, because non-parallelism does not have an existential definition. In cases where E *is* functorial on $\mathrm{Mod}^{\subseteq}(T)$, and the diagram above commutes whenever h embeds $\mathfrak{A}$ in $\mathfrak{A}^*$, so that R is an equivalence of $\mathrm{Mod}^{\subseteq}(T')$ and $\mathrm{Mod}^{\subseteq}(T)$,—in such cases, we may say that R is simply a **conservative** reduction.

Now the statement of Theorem 1.1 makes sense. For the proof, I first note that reduction and expansion are instances of *interpretation.* In the account of Wilfrid Hodges [7, §5.3, p. 212], an **interpretation** of a (one-sorted) structure $\mathfrak{A}$ whose signature is $\mathscr{L}$ in a structure $\mathfrak{B}$ whose signature is $\mathscr{L}'$ consists, for some positive integer n, of:

1. a function $\varphi \mapsto \varphi_I$ converting each k-ary unnested atomic formula of $\mathscr{L}$, for each k in ω, into an nk-ary formula of $\mathscr{L}'$,
2. an n-ary formula δ_I of $\mathscr{L}'$, and
3. a surjective function $f_I^{\mathfrak{B}}$ from ${\delta_I}^{\mathfrak{B}}$ to A, such that
$$\mathfrak{B} \models \varphi_I(\boldsymbol{b}) \iff \mathfrak{A} \models \varphi(f_I^{\mathfrak{B}}(\boldsymbol{b})) \tag{1.4}$$
for all $\boldsymbol{b}$ from ${\delta_I}^{\mathfrak{B}}$.

We may assume that the universe A of $\mathfrak{A}$ is literally the set ${\delta_I}^{\mathfrak{B}}$, *modulo* the equivalence-relation defined in $\mathfrak{B}$ by $(x = y)_I$. Then $f_I^{\mathfrak{B}}$ is the quotient-map, and $\mathfrak{A}$ is uniquely determined by $\mathfrak{B}$ and $\varphi \mapsto \varphi_I$ and δ_I [7, 5.3.3, p. 216]: we may write $\mathfrak{A} = I(\mathfrak{B})$. Thus we have a function I converting structures of $\mathscr{L}'$ into structures of $\mathscr{L}$.

In case $\mathfrak{A}$ has several sorts, composing a tuple $(A_\alpha \colon \alpha \in S)$ perhaps, then the definition must be modified: For each α in S, for some positive integer n_α, there will be an formula $\delta_{I,\alpha}$ in some n_α-tuple of variables (each variable being assigned to a sort), along with a function $f_{I,\alpha}^{\mathfrak{B}}$ from ${\delta_{I,\alpha}}^{\mathfrak{B}}$ onto A_α, so that a variant of (1.4) holds, namely,

$$\mathfrak{B} \models \varphi_I(\boldsymbol{b}_{\alpha(0)}, \dots, \boldsymbol{b}_{\alpha(k-1)}) \iff \mathfrak{A} \models \varphi(f_{I,\alpha(0)}^{\mathfrak{B}}(\boldsymbol{b}_{\alpha(0)}), \dots, f_{I,\alpha(k-1)}^{\mathfrak{B}}(\boldsymbol{b}_{\alpha(k-1)})) \tag{1.5}$$

where $\boldsymbol{b}_{\alpha(j)} \in {\delta_{I,\alpha(j)}}^{\mathfrak{B}}$.

The function $\varphi \mapsto \varphi_I$ extends so that its domain comprises all unnested formulas of $\mathscr{L}$, and (1.4) or (1.5) still holds [7, 5.3.2, p. 214]: the extension takes $\varphi \wedge \psi$ to $\varphi_I \wedge \psi_I$, and $\neg\varphi$ to $\neg(\varphi_I)$, and $\exists x\ \varphi$ to $\exists(x^0, \dots, x^{n-1})\ (\varphi_I \wedge \delta_I(x^0, \dots, x^{n-1}))$ (with appropriate modifications in the many-sorted case).

If there are theories T of $\mathscr{L}$ and T' of $\mathscr{L}'$ such that $I(\mathfrak{B}) \models T$ whenever $\mathfrak{B} \models T'$, then let us say that I is an interpretation from T' to T. In this case, we can understand I as a functor from $\mathrm{Mod}^{\preccurlyeq}(T')$ to $\mathrm{Mod}^{\preccurlyeq}(T)$, where, if $\mathfrak{B}$ and $\mathfrak{B}^*$ are models of T', and h is an elementary embedding of $\mathfrak{B}$ in $\mathfrak{B}^*$, then $I(h)$ is the well-defined elementary embedding of $I(\mathfrak{B})$ in $I(\mathfrak{B}^*)$ given by

$$I(h)(f_I^{\mathfrak{B}}(\boldsymbol{b})) = f_I^{\mathfrak{B}^*}(h(\boldsymbol{b})) \tag{1.6}$$

[7, 5.3.4(a), p. 217]. If the formulas involved in I are simple enough, then more follows:

Lemma 1.1 *If I is an interpretation from T' to T (and therefore a functor from* $\mathrm{Mod}^{\preccurlyeq}(T')$ *to* $\mathrm{Mod}^{\preccurlyeq}(T)$*), then I is also a functor from* $\mathrm{Mod}^{\subseteq}(T')$ *to* $\mathrm{Mod}^{\subseteq}(T)$, provided *both that the formula δ_I is existential, and also that the formula φ_I is existential whenever φ is an unnested atomic formula of $\mathscr{L}$, or is $\neg(x = y)$, or is $\neg Rx^0 \cdots x^{k-1}$ for some predicate R in $\mathscr{L}$.*

Proof The claim is a variant of [7, 5.3.4(b), p. 217] and a refinement of [7, §5.4, exercise 3, p. 225].

An interpretation I from T' to T, paired with an interpretation J from T to T', is a **bi-interpretation** of T' and T if there are formulas χ_{IJ} of $\mathscr{L}$, and χ_{JI} of $\mathscr{L}'$, such that, whenever $\mathfrak{A} \models T$, then the set

$$\{(a, f_I^{J(\mathfrak{A})}(f_J^{\mathfrak{A}}(\boldsymbol{b}_0), \dots, f_J^{\mathfrak{A}}(\boldsymbol{b}_{n-1}))) \colon \mathfrak{A} \models \chi_{IJ}(a, \boldsymbol{b}_0, \dots, \boldsymbol{b}_{n-1})\} \tag{1.7}$$

is (the graph of) an isomorphism $\sigma_{IJ}^{\mathfrak{A}}$ from $\mathfrak{A}$ to $I(J(\mathfrak{A}))$, and similarly for all models of T', with the places of I and J reversed (Hodges [7, §5.4(c), p. 222], generalizing Ahlbrandt and Ziegler [1, p. 67]; in the case of several sorts, there will be several formulas $\chi_{IJ,\alpha}$, indexed with the sorts, and the graph of $\sigma_{IJ}^{\mathfrak{A}}$ will have components of the form $\{(a, f_{I,\alpha}^{J(\mathfrak{A})}(f_{J,\beta(0)}^{\mathfrak{A}}(\boldsymbol{b}_0), \dots, f_{J,\beta(n_\alpha-1)}^{\mathfrak{A}}(\boldsymbol{b}_{n_\alpha-1}))) \colon \mathfrak{A} \models \chi_{IJ,\alpha}(a, \boldsymbol{b}_0, \dots, \boldsymbol{b}_{n_\alpha-1})\}$). A special case was alluded to above, where reduction can be 'undone' by an expansion.

Lemma 1.2 *If (I,J) is a bi-interpretation of T' and T, then I is an equivalence of the categories* $\mathrm{Mod}^{\preccurlyeq}(T')$ *and* $\mathrm{Mod}^{\preccurlyeq}(T)$*; if I and J are also functorial between* $\mathrm{Mod}^{\subseteq}(T')$ *and* $\mathrm{Mod}^{\subseteq}(T)$*, then these categories too are equivalent, provided that the formulas χ_{IJ} and χ_{JI} are existential.*

Proof To prove the first claim, if $\mathfrak{A}$ and $\mathfrak{A}^*$ are models of T, and h is an elementary embedding of $\mathfrak{A}$ in $\mathfrak{A}^*$, then we show that the following diagram commutes (and likewise for models of T'):

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{h} & \mathfrak{A}^* \\ {\scriptstyle\sigma_{IJ}^{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle\sigma_{IJ}^{\mathfrak{A}^*}} \\ I(J(\mathfrak{A})) & \xrightarrow[I(J(h))]{} & I(J(\mathfrak{A}^*)) \end{array}$$

Suppose $a \in A$ (and all models are one-sorted; extra sorts would be a challenge only to the notation.) Then (abbreviating the notation in (1.7)) we have $\sigma_{IJ}^{\mathfrak{A}}(a) = f_I^{J(\mathfrak{A})}(f_J^{\mathfrak{A}}(\boldsymbol{b}))$ for some $\boldsymbol{b}$ such that $\mathfrak{A} \models \chi_{IJ}(a, \boldsymbol{b})$. Then $\mathfrak{A}^* \models \chi_{IJ}(h(a), h(\boldsymbol{b}))$ since h is elementary, so by (1.6)

$$\begin{aligned} \sigma_{IJ}^{\mathfrak{A}^*}(h(a)) = f_I^{J(\mathfrak{A}^*)}(f_J^{\mathfrak{A}^*}(h(\boldsymbol{b}))) &= f_I^{J(\mathfrak{A}^*)}(J(h)(f_J^{\mathfrak{A}}(\boldsymbol{b}))) \\ &= I(J(h))(f_I^{J(\mathfrak{A})}(f_J^{\mathfrak{A}}(\boldsymbol{b})) = I(J(h))(\sigma_{IJ}^{\mathfrak{A}}(a)). \end{aligned}$$

So the diagram does commute, and the first claim is proved. If h is merely an embedding, but $I(h)$ is still well-defined as an embedding by (1.6), then the same argument works, provided also χ_{IJ} is existential. □

Now let us return to the special case of reductions and expansions. The following lemma singles out some of the basic properties of vector-spaces to be used in proving Theorem 1.1.

Lemma 1.3 *Suppose* $(V, K, *, \|) \models \mathrm{VS}_2^{\mathrm{m}}$.

1. *Say* $\boldsymbol{x},\boldsymbol{y}\in V$, *and* $\boldsymbol{x}\neq 0$. *Then*
$$\boldsymbol{x}\parallel\boldsymbol{y}\iff\exists u\,u*\boldsymbol{x}=\boldsymbol{y}\iff\exists!\,u\,u*\boldsymbol{x}=\boldsymbol{y}.$$
2. *The relation* $\parallel$, *restricted to* $V\smallsetminus\{\mathbf{0}\}$, *is an equivalence-relation.*
3. *The set* $\{\boldsymbol{y}\in V:\boldsymbol{x}\parallel\boldsymbol{y}\}$ *is a subgroup of* V, *for all* $\boldsymbol{x}$ *in* V.
4. *Let* $\boldsymbol{x}$, $\boldsymbol{z}$, *and* $\boldsymbol{w}$ *be vectors in* V *such that* $\boldsymbol{x}\nparallel\boldsymbol{z}$; *let* $u\in K$; *and let* $\boldsymbol{y}=u*\boldsymbol{x}$. *Then*
$$u*\boldsymbol{z}=\boldsymbol{w}\iff\boldsymbol{z}\parallel\boldsymbol{w}\ \&\ \boldsymbol{x}-\boldsymbol{z}\parallel\boldsymbol{y}-\boldsymbol{w}.$$

Proof I write a proof for item 4 only, assuming items 1, 2, and 3. Assume $\boldsymbol{x}\nparallel\boldsymbol{z}$ and $\boldsymbol{y}=u*\boldsymbol{x}$. Part of item 3 is that $\mathbf{0}$ is parallel to every vector; so $\boldsymbol{z}\neq\mathbf{0}$. Similarly, but with the help of item 2, we have $\boldsymbol{z}\nparallel\boldsymbol{x}-\boldsymbol{z}$, so $\boldsymbol{x}-\boldsymbol{z}\neq\mathbf{0}$. Hence, if $u*\boldsymbol{z}=\boldsymbol{w}$, then $\boldsymbol{z}\parallel\boldsymbol{w}$ by item 1, and also $\boldsymbol{y}-\boldsymbol{w}=u*(\boldsymbol{x}-\boldsymbol{z})$, which is parallel to $\boldsymbol{x}-\boldsymbol{z}$ for the same reason. Conversely, say $\boldsymbol{z}\parallel\boldsymbol{w}$. Then $\boldsymbol{w}=t*\boldsymbol{z}$ for some t in K, so that $\boldsymbol{y}-\boldsymbol{w}=u*\boldsymbol{x}-t*\boldsymbol{z}=u*(\boldsymbol{x}-\boldsymbol{z})+(u-t)*\boldsymbol{z}$. If also $\boldsymbol{x}-\boldsymbol{z}\parallel\boldsymbol{y}-\boldsymbol{w}$, then $(u-t)*\boldsymbol{z}\parallel\boldsymbol{x}-\boldsymbol{z}$ by item 3, so $u=t$. □

Theorem 1.1 *Reduction from* $\mathrm{VS}_2^{\mathrm{m}}$ *to* $\mathrm{VS}_2^{\mathrm{r}}$ *is conservative.*

Proof We have R from $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_2^{\mathrm{m}})$ to $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_2^{\mathrm{r}})$. To go the other way, we want to interpret an arbitrary model $(V,K,*,\parallel)$ of $\mathrm{VS}_2^{\mathrm{m}}$ in its reduct, $(V,\parallel)$, obtaining an expansion-functor E. (It will follow that the reducts $(V,\parallel)$ compose an elementary class.) In the notation for interpretations given earlier, we shall need formulas $\delta_{E,\mathrm{vec}}$ and $\delta_{E,\mathrm{sca}}$; these will be $\boldsymbol{x}=\boldsymbol{x}$ and
$$\boldsymbol{x}_0\neq\mathbf{0}\wedge\boldsymbol{x}_0\parallel\boldsymbol{x}_1,$$
respectively. Then $f_{E,\mathrm{vec}}^{(V,\parallel)}$ will be the identity on V, while $f_{E,\mathrm{sca}}^{(V,\parallel)}$ will be given by
$$f_{E,\mathrm{sca}}^{(V,\parallel)}(\boldsymbol{a}_0,\boldsymbol{a}_1)=b\iff(V,K,*,\parallel)\models b*\boldsymbol{a}_0=\boldsymbol{a}_1.\tag{1.8}$$
Indeed, the function $f_{E,\mathrm{sca}}^{(V,\parallel)}$ is well defined on $\delta_{E,\mathrm{sca}}{}^{(V,\parallel)}$, and is surjective onto K, by Lemma 1.3 (item 1):

1. If $(V,\parallel)\models\delta_{E,\mathrm{sca}}(\boldsymbol{a}_0,\boldsymbol{a}_1)$, then there is a unique b in K such that $(V,K,*,\parallel)\models b*\boldsymbol{a}_0=\boldsymbol{a}_1$.
2. If $b\in K$, and $\boldsymbol{a}\neq\mathbf{0}$, then $(V,\parallel)\models\delta_{E,\mathrm{sca}}(\boldsymbol{a},b*\boldsymbol{a})$.

Let $f_{E,\mathrm{sca}}^{(V,\parallel)}$ be denoted by
$$(\boldsymbol{x},\boldsymbol{y})\mapsto[\boldsymbol{x}:\boldsymbol{y}].$$
We have to define the function $\varphi\mapsto\varphi_E$. This will be the identity at all formulas with no scalar-variables. It is now enough to suppose that φ is one of
$$x=y,\quad x+y=z,\quad x\circ y=z,\quad x*\boldsymbol{y}=\boldsymbol{z}.$$
The formula $(x=y)_E$ will be the equation
$$[\boldsymbol{x}_0:\boldsymbol{x}_1]=[\boldsymbol{y}_0:\boldsymbol{y}_1],\tag{1.9}$$

expressed in $\{+,-,0,\|\}$. To obtain this expression, first note that

$$\boldsymbol{y}_0 \neq \mathbf{0} \;\&\; [\boldsymbol{x}_0 : \boldsymbol{x}_1] * \boldsymbol{y}_0 = \boldsymbol{y}_1 \iff [\boldsymbol{x}_0 : \boldsymbol{x}_1] = [\boldsymbol{y}_0 : \boldsymbol{y}_1]. \tag{1.10}$$

Now let $\psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}_0, \boldsymbol{y}_1)$ denote the formula

$$\boldsymbol{x}_0 \nparallel \boldsymbol{y}_0 \wedge \boldsymbol{x}_0 \parallel \boldsymbol{x}_1 \wedge \boldsymbol{y}_0 \parallel \boldsymbol{y}_1 \wedge \boldsymbol{x}_0 - \boldsymbol{y}_0 \parallel \boldsymbol{x}_1 - \boldsymbol{y}_1.$$

(The situation can be illustrated by Fig. 0.2 of §0, with (A,B,C,D,E) replaced with $(\boldsymbol{x}_0, \mathbf{0}, \boldsymbol{y}_0, \boldsymbol{x}_1, \boldsymbol{y}_1)$.) This formula ψ defines a subset of $\delta_{E,\mathrm{sca}}{}^{(V,\|)} \times \delta_{E,\mathrm{sca}}{}^{(V,\|)}$; that is, it defines a binary relation on $\delta_{E,\mathrm{sca}}{}^{(V,\|)}$. This relation is symmetric, though not reflexive, and not quite transitive. However, by Lemma 1.3 (item 4) and (1.10), and by the assumption that $\dim_K V \geqslant 2$, we have

$$\begin{aligned}
&[\boldsymbol{x}_0 : \boldsymbol{x}_1] = [\boldsymbol{y}_0 : \boldsymbol{y}_1] \;\&\; \boldsymbol{x}_0 \nparallel \boldsymbol{y}_0 \iff \psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}_0, \boldsymbol{y}_1);\\
&[\boldsymbol{x}_0 : \boldsymbol{x}_1] = [\boldsymbol{y}_0 : \boldsymbol{y}_1] \;\&\; \boldsymbol{x}_0 \parallel \boldsymbol{y}_0 \iff \boldsymbol{x}_0 \parallel \boldsymbol{y}_0 \;\&\; \exists \boldsymbol{z}_0\, \exists \boldsymbol{z}_1\\
&\qquad\qquad\qquad\qquad\qquad\qquad (\psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{z}_0, \boldsymbol{z}_1) \;\&\; \psi(\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{y}_0, \boldsymbol{y}_1)).
\end{aligned}$$

From these observations, we obtain (1.9) as a formula of $\{+,-,0,\|\}$. A slight simplification is possible: $(x = y)_E$ can be the existential formula

$$\exists \boldsymbol{z}_0\, \exists \boldsymbol{z}_1\, (\psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}_0, \boldsymbol{y}_1) \vee (\psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{z}_0, \boldsymbol{z}_1) \wedge \psi(\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{y}_0, \boldsymbol{y}_1))). \tag{1.11}$$

We obtain the remaining formulas φ_E by means of (1.10) and the following identities:

$$\begin{aligned}
[\boldsymbol{x} : \boldsymbol{y}] + [\boldsymbol{x} : \boldsymbol{z}] &= [\boldsymbol{x} : \boldsymbol{y} + \boldsymbol{z}];\\
[\boldsymbol{y} : \boldsymbol{z}] \circ [\boldsymbol{x} : \boldsymbol{y}] &= [\boldsymbol{x} : \boldsymbol{z}].
\end{aligned}$$

In particular, writing (1.9) as an abbreviation for (1.11), we can tabulate the results:

φ	φ_E
$x + y = z$	$\exists \boldsymbol{w}\, ([\boldsymbol{x}_0 : \boldsymbol{w}] = [\boldsymbol{y}_0 : \boldsymbol{y}_1] \wedge [\boldsymbol{x}_0 : \boldsymbol{x}_1 + \boldsymbol{w}] = [\boldsymbol{z}_0 : \boldsymbol{z}_1])$
$x \circ y = z$	$\exists \boldsymbol{w}\, ([\boldsymbol{y}_1 : \boldsymbol{w}] = [\boldsymbol{x}_0 : \boldsymbol{x}_1] \wedge [\boldsymbol{y}_0 : \boldsymbol{w}] = [\boldsymbol{z}_0 : \boldsymbol{z}_1])$
$x * \boldsymbol{y} = \boldsymbol{z}$	$[\boldsymbol{x}_0 : \boldsymbol{x}_1] = [\boldsymbol{y} : \boldsymbol{z}] \vee (\boldsymbol{y} = \mathbf{0} \wedge \boldsymbol{z} = \mathbf{0})$.

These formulas φ_E are existential. Also, $\neg((x = y)_E)$ is equivalent to the existential formula

$$\exists \boldsymbol{w}\, ([\boldsymbol{x}_0 : \boldsymbol{x}_1] = [\boldsymbol{y}_0 : \boldsymbol{w}] \wedge \boldsymbol{w} \neq \boldsymbol{y}_1).$$

Now we have E as a functor from $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_2^{\mathrm{r}})$ to $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_2^{\mathrm{m}})$, by Lemma 1.1. The composition $R \circ E$ is the identity, and $\sigma_{RE}^{(V,\|)}$ is the identity, and χ_{RE} is $\boldsymbol{x} = \boldsymbol{y}$. For the other side, $\sigma_{ER}^{(V,K,*,\|)}$ is the identity on V, while on K it takes y to the equivalence-class $\{(\boldsymbol{x}_0, \boldsymbol{x}_1) \colon \boldsymbol{x}_0 \neq \mathbf{0} \;\&\; y * \boldsymbol{x}_0 = \boldsymbol{x}_1\}$; correspondingly, $\chi_{ER,\mathrm{vec}}$ is $\boldsymbol{x} = \boldsymbol{y}$, while $\chi_{ER,\mathrm{sca}}$ is $\boldsymbol{x}_0 \neq \mathbf{0} \;\&\; y * \boldsymbol{x}_0 = \boldsymbol{x}_1$. By Lemma 1.2, the proof is complete. □

2 Existentially closed vector-spaces

It was said in §0 that, if U is a *complete* theory of fields, then the theory, in $\{+,-,\mathbf{0},\circ,0,1,*\}$, of n-dimensional vector-spaces whose scalar-fields are models of U is complete, but not model-complete or even inductive, if $n>1$. Indeed, suppose V is a vector-space with basis $(\boldsymbol{b}_0,\dots,\boldsymbol{b}_{n-1})$ over a model K of U, and $n>1$. If $K\subset K'$, then the $\boldsymbol{b}_i$ span a vector-space W over K', but we can have $\boldsymbol{b}_0=a*\boldsymbol{b}_1$ for some a in $K'\smallsetminus K$, so that $\dim_{K'}W\leqslant n-1$. Still, $(V,K,*)$ is a substructure of $(W,K',*)$. We can have $K'\models U$, and we can extend W to an n-dimensional space V' over K'. So $(V,K,*)$ is a substructure of $(V',K',*)$, and both are models of $T_n\cup U$. Continuing, we form a chain

$$(V,K,*)\subset(V',K',*)\subset\cdots\subset(V^{(k)},K^{(k)},*)\subset\cdots \tag{2.1}$$

of models of $T_n\cup U$; the union of this chain has dimension at most $n-1$. The chain *can* have dimension one, even if $n=\infty$: for each k, let $V^{(k)}$ have basis $(\boldsymbol{b}_k,\boldsymbol{b}_{k+1},\dots)$, and assume $\boldsymbol{b}_k=a_k*\boldsymbol{b}_{k+1}$ for some a_k in $K^{(k+1)}\smallsetminus K^{(k)}$.

However, when the structures in (2.1) expand to models of $\mathrm{VS}_2^{\mathrm{m}}$, they no longer form a chain: non-parallelism is not preserved in super-structures. Nonetheless, we shall see that there are chains where non-parallelism *is* preserved in going up, but the union has dimension two. The conclusion will be that the existentially closed models of $\mathrm{VS}_2^{\mathrm{m}}$ are two-dimensional. There is a generalization involving chains that preserve n-ary linear independence.

If $n>0$, let P^n be an n-ary predicate for linear dependence. (So $P^2\boldsymbol{x}\boldsymbol{y}$ means $\boldsymbol{x}\parallel\boldsymbol{y}$; but P^1 is superfluous, since $P^1\boldsymbol{x}$ is equivalent to $\boldsymbol{x}=\mathbf{0}$.) For each positive n, there is a theory VS_n of vector-spaces in the signature $\{+,-,\mathbf{0},\circ,0,1,*,P^n\}$, axiomatized by the usual vector-space axioms, along with

$$P^n\boldsymbol{x}_0\cdots\boldsymbol{x}_{n-1}\leftrightarrow\exists y^0\cdots\exists y^{n-1}\ \Bigl(\sum_{i<n}y^i*\boldsymbol{x}_i=\mathbf{0}\wedge\bigvee_{i<n}y^i\neq 0\Bigr). \tag{2.2}$$

Having the form of (1.2), the axiom (2.2) is equivalent to an $\forall\exists$-sentence. So VS_n is inductive. Models of VS_n have two sorts. But let $\mathrm{VS}_n^{\mathrm{m}}$ be the theory of models of VS_n of dimension at least n; so $\mathrm{VS}_n^{\mathrm{m}}$ has the additional axiom

$$\exists\boldsymbol{x}_0\cdots\exists\boldsymbol{x}_{n-1}\ \neg P^n\boldsymbol{x}_0\cdots\boldsymbol{x}_{n-1}.$$

In this theory, when $n\geqslant 2$, we can define non-parallelism and parallelism in $\mathrm{VS}_n^{\mathrm{m}}$ by means of the existential formulas

$$\exists(\boldsymbol{x}_2,\dots,\boldsymbol{x}_{n-1})\ \neg P^n\boldsymbol{x}_0\cdots\boldsymbol{x}_{n-1},$$

$$\exists(\boldsymbol{x}_2,\dots,\boldsymbol{x}_n)\ \Bigl(\boldsymbol{x}_1=0\vee\bigl(\neg P^n\boldsymbol{x}_1\cdots\boldsymbol{x}_n\wedge\bigwedge_{j=2}^{n}P^n\boldsymbol{x}_0\cdots\boldsymbol{x}_{j-1}\boldsymbol{x}_{j+1}\cdots\boldsymbol{x}_n\bigr)\Bigr).$$

Indeed, if $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)$ is linearly independent, but $(\boldsymbol{a}_0,\dots,\boldsymbol{a}_{i-1},\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)$ is not, then $\boldsymbol{a}_0$ is a unique linear combination of $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)$, and this combination does not use $\boldsymbol{a}_j$. If that is so, whenever $2\leqslant j\leqslant n$, then $\boldsymbol{a}_0$ must be a multiple of $\boldsymbol{a}_1$. If we reduce a model of $\mathrm{VS}_n^{\mathrm{m}}$ to the sort of vectors, in the signature $\{+,-,\mathbf{0},P^n\}$, the reduct might be termed an **abelian group with n-ary linear dependence.** As a porism from Theorem 1.1, we have that the reduction from $\mathrm{VS}_n^{\mathrm{m}}$ to the theory of abelian groups with n-ary linear dependence is conservative. By Theorem 2.3 below, the model-companion of VS_n—hence, of $\mathrm{VS}_n^{\mathrm{m}}$—is the theory $\mathrm{VS}_n{}^*$ of n-dimensional vector-spaces over algebraically closed fields. Hence the theory of abelian groups with n-ary linear dependence is likewise companionable.

To warm up for the proof of Theorem 2.3, I consider first VS_1, the theory of non-trivial vector-spaces. I shall generally write models as (V,K), rather than the full $(V,K,*)$:

Theorem 2.1 *The theory* VS_1 *is companionable; its model-companion is the theory* $\mathrm{VS}_1{}^*$. *The model-companion is* not *a model-*completion; *much less does it admit quantifier-elimination; but it is the model-completion of the theory of vector-spaces of dimension at most* 1.

Proof Suppose $(V,K) \models \mathrm{VS}_1$ and has basis B. We may assume that B is a subset of a field extending K (while still being linearly independent over K). Then we can form $(K(B)^{\mathrm{alg}}, K(B)^{\mathrm{alg}})$, a model of $\mathrm{VS}_1{}^*$ in which (V,K) embeds.

Suppose $(V,K) \models \mathrm{VS}_1{}^*$ and is a substructure of the models (W,L) and (W',L'), which have equal cardinality greater than $|K|$. Then $|L| = |L'|$, so $L \cong L'$ over K; hence

$$(W,L) \cong (L \otimes_K V, L) \cong (L' \otimes_K V, L') \cong (W', L')$$

over (V,K). So $\mathrm{VS}_1{}^* \cup \operatorname{diag}(V,K)$ is categorical in powers greater than $|K|$; having also no finite models, the theory is complete. Therefore $\mathrm{VS}_1{}^*$ is model-complete, so it is the model-companion of VS_1.

In the foregoing argument, it is enough to suppose merely that (V,K) is one-dimensional; K need not be algebraically closed. Even if $V = \{\mathbf{0}\}$, the theory $\mathrm{VS}_1{}^* \cup \operatorname{diag}(V,K)$ is complete. This shows that $\mathrm{VS}_1{}^*$ is the model-completion of the theory of vector-spaces of dimension at most 1.

The model $(\mathbb{Q}^{\mathrm{alg}}, \mathbb{Q}^{\mathrm{alg}})$ of $\mathrm{VS}_1{}^*$ has the two substructures $(\langle 1, \sqrt{2}\rangle, \mathbb{Q})$ and $(\langle 1, \sqrt{3}\rangle, \mathbb{Q})$, which are two-dimensional isomorphic models of VS_1, although the formula

$$\exists z\, (z \circ z = 1 + 1 \wedge z * \boldsymbol{x}_0 = \boldsymbol{x}_1)$$

is satisfied in $(\mathbb{Q}^{\mathrm{alg}}, \mathbb{Q}^{\mathrm{alg}})$ by $(1, \sqrt{2})$, but not by $(1, \sqrt{3})$. Therefore the theory $\mathrm{VS}_1{}^* \cup \operatorname{diag}(\langle 1, \sqrt{2}\rangle, \mathbb{Q})$ is the same as $\mathrm{VS}_1{}^* \cup \operatorname{diag}(\langle 1, \sqrt{3}\rangle, \mathbb{Q})$ (as long as the same symbol denotes $\sqrt{2}$ and $\sqrt{3}$ respectively), but this theory is not complete. Thus $\mathrm{VS}_1{}^*$ is not a model-completion of VS_1. □

Proving the more general Theorem 2.3 below requires some preliminary work on matrices. I shall understand an $m \times n$ matrix (m rows, n columns) as a function on $m \times n$, that is, on $\{(i,j) \in \omega^2 \colon i < m \ \&\ j < n\}$. Such a function can be denoted by

$$(u^i_j)^{i<m}_{j<n}\,;$$

its transpose is $(u^i_j)^{j<n}_{i<m}$. I shall consider elements of a Cartesian power like K^n as row-vectors, that is, $1 \times n$ matrices.

Lemma 2.1 *For all positive integers n, if U is an $n \times n$ matrix (over some field), and $\boldsymbol{u}$ is $n \times 1$, and $\boldsymbol{a}$ is $1 \times n$, then*

$$\det \left(\begin{array}{c|c} U & \boldsymbol{u} \\ \hline \boldsymbol{a} & 1 \end{array} \right) = \det(U - \boldsymbol{u} \cdot \boldsymbol{a}). \tag{2.3}$$

Proof Start with the identity

$$\left(\begin{array}{c|c} U & \boldsymbol{u} \\ \hline \boldsymbol{a} & 1 \end{array} \right) \cdot \left(\begin{array}{c|c} \mathrm{I}_n & \mathbf{0} \\ \hline -\boldsymbol{a} & 1 \end{array} \right) = \left(\begin{array}{c|c} U - \boldsymbol{u} \cdot \boldsymbol{a} & \boldsymbol{u} \\ \hline \mathbf{0} & 1 \end{array} \right)$$

and take the determinant of either side. □

The lemma can be interpreted as follows. Let K be a field, and let V be K^{n+1}. The space of alternating n-forms on V has dimension $\binom{n+1}{n}$, or $n+1$. Now understand V^n as comprising the $n \times (n+1)$ matrices over K. The function $\left(U \middle| \boldsymbol{u} \right) \mapsto \det \left(\begin{array}{c|c} U & \boldsymbol{u} \\ \hline \boldsymbol{a} & 1 \end{array} \right)$ from V^n to K is an alternating n-form on V that takes $\left(\mathrm{I}_n \middle| \mathbf{0} \right)$ to 1, and that takes $\left(U \middle| \boldsymbol{u} \right)$ to 0 if one of its rows is $\left(\boldsymbol{a} \middle| 1 \right)$. There is only one such function; the function $\left(U \middle| \boldsymbol{u} \right) \mapsto \det(U - \boldsymbol{u} \cdot \boldsymbol{a})$ has the same properties; (2.3) follows.

Suppose further π_i is $\boldsymbol{x} \mapsto x_i$ on V, so $\pi_i \in V^*$. If $B \in V^n$, let ΦB be the function $\det \left(\begin{array}{ccc} & B & \\ \hline \pi_0 & \cdots & \pi_n \end{array} \right)$: this is an element $\sum_{i=0}^{n} c^i \cdot \pi_i$ of V^*. Then $(c^0, \dots, c^n)$ is the **cross-product** [16, pp. 83 f.] of (the rows of) B: it is the element $\times B$ of V such that $\boldsymbol{x} \cdot (\times B) = \Phi B \boldsymbol{x}$ for all $\boldsymbol{x}$ in V.

Lemma 2.2 *Let K be a field, $V = K^{n+1}$, and $B \in V^n$. Say $K \subset L$, and $\boldsymbol{a} \in L^{n+1}$, and $\boldsymbol{a}$ is linearly independent over K. The following are equivalent:*

1. $\times B = 0$;
2. $\det \left(\begin{array}{c} B \\ \hline \boldsymbol{a} \end{array} \right) = 0$;
3. *The rows of B are linearly dependent over K;*
4. *The rows of B are linearly dependent over L.*

Proof The equivalence of (1) and (2) follows from the definition of $\times B$ and the linear independence of $\boldsymbol{a}$ over K. The equivalence of (3) and (4) follows from the observation that if the equation $\boldsymbol{x} \cdot B = \mathbf{0}$ has a non-trivial solution at all, it has one in K^n. Finally, $\boldsymbol{a}$ is not in the span over L of the rows of B: if it were, then $\boldsymbol{c} \cdot B = \boldsymbol{a}$ for some $\boldsymbol{c}$ in L^n, which would mean that the $n+1$ entries of $\boldsymbol{a}$ belonged to the span over K of the n entries of $\boldsymbol{c}$. Hence (2) and (4) are equivalent. □

Theorem 2.2 *Suppose L/K is a field-extension, and $[L : K] \geqslant n+1$. Then the vector-space (K^{n+1}, K) embeds in (L^n, L) so as to preserve linear independence of n-tuples: that is, the embedding is of the structure (K^{n+1}, K, P^n) in (L^n, L, P^n). One such embedding is*

$$\boldsymbol{x} \longmapsto \boldsymbol{x} \cdot \left(\begin{array}{c} \mathrm{I}_n \\ \hline -\boldsymbol{a} \end{array} \right), \tag{2.4}$$

where $\boldsymbol{a}$ in L^n is such that $(a_0, \dots, a_{n-1}, 1)$ is linearly independent over K.

Proof Assume that $(a_0,\dots,a_{n-1},1)$ is linearly independent over K. Treating n vectors from K^{n+1} as the rows of an $n\times(n+1)$ matrix, we can write the result as $\big(U\big|\boldsymbol{u}\big)$. By Lemma 2.2, the rows of $\big(U\big|\boldsymbol{u}\big)$ are linearly dependent over K if and only if

$$\det\left(\begin{array}{c|c} U & \boldsymbol{u} \\ \hline \boldsymbol{a} & 1 \end{array}\right)=0. \tag{2.5}$$

The images of those rows under the transformation in (2.4) are the rows of the product $\big(U\big|\boldsymbol{u}\big)\cdot\left(\begin{array}{c} \mathrm{I}_n \\ \hline -\boldsymbol{a} \end{array}\right)$. This product is $U-\boldsymbol{u}\cdot\boldsymbol{a}$. So the images are linearly dependent over L if and only if

$$\det(U-\boldsymbol{u}\cdot\boldsymbol{a})=0. \tag{2.6}$$

By Lemma 2.1, equations (2.5) and (2.6) are equivalent. Hence the transformation in (2.4) is an embedding that preserves independence of n-tuples. □

Not every embedding of (K^{n+1},K) in (L^n,K) preserves independence of n-tuples: Just use an embedding as in (2.4), but replace some—not all—entries of $\boldsymbol{a}$ with 0.

Theorem 2.3 *For each positive n, the theory* $\mathrm{VS}_n{}^*$ *is the model-companion of* VS_n*; it is not a model-completion of* VS_n *or even of* $\mathrm{VS}_n^{\mathrm{m}}$*, but is the model-completion of the theory of n-dimensional vector-spaces.*

Proof The proof is, in part, as for Theorem 2.1. Since VS_n is inductive, every model embeds in an existentially closed model. Such models are models of $\mathrm{VS}_n{}^*$. Indeed, every model (V,K,P^n) of VS_n embeds in $(K^{\mathrm{alg}}\otimes_K V,K^{\mathrm{alg}},P^n)$; so existentially closed models must have algebraically closed scalar-fields. Also, if $\dim_K V=m<n$, then (V,K,P^n) fails to have a solution to

$$\neg P^n\boldsymbol{x}_0\cdots\boldsymbol{x}_{n-1}; \tag{2.7}$$

but (V,K,P^n) embeds in $(V\oplus K^{n-m},K,P^n)$, where (2.7) does have a solution; so (V,K,P^n) was not existentially closed. Finally, say $\dim_K V>n$, so that there is a linearly independent $(n+1)$-tuple $(\boldsymbol{a}^0,\dots,\boldsymbol{a}^n)$ of vectors in V. Then (V,K,P^n) has no solution to

$$\sum_{i=0}^{n}x^i*\boldsymbol{a}_i=\mathbf{0}\wedge\bigvee_{i=0}^{n}x^i\neq 0. \tag{2.8}$$

But analyse V as $V_0\oplus V_1$, where V_0 is spanned by the vectors $\boldsymbol{a}^i$. By Theorem 2.2, there is a model (W,L,P^n) of VS_n in which (V_0,K,P^n) embeds, but which has a solution to (2.8). Then (V,K,P^n) embeds in $(W\oplus L\otimes_K V_1,L,P^n)$, and the latter has a solution to (2.8). So again (V,K,P^n) was not existentially closed.

So the existentially closed models of VS_n are models of $\mathrm{VS}_n{}^*$; in particular, every model of VS_n embeds in a model of $\mathrm{VS}_n{}^*$. If (V,K,P^n) is an n-dimensional model of VS_n, then $\mathrm{VS}_n{}^*\cup\operatorname{diag}(V,K,P^n)$ is complete, by a categoricity argument as in the proof of Theorem 2.1. Therefore $\mathrm{VS}_n{}^*$ is the model-completion of the theory of n-dimensional models of VS_n.

However, $\mathrm{VS}_n{}^*$ is not the model-completion of $\mathrm{VS}_n^{\mathrm{m}}$ (much less of VS_n). Indeed, let $(a_0,\dots,a_n)$ be an $(n+1)$-tuple of algebraic numbers such that the $(n+2)$-tuple $(a_0,\dots,a_n,1)$ is linearly independent over $\mathbb{Q}$. By Theorem 2.2, the row-spaces of the matrices

$$\left(\frac{\mathrm{I}_n}{a_0\ \cdots\ a_{n-1}}\right) \quad \text{and} \quad \left(\frac{\mathrm{I}_n}{a_1\ \cdots\ a_n}\right)$$

are substructures $\mathfrak{A}$ and $\mathfrak{B}$ of $((\mathbb{Q}^{\mathrm{alg}})^n,\mathbb{Q}^{\mathrm{alg}},P^n)$. Being $(n+1)$-dimensional, they are isomorphic models of $\mathrm{VS}_n^{\mathrm{m}}$; but no automorphism of $((\mathbb{Q}^{\mathrm{alg}})^n,\mathbb{Q}^{\mathrm{alg}},P^n)$ takes $\mathfrak{A}$ to $\mathfrak{B}$. □

We can let VS_∞ be the union of the theories VS_n; so it is the theory of vector-spaces in the signature $\{+,-,\mathbf{0},\circ,0,1,*,P^1,P^2,P^3,\dots\}$. Let $\mathrm{VS}_\infty{}^*$ be the theory, in the same signature, of infinite-dimensional vector-spaces over algebraically closed fields.

Theorem 2.4 *The theory* $\mathrm{VS}_\infty{}^*$ *is the model-completion of* VS_∞*, but does not admit quantifier-elimination.*

Proof Every model $(V,K,P^1,\dots)$ of VS_∞ embeds in the model

$$(K^{\mathrm{alg}}\otimes_K V\oplus(K^{\mathrm{alg}})^\omega,K^{\mathrm{alg}},P^1,\dots)$$

of $\mathrm{VS}_\infty{}^*$. Also, let $(V,K,P^1,\dots)\models\mathrm{VS}_\infty$, and $\kappa=\aleph_0+|V|+|K|$. Then $\mathrm{VS}_\infty{}^*\cup\operatorname{diag}(V,K,P^1,\dots)$ is complete, since all of its models (W,L,P^1) are isomorphic, provided $\text{tr-deg}(L/K)=\kappa$ and $\dim_L(W/(L\otimes_K V))=\kappa$; but these are just the models of size κ that realize certain types. So $\mathrm{VS}_\infty{}^*$ is the model-completion of VS_∞.

Finally, the example at the end of the proof of Theorem 2.1 shows that $\mathrm{VS}_\infty{}^*$ does not admit elimination of quantifiers. Indeed, $(\langle 1,\sqrt{2}\rangle,\mathbb{Q})$ expands to a substructure of a model of $\mathrm{VS}_\infty{}^*$ in which $1\parallel\sqrt{2}$, but $\mathrm{VS}_\infty{}^*\cup\operatorname{diag}(\langle 1,\sqrt{2}\rangle,P^1,P^2,\dots)$ is not complete, since it does not specify which scalars ensure that $1\parallel\sqrt{2}$. □

Theorem 2.5 *The completions of the model-complete theories* $\mathrm{VS}_n{}^*$ *are obtained by specifying a characteristic for the scalar field. The completions are* ω*-stable.*

Proof Let $\mathrm{VS}_{n,p}{}^*$ be the theory of models of $\mathrm{VS}_n{}^*$ whose scalar fields have characteristic p (positive or zero). If n is finite, then $\mathrm{VS}_{n,p}{}^*$ is uncountably categorical, hence complete and ω-stable.

All models of $\mathrm{VS}_{\infty,p}{}^*$ have a substructure isomorphic to $(\{\mathbf{0}\},\mathbb{F}_p)$, where $\mathbb{F}_p$ is a prime field of characteristic p. Hence $\mathrm{VS}_{\infty,p}{}^*\vdash\operatorname{diag}(\{\mathbf{0}\},\mathbb{F}_p)$. But here $(\{\mathbf{0}\},\mathbb{F}_p)\models\mathrm{VS}_\infty$, so the theory $\mathrm{VS}_\infty{}^*\cup\operatorname{diag}(\{\mathbf{0}\},\mathbb{F}_p)$ is complete by Theorem 2.4. Therefore $\mathrm{VS}_{\infty,p}{}^*$ is complete.

Finally, let (V,K) be a countable, *definably closed* substructure of a big model of $\mathrm{VS}_\infty{}^*$. This implies that, if some vectors in V are linearly dependent, then scalars witnessing this can be found in K. A complete type of $\boldsymbol{x}$ over (V,K) says either that $\boldsymbol{x}$ is linearly independent from V, or—for some $\boldsymbol{a}_i$ in V—that $\boldsymbol{x}=\sum_{i<n}t^i*\boldsymbol{a}_i$ for some scalars t^i. In the latter case, the type also specifies a variety over K of which $(t^0,\dots,t^{n-1})$ is a generic point. If $(u^0,\dots,u^{n-1})$ is another generic point of

the same variety, there is an automorphism of the big model, over (V,K), taking $\sum_{i<n} t^i * \boldsymbol{a}_i$ to $\sum_{i<n} u^i * \boldsymbol{a}_i$. Thus there are just countably many types of $\boldsymbol{x}$ over (V,K). Similarly for types of a scalar variable t: If $V = \{\mathbf{0}\}$, then these types correspond to types over K, of which there are countably many. If V contains a non-zero vector $\boldsymbol{a}$, then for every type $p(t)$ over (V,K), there is a type of $\boldsymbol{x}$ that includes $\{\exists t\ (\boldsymbol{x} = t * \boldsymbol{a} \wedge \varphi(t) \colon \varphi \in p\}$; but the union of any two such sets is inconsistent.

$\square$

Acknowledgements I read Euclid and Descartes as a student at St John's College, Annapolis and Santa Fe. At Logic Colloquium 2006 in Nijmegen, Ehud Hrushovski pointed out an error in an earlier version of this work (and informed me of Kamensky's work [9]). The anonymous referee offered some helpful suggestions and corrections.

References

1. Ahlbrandt, G., Ziegler, M.: Quasi-finitely axiomatizable totally categorical theories. Ann. Pure Appl. Logic **30**(1), 63–82 (1986). Stability in model theory (Trento, 1984)
2. Descartes: The Geometry of René Descartes. Dover Publications, Inc., New York (1954). Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition of 1637
3. Descartes: The Philosophical Writings of Descartes, vol. I. Cambridge University Press (1985). Translated by John Cottingham, Robert Stoothoff, and Dugald Murdoch
4. Descartes: Règles pour la direction de l'esprit. Classiques de la philosophie. Le Livre de Poche (2002). Traduction et notes par Jacques Brunschwig. Préface, dossier et glossaire par Kim Sang Ong-Van-Cung
5. Euclid: The thirteen books of Euclid's Elements translated from the text of Heiberg. Vol. I: Introduction and Books I, II. Vol. II: Books III–IX. Vol. III: Books X–XIII and Appendix. Dover Publications Inc., New York (1956). Translated with introduction and commentary by Thomas L. Heath, 2nd ed
6. Hartshorne, R.: Geometry: Euclid and beyond. Undergraduate Texts in Mathematics. Springer-Verlag, New York (2000)
7. Hodges, W.: Model theory, *Encyclopedia of Mathematics and its Applications*, vol. 42. Cambridge University Press, Cambridge (1993)
8. Jensen, C.U., Lenzing, H.: Model-theoretic algebra with particular emphasis on fields, rings, modules, *Algebra, Logic and Applications*, vol. 2. Gordon and Breach Science Publishers, New York (1989)
9. Kamensky, M.: The model completion of the theory of modules over finitely generated commutative algebras. Http://arxiv.org/abs/math/0607418
10. Kuzichev, A.A.: Elimination of quantifiers over vectors in some theories of vector spaces. Z. Math. Logik Grundlag. Math. **38**(5-6), 575–577 (1992)
11. Macintyre, A.: Model theory: geometrical and set-theoretic aspects and prospects. Bull. Symbolic Logic **9**(2), 197–212 (2003). New programs and open problems in the foundation of mathematics (Paris, 2000)
12. Marker, D.: Model theory: an introduction, *Graduate Texts in Mathematics*, vol. 217. Springer-Verlag, New York (2002)
13. Pillay, A.: Geometric stability theory, *Oxford Logic Guides*, vol. 32. The Clarendon Press Oxford University Press, New York (1996). Oxford Science Publications
14. Robinson, A.: Complete theories, second edn. North-Holland Publishing Co., Amsterdam (1977). With a preface by H. J. Keisler, Studies in Logic and the Foundations of Mathematics, first published 1956
15. Rothmaler, P.: Introduction to model theory, *Algebra, Logic and Applications*, vol. 15. Gordon and Breach Science Publishers, Amsterdam (2000). Prepared by Frank Reitmaier, Translated and revised from the 1995 German original by the author
16. Spivak, M.: Calculus on manifolds. A modern approach to classical theorems of advanced calculus. Addison-Wesley Publishing Company, Reading, Massachusetts (1968). Corrected reprint of 1965 edition

17. Tarski, A.: What is elementary geometry? In: The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), Studies in Logic and the Foundations of Mathematics, pp. 16–29. North-Holland Publishing Co., Amsterdam (1959)
18. Tarski, A., Givant, S.: Tarski's system of geometry. Bull. Symbolic Logic **5**(2), 175–214 (1999)