Abscissas and Ordinates

David Pierce

October 7, 2014
Mathematics Department
Mimar Sinan Fine Arts University
Istanbul
dpierce@msgsu.edu.tr
http://mat.msgsu.edu.tr/~dpierce/

Abstract

In the manner of Apollonius of Perga, but hardly any modern book, we investigate conic sections *as such.* We thus discover why Apollonius calls a conic section a parabola, an hyperbola, or an ellipse; and we discover the meanings of the terms abscissa and ordinate. In an education that is liberating and not simply indoctrinating, the student of mathematics will learn these things.

Contents

List of Figures

1 The liberation of mathematics

In the third century before the Common Era, Apollonius of Perga (near Antalya in Turkey) wrote eight books on conic sections. The first four of these books survive in the original Greek; the next three survive in Arabic translation only. The last book is lost. Lucio Russo [37, page 8] uses this and other examples to show that we cannot expect all of the best ancient work to have come down to us.

In the first book of the *Conics* [4], Apollonius derives properties of the conic sections that can be used to write their equations in rectangular or oblique coordinates. The present article reviews these properties, because (1) they have intrinsic mathematical interest, (2) they are the reason why Apollonius gave to the three conic sections the names that they now have, and (3) the vocabulary of Apollonius is a source for a number of our other technical terms as well.

In a modern textbook of analytic geometry, the two coordinates of a point in the so-called Cartesian plane may be called the "abscissa" and "ordinate." Probably the book will not explain why. But the reader deserves an explanation. The student should not have to learn meaningless words, for the same reason that she should not be expected to memorize the quadratic formula without seeing a derivation of it. True education is not indoctrination, but liberation. I elaborate on this point, if only obliquely, in an article on my college [34]. Mathematics is liberating when it teaches us our own power to decide what is true. This power comes with a responsibility to justify our decisions to anybody who asks; but this is a responsibility that must be shared by all of us who do mathematics.

Mathematical terms *can* be assigned arbitrarily. This is permissible, but it is not desirable. The terms "abscissa" and "ordinate" arise quite naturally in Apollonius's development of the conic sections. This development should be better known, especially by anybody who teaches analytic geometry. This is why I write.

2 Lexica and registers

Apollonius did not create his terms: they are just ordinary words, used to refer to mathematical objects. When we do not *translate* Apollonius, but simply transliterate his words, or use their Latin translations, then we put some distance between ourselves and the mathematics. When I read in high school that a conic section had a *latus rectum,* I had a sense that there was a whole theory of conic sections that was not being revealed, although its existence was hinted at by this peculiar Latin term. If we called the *latus rectum* by its English name of "upright side," then the student could ask, "What is upright about it?" In turn, textbook writers might feel obliged to answer this question. In any case, I am going to answer it here. Briefly, it is called upright because, for good reason, it is to be conceived as having one endpoint on the vertex of the conic section, but as sticking out from the plane of the section.

English does borrow foreign words freely: this is a characteristic of the language. A large lexicon is not a bad thing. A choice from among two or more synonyms can help establish the register of a piece of speech. In the 1980s, a book called *Color Me Beautiful* was on the American bestseller lists. The *New York Times* blandly said the book provided "beauty tips for women"; the *Washington Post* described it as offering "the color-wheel approach to female pulchritude." (The quotations are from my memory only.) By using an obscure synonym for beauty, the *Post* mocked the book.

If distinctions between near-synonyms are maintained, then subtleties of expression are possible. "Circle" and "cycle" are Latin and Greek words for the same thing, but the Greek word is used more abstractly in English, and it would be bizarre to refer to a finite group of prime order as being circular rather than cyclic.

To propose or maintain distinctions between near-synonyms is a *raison d'être* of works like Fowler's *Dictionary of Modern English Usage* [18]. Fowler laments, for example, the use of the Italian word *replica* to refer to any copy of an art-work, when the word properly refers to a copy *made by the same artist.* In his article on synonyms, Fowler sees in language the kind of liberation, coupled with responsibility, that I ascribed to mathematics:

> Synonym books in which differences are analysed, engrossing as they may have been to the active party, the analyst, offer to the passive party, the reader, nothing but boredom. Every reader must, for the most part, be his own analyst; & no-one who does not expend, whether expressly & systematically or as a half-conscious accompaniment of his reading & writing, a good deal of care upon points of synonymy is likely to write well.

Fowler's own book *is,* in part, one of the synonym books that he denigrates here; so I suppose he is saying, "You will not be a good writer, just by reading me: you must read, and *think,* for yourself." What other synonym books does he have in mind? Perhaps books like *Roget's Thesaurus,* which I do find occasionally useful, but which is *not* the "indispensible guide to the English language" that the back cover of "my" edition [9] claims it to be. Fowler's own book is closer to that description, for the example it sets of sound thinking. This is why I prefer to read his original work, rather than the posthumous second edition, edited and updated by Gowers [17].

The boredom, described by Fowler, of the reader of a *mere* book of synonyms is comparable to that of the reader of a mathematics textbook that begins with a bunch of strange words like "abscissa" and "ordinate."

Mathematics can be done in any language. Greek does mathematics without a specialized vocabulary. It is worthwhile to consider what this is like.

I shall take Apollonius's terminology from Heiberg's edition [2]—actually a printout of a `pdf` image downloaded from the Wilbour Hall website, `wilbourhall.org`. Meanings are checked with the big Liddell–Scott–Jones lexicon, the "LSJ" [25]. The articles of the LSJ are available from the Perseus Digital Library, `perseus.tufts.edu`, though I myself splurged on the print version of the whole book. In fact the great majority of works in the references, including all of the language books, are from the shelves of my personal library; some might not be the most up-to-date references, but they are the ones that I am pleased to have at hand.

I am going to write out Apollonius's terms in Greek letters, using the NeoHellenic font made available by the Greek Font Society. I shall use the customary minuscule forms—today's "lower case"—developed in the Middle Ages. Apollonius himself would have used only the letters that we now call capital; but modern mathematics uses minuscule Greek letters freely, and the reader ought to be able to make sense of them, even without studying the Greek language as such. I have heard a plausible rumor that it is actually *beneficial* for calculus students to memorize the Greek alphabet.

3 The gendered article

Apollonius's word for **cone** is ὁ κῶνος, meaning originally "pinecone." Evidently our word "cone" comes ultimately from Apollonius's word (and this is confirmed by such resources as the *Concise Oxford Dictionary of English Etymology* [20]). I write ὁ in front of κῶνος to indicate its gender: ὁ is the masculine definite article. The feminine definite article is ἡ. In each case, the sign over the vowel is the "rough breathing" mark, indicating the prefixed sound that is spelled in English with the letter H. The other accents (ώ, ῶ, ὼ) can be ignored; I reproduce them because they are in the modern texts (and are occasionally useful in parsing).

In the specific terminology of Apollonius, all of the nouns that we shall look at will be feminine or masculine. Greek does however have a neuter gender as well, and the neuter definite article is τό. English retains the threefold gender distinction in the pronouns "he, she, it." English *formerly* had the distinction in its definite article as well [39, pages 28–9], with *se, seo, ðæt* corresponding to the Greek ὁ, ἡ, τό; the masculine *se* became *ðe,* and then our "the," while the neuter *ðæt* became our demonstrative "that" [38, pages 550–1]. Thus residues of gender exist in English today. However, English no longer allows the economy of mathematical expression that is made possible by a gendered article as such.

In Greek mathematics, **point** is τὸ σημεῖον, neuter, while **line** is ἡ γραμμή, feminine. The feminine ἡ στιγμή can also be used for a mathematical point. However, in *The Shaping of Deduction in Greek Mathematics,* Netz argues [30, page 113] that mathematicians dropped στιγμή in favor of σημεῖον, so that an expression like ἡ Α could unambiguously designate a *line* labelled as Α in a diagram, while τὸ Α would designate a point.

Proposition 43 may be the most portentous of the 48 propositions of Book I of Euclid's *Elements.* It makes possible the definition of an algebra of line segments such as Descartes would define, almost two millenia later. In particular, it makes possible the definition of the product of two line segments, once a unit segment is chosen. (In fact, for this definition, Descartes uses not Proposition I.43, but the theory of proportion that Euclid will not develop until Book V. My article [33] features a contemporary theoretical development of Descartes's idea.)

Euclid's Proposition I.43 establishes the *equality* of certain dissimilar parallelograms having the same angle. It bothers some of my students to say "equality" here; they apparently think equality must be congruence, if not identity, even though they will allow that the fractions $2/3$ and $4/6$ are equal. They want to say that the equality in Proposition I.43 is an equality of *areas.* But they cannot say what area is, except that it is some real number of units. This is a doctrine that they have been taught without justification; the justification starts in Euclid, and this is a reason why my department asks students to read him.

Euclid's proposition is specifically that, if two straight lines parallel to sides of a given parallelogram intersect on a diagonal of that parallelogram, then the "complements" of the two smaller parallelograms formed about the diagonal are equal to one another. That is, parallelograms ΒΕΚΗ and ΚΖΔΘ in Figure 1 are equal to one

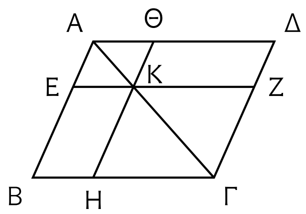

Figure 1 Complementary parallelograms

another. The diagram is Euclid's, at least as it is reproduced in Heiberg's text [12, page 100]. But Euclid can express the equation of parallelograms more tersely. Heath's translation [13, 14] of Euclid's proof begins as follows, if we put Euclid's Greek letters back in for Heath's Latin replacements:

> Let ΑΒΓΔ be a parallelogram, and ΑΓ its diameter; and about ΑΓ let ΕΘ, ΖΗ be parallelograms, and ΒΚ, ΚΔ the so-called complements; I say that the complement ΒΚ is equal to the complement ΚΔ. For, since ΑΒΓΔ is a parallelogram, and ΑΓ is its diameter, the triangle ΑΒΓ is equal to the triangle ΑΓΔ...

Euclid can refer to the parallelogram ΑΕΚΘ simply as τὸ ΕΘ, using the neuter article τό, because παραλληλόγραμμον is neuter. The Greek reader cannot think that τὸ ΕΘ is a line; the line would be ἡ ΕΘ, with the feminine article, just as the diagonal of the big parallelogram is ἡ ΑΓ. The parallelograms about this diagonal are τὰ ΕΘ, ΖΗ, with the plural neuter article τά. Their complements, which are also parallelograms, are τὰ ΒΚ, ΚΔ, likewise with the plural neuter article, because παραπλήρωμα is neuter, as παραλληλόγραμμον is. Perhaps it is no coincidence that Euclid's word for "complement" has the same gender as "parallelogram," but Euclid imposed this as a necessary condition on his choice of the word. The LSJ lexicon cites only Euclid for the geometrical meaning of παραπλήρωμα; its other meanings are "expletive" and "*sagina,*" the latter being a Latin term for "meat to cram fowls" [44].

The concision that is possible for Euclid is confusing to the reader in English—and in Turkish, to name another language that is free of gender. This is the language that my students are reading Euclid in [15]. I said they could be uneasy that Proposition I.43 was about equality of incongruent parallelograms; but in fact they have already had to get used to this kind of equality in Proposition 35:

> Parallelograms on the same base and being in the same parallels are equal to one another.

When they get to Proposition 43, some students think that what is going to be proved is the equality of the *straight lines* ΒΚ and ΚΔ. A sufficiently careful reader in any language may avoid this mistake, especially because those straight lines have not actually been drawn in Euclid's diagram; but *no* reader in Greek can make the mistake. This is one of the few cases where gender in a language is actually useful.

Greek has no *indefinite* article. English and Turkish do have one, derived from the word "one." In Turkish it is still the same word, but positioned differently: *bir güzel gün* "one fine day," but *güzel bir gün* "a fine day." Turkish has no definite article.

Today, perhaps oddly, we may use the indefinite article where Greek uses the definite. Heath translates the enunciation of Proposition I.1 of Euclid as,

> On a given finite straight line to construct an equilateral triangle.

But the Greek says, "*the* given finite straight line." One might read the definite article here as the *generic* article, seen for example in the seventh verse of Wordsworth's little poem [45, page 85],

MY HEART LEAPS UP.

My heart leaps up when I behold
  A rainbow in the sky:
So was it when my life began;
So is it now I am a man;
So be it when I shall grow old,
  Or let me die!
The child is the father of the man;
And I could wish my days to be
Bound each to each by natural piety.

The old *Descriptive English Grammar* of Harman and House [22, page 76] gives Wordsworth's verse as an example, though without actually naming the poet. Devoting little more than a page to "the," the authors say of it only that (1) it is derived from the masculine demonstrative pronoun *ðe* and has its meaning, weakened; (2) it also has a generic meaning, as in "**The** child is the father of the man"; (3) it is used in names and titles, such as "**The** Soviet Union." By contrast, the colossal *Cambridge Grammar of the English Language* [23], in its main section on the definite article as such (5.6.1, pages 368–71), does *not* mention the generic meaning of "the": this comes later, in "Class uses of the definite article" (5.8.4(a), page 407), with examples like, "The African elephant will soon be extinct."

We can say, "The isosceles triangle has its base angles equal" (*Elements* I.5), meaning "Every isosceles triangle"; we can also say "An isosceles triangle," with the same meaning. But the explanation for Euclid's use of the definite article in Proposition I.1 may be simpler. He is referring to the straight line that the reader or listener *can already see* in the diagram that Euclid has put on display. Netz points out [30, pages 14–16] that ancient schools would not have had our blackboards and whiteboards, on which diagrams could be drawn during the course of a lecture; diagrams were most likely prepared beforehand, perhaps by being graven in wax.

4 Applications

There is no word in Greek for what we now label as *propositions* in Euclid and Apollonius. Writing hundreds of years later than those mathematicians, Pappus of Alexandria [43, pages 566–7] distinguishes between *theorems* and *problems.* In a theorem (τὸ θεώρημα), something *is looked at* (θεωρεῖται), while in a problem (τὸ πρόβλημα), a construction *is proposed* (προβάλλεται). Euclid's Proposition I.43 is a theorem, but it is used for Proposition I.44, which (like Proposition I.1) is a problem. The problem of I.44 is to construct on a given straight line, and in a given angle, a parallelogram equal to a given triangle. (Euclid's straight lines are our "line segments.")

The verb προβάλλω from which τὸ πρόβλημα is derived has the root meaning of "throw forward." Changing the prefix makes παραβάλλω, "throw alongside," which is just what is proposed in Proposition I.44: to throw a parallelogram alongside a given straight line. An English term (derived from Latin) for this activity is *application.* This is a translation of ἡ παραβολή, which will be Apollonius's term for the *parabola,* because the parabola can be described in terms of applications in the sense just defined, as we shall see.

Euclid's Proposition I.44 is in turn a lemma for Proposition 45, which is another *problem*: given a figure with any number of straight sides, to construct a rectangle (or a parallelogram in any given angle) that is equal to this figure. This problem is the climax of Book I of the *Elements.* Implicitly, by I.43 and 44, the rectangle can have a predetermined base, and then its height can be taken as a measure of the area of the original figure. *Now* we are a step on the way to justifying my students' unexamined conviction that areas are *numbers.*

Euclid's proposition I.45 would seem to corroborate the judgment of Herodotus: that the Greeks learned geometry from the Egyptians, who had to be able to measure land lost in the annual flooding of the Nile [19, II.109]. It is probably better to say that the Greeks learned *surveying* from the Egyptians: this is the root meaning of ἡ γεωμετρία. It appears that Egyptian surveyors defined the area of a four-sided field as the product of the averages of the opposite sides. Thus they overestimated the area, unless the field was a perfect rectangle. In *The Mathematics of Plato's Academy* [16], Fowler reports this, while doubting that the Greeks did "discover geometry from Egyptian land measurement" (pages 232, 281). It would seem to me that Euclid's Proposition I.45 could well be the result of attempts to improve the Egyptian formula for quadrilateral areas; but my interest now is in a passage that Fowler quotes (page 279) together with the passage of Herodotus mentioned above.

The passage is by Proclus in his *Commentary on the First Book of Euclid's Elements* [35, 36, pages 64–5]:

> ... we say, as have most writers of history, that geometry (ἡ γεωμετρία) was first discovered among the Egyptians and originated in the remeasuring of their lands. This was necessary (ἀναγκαία) for them because the Nile overflows and obliterates the boundary lines between their properties. It is not surprising that the discovery of this and the other sciences had its origin in necessity (ἡ χρεία) ... Just as among the Phoenicians the necessities of trade and exchange gave the impetus to the accurate study of number, so also among the Egyptians the invention of geometry came from the cause (ἡ αιτία) mentioned.

The last clause but one would be more literally translated as,

> Among the Phoenicians, through trade and exchange, the accurate study of numbers took a beginning (ἡ ἀρχή).

There is no specific mention of necessity, except that ἀρχη itself can be translated as "cause."

While it may be *inspired* by the physical world, I want to propose that mathematics has no external *cause.* Any necessity in its development is internal. Apparently the Egyptians were *not* required by their physical conditions to find a formula for quadrilateral areas that we would recognize today as strictly correct. Euclid did happen to develop a precise theoretical understanding of areas, as presented in Book I of the *Elements*; but nothing *made* him do it but his own internal drive.

The Devil says to the Angel in Blake's *Marriage of Heaven and Hell* [7, plate 23],

> bray a fool in a mortar with wheat
> yet shall not his folly be beaten out of him.

Nothing external can separate the fool from his folly. This might be contradicted by the experience of George Orwell, as recounted in the essay called "Such, such were the joys . . . " [31]. The essay happens to take its ironic title from the middle stanza of "The Ecchoing Green," one of Blake's *Songs of Innocence* [6]:

> Old John, with white hair,
> Does laugh away care,
> Sitting under the oak,
> Among the old folk.
> "Such, such were the joys
> When we all, girls and boys,
> In our youth time were seen
> On the Ecchoing Green."

At a boarding school at the age of eight, Orwell was not pounded by a pestle, but he was whipped with a riding crop until he stopped wetting his bed. Since he did in fact stop, "perhaps this barbarous remedy does work, though at a heavy price, I have no doubt."

During a lesson, Orwell might be taken out for a beating, right in the middle of construing a Latin sentence; then he would be brought back in, "red-wealed and smarting," to continue.

> It is a mistake to think that such methods do not work. They work very well for their special purpose. Indeed, I doubt whether classical education ever has been or can be successfully carried on without corporal punishment. The boys themselves believed in its efficacy.

Orwell mentions a boy who wished he had been beaten more before an examination that he failed. The student did fail though, and I doubt a whipping would have helped him. Orwell succeeded at examinations; but he was at school as a "scholarship boy" only because he had been thought likely to be successful. Moreover, this success was of doubtful value, at least given its price:

> Over a period of two or three years the scholarship boys were crammed with learning as cynically as a goose is crammed for Christmas. And with what learning! This business of making a gifted boy's career depend on a competitive examination, taken when he is only twelve or thirteen, is an evil thing at best, but there do appear to be preparatory schools which send scholars to Eton, Winchester, etc., without teaching them to see everything in terms of marks. At Crossgates the whole process was frankly a preparation for a sort of confidence trick. Your job was to learn exactly those things that would give the examiner the impression that you knew more than you did know, and as far as possible to avoid burdening your brain with anything else.

I quote all of this because it sounds as if American schools today are becoming like Orwell's "Crossgates," at least with their standardized examinations. There may be no corporal punishment, though the threat of school closure or reduced funding might be comparable to it.

I return to the Devil's assertion that folly cannot be beaten out of you. Neither can wisdom—or true academic success—be beaten *into* you, be it by a stick, or the flooding of the Nile, or frequent examinations for that matter. If the weapons seem efficacious, it is only because their victims had it in themselves to perform. Is there really no better way to encourage this performance? As one of Blake's "Proverbs of Hell" runs [7, plate 7],

> If the fool would persist in his folly he would become wise.

In the academic context, I interpret this as recommending *attention,* the avoidance of distractions: distractions such as certain portable electronic devices are *designed* to be. Another word for what is wanted is—*application.*

After the finding of the area of an arbitrary polygon by triangulation and application, the *dénouement* of Book I of Euclid is Propositions 47 and 48: the Pythagorean Theorem and its converse.

5 The cone

Again, the **cone** of Apollonius is ὁ κῶνος, a "pine-cone." It is a solid figure determined by (1) a **base** (ἡ βάσις), which is a circle, and (2) a **vertex** (ἡ κορυφή "summit"), which is a point that is not in the plane of the base. The surface of the cone contains all of the straight lines drawn from the vertex to the circumference of the base. A **conic surface** consists of such straight lines, not bounded by the base or the vertex, but extended indefinitely in both directions. The Greek for "conic surface" is ἡ κωνικὴ ἐπιφάνεια, the last word meaning originally "appearance" and being the source of the English "epiphany."

The straight line drawn from the vertex of a cone to the center of the base is the **axis** (ὁ ἄξων "axle") of the cone. If the axis is perpendicular to the base, then the cone is **right** (ὀρθός); otherwise it is **scalene** (σκαληνός "uneven"). Apollonius considers both kinds of cones indifferently.

A plane containing the axis intersects the cone in a triangle. Suppose a cone with vertex A has the axial triangle ABC shown in Figure 2. Then the base BC of this triangle is a diameter of the

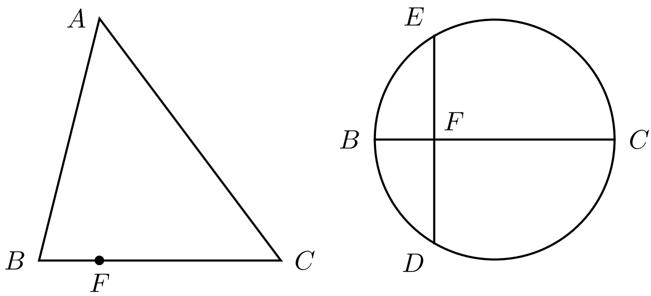

Figure 2   Axial triangle and base of a cone

base of the cone. Let an arbitrary chord DE of the base of the cone cut the base BC of the axial triangle at right angles at a point F, again as in Figure 2. We are going to cut the cone itself with a plane that contains this chord DE, thus obtaining a *conic section.*

Figure 2 consists of a "cross section" and "floor plan" of a cone, as if in an architectural drawing. However, it is important to note that the cross section may not be strictly vertical. I shall not attempt an "axonometric projection," showing the cone itself, with a conic section drawn along its surface. I am reluctant to spend my own time figuring out how to draw such a projection, using PSTricks (as for the present figures) or another program.

I am not *opposed* to making such projections. I should not mind being able to write a code for drawing them. I have a dream that a sculptor can be inspired to make three-dimensional diagrams for the propositions of Apollonius, using heavy gauge wires perhaps. I have made them myself, using cardboard, as in Figure 3. But there may be pedagogic value in having to construct, in one's own mind,

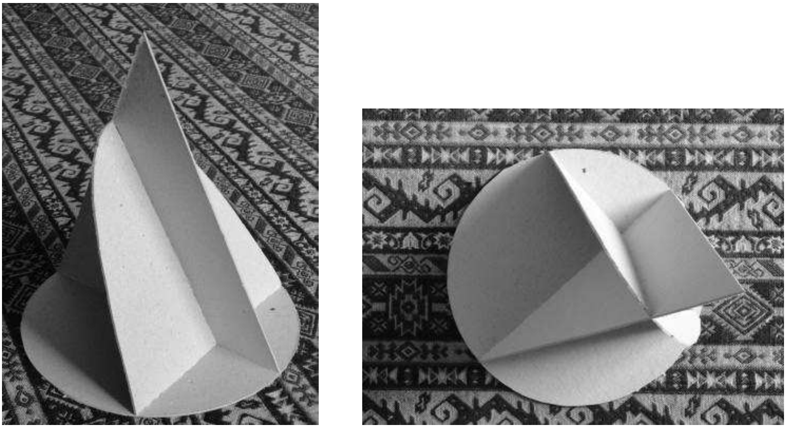

Figure 3 Cardboard model

the vision of a cone and its sections.

One can find wooden models of cones cut by various planes; but I have seen only *right* cones in this way. The beauty of Apollonius is that his cones can be scalene. He is not doing projective geometry though: there is still a distance between any two points, and any two distances have a ratio. Earlier mathematicians knew the ratios that arose from sections of *right* cones; Apollonius understood that the restriction was not needed.

We do not know what Apollonius's own diagrams looked like. This frees the editors of the *Green Lion* English edition [4] to provide the best diagrams according to their own judgment. However, Netz recognizes the possibility and value of figuring out what the original diagrams of the Greek mathematicians looked like. They do not necessarily look like what we might draw. Netz himself has initiated the recovery of the diagrams of Archimedes [5]. Given that Heiberg's edition of Apollonius [2] does *not* give diagrams in the architectural style of Figure 2, we can probably assume that Apollonius did not draw them this way. Nonetheless, this manner of drawing may clarify some points.

We drew the chord DE of the base of a cone. Apollonius uses no word for a chord as such, even though he proves in his Proposition I.10 that the straight line joining any two points of a conic section *is* a chord, in the sense that it falls within the section. The English words "cord" and "chord" are derived from the Greek ἡ χορδή [20]; but this word means gut, hence *anything* made with gut, be it a lyre-string or a sausage [25].

In the axial triangle in our figure, let a straight line FG be drawn from the base to the side AC. Some possibilities are shown in Figure 4. This straight line FG may, but need not, be parallel to the side

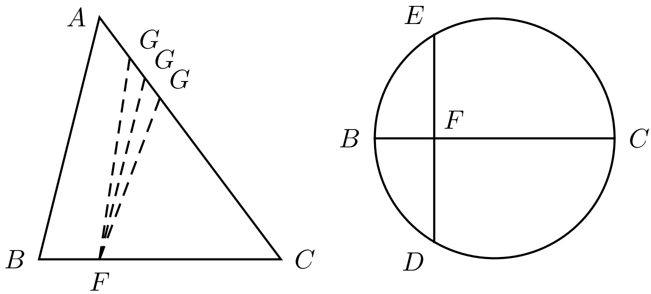

Figure 4 Axial triangle—now cut—and base of a cone

BA. It is not at right angles to the chord DE of the base of the cone, unless the plane of the axial triangle is at right angles to the plane of the base of the cone. In any case, the two straight lines FG and DE, meeting at F, are not in a straight line with one another, and so they determine a plane. This plane cuts the surface of the cone in such a curve DGE as is shown in Figure 5. So now we have another cross section of our cone, but again, perhaps not a stictly vertical

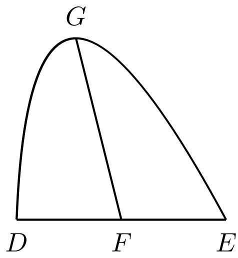

Figure 5 A conic section

one. Apollonius refers to a curve like DGE as a section (ἡ τομή) in the *surface* of the cone, in Proposition I.7; later, in I.10, it is just a section of a cone. All of the chords of this section that are parallel to DE are bisected by the straight line GF. Therefore Apollonius calls this straight line a **diameter** (ἡ διάμετρος [γραμμή]) of the section. The associated verb is διαμετρέ-ω "measure through"; this is the verb used in Homer's *Iliad* [21, III.315] for what is done to prepare for the single combat of Paris and Menelaus (the reference is in the LSJ lexicon):

> But Hector, Priam's son, and goodly Odysseus first *measured out* (διεμέτρεον) a space, and thereafter took the lots and shook them in the bronze-wrought helmet, to know which of the twain should first let fly his spear of bronze.

The parallel chords bisected by the diameter of a conic section are said to be drawn to the diameter **in an orderly way.** Our prepositional phrase stands for the Greek adverb τεταγμένως, from the verb τάσσω, which has meanings like "to draw up in order of battle" or "marshall"—what Cyrus the Great did in posting his troops outside the walls of Babylon, where the Euphrates entered and exited the city, so that when he had diverted enough of the river's waters, the troops might enter the city and take it [19, I.191].

A Greek noun derived from τάσσω is τάξις, which is found in English technical terms like "taxonomy" and "syntax" [26]. The Latin adverb corresponding to the Greek τεταγμένως is *ordinate* (four syllables), from the verb *ordino.* From the Greek expression for "the straight line drawn in an orderly way," Apollonius will elide the middle part, leaving "the in-an-orderly-way." This term will refer to *half* of a chord bisected by a diameter. Similar elision in the Latin leaves us with the word **ordinate** for this half-chord [28]. In the *Geometry,* Descartes refers to ordinates as *[lignes] qui s'appliquent par ordre [au] diametre* [10, page 328].

Heath [1, page clxi] translates τεταγμένως as "ordinate-wise"; Taliaferro [4, page 3], as "ordinatewise." But this usage strikes me as anachronistic. The term "ordinatewise" seems to mean "in the manner of an ordinate"; but ordinates are just what we are trying to define when we translate τεταγμένως.

I do not know whether the classical *orders* of architecture—the Doric, Ionic, and Corinthian orders—are so called because of the mathematical use of the word "ordinate." But we may compare the ordinates of a conic section as in Figure 6 with the row of columns

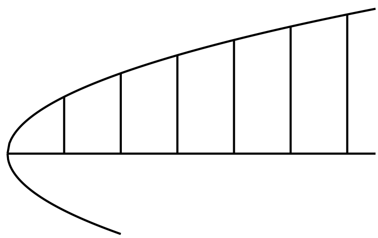

Figure 6 Ordinates of a conic section

of a Greek temple, as in Figure 7.

Back in Figure 5, the point G at which the diameter GF cuts the conic section DGE is called a **vertex** (κορυφής as before). The seg-

Figure 7 Columns in the Ionic order, at Priene, Söke, Aydın, Turkey

ment of the diameter between the vertex and an ordinate has come to be called in English an **abscissa;** but this just the Latin translation of Apollonius's Greek for "being cut off": ἀπολαμβανομένη. This participle is used in Proposition I.11 [2, page 38], and its general usage for what we translate as *abscissa* is confirmed in the LSJ lexicon; however, the root sense of the verb is actually not of cutting, but of taking.

Apollonius will show that every point of a conic section is the vertex for some unique diameter. If the ordinates corresponding to a particular diameter are at right angles to it, then the diameter will be an **axis** of the section. Meanwhile, in describing the relation between the ordinates and the abscissas of conic section, there are three cases to consider.

6 The parabola

Suppose the diameter of a conic section is parallel to a side of the corresponding axial triangle. For example, suppose in Figure 8 that

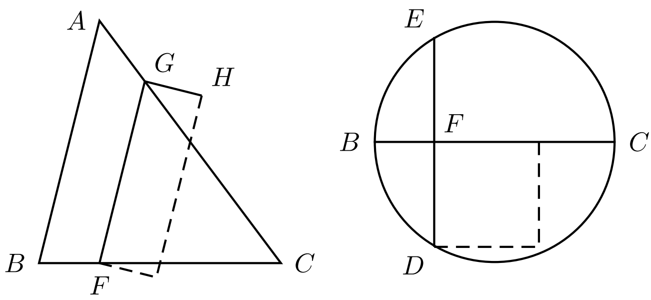

Figure 8 The parabola in the cone

FG is parallel to BA. The square on the ordinate DF is equal to the rectangle whose sides are BF and FC (by Euclid's Proposition III.35). More briefly,

$$DF^2 = BF \cdot FC.$$

But BF is independent of the choice of the point D on the conic section. That is, for any such choice (aside from the vertex of the section), a plane containing the chosen point and parallel to the base of the cone cuts the cone in another circle, and the axial triangle cuts this circle along a diameter, and the plane of the section cuts this diameter at right angles into two pieces, one of which is equal to BF.

The square on DF thus varies as FC, which varies as FG. That is, the square on an ordinate varies as the abscissa (Apollonius I.20). By means of Euclid's Proposition I.43, used as in I.44 as discussed earlier, we obtain a straight line GH such that

$$DF^2 = FG \cdot GH;$$

and GH is independent of the choice of D.

This straight line GH can be conceived as being drawn at right angles to the plane of the conic section DGE. Therefore Apollonius calls GH the **upright side** (ὀρθία [πλευρά]), and Descartes accordingly calls it *le costé droit* [10, page 329]. Apollonius calls the conic section itself ἡ παραβολή; we transliterate this as **parabola.** The Greek word is also the origin of the English "parable," but can have various related meanings, like "juxtaposition, comparison, conjunction, application." The word is self-descriptive, being itself a juxtaposition of the preposition παρά "along, beside" and the noun ἡ βολή "throw." Alternatively, παραβολή is a noun derived from the verb παραβάλλω, which as suggested earlier is παρά plus βάλλω "throw." In the parabola of Apollonius, the rectangle FH bounded by the abscissa FG and the upright side GH is the result of *applying,* to the upright side, the square whose side is the ordinate DF.

7 The *latus rectum*

The Latin term for the upright side is *latus rectum.* This term is also used in English. In the *Oxford English Dictionary* [28], the earliest quotation illustrating the use of the term is from a mathematical dictionary published in 1702. Evidently the quotation refers to Apollonius and gives his meaning:

> App. Conic Sections 11 In a Parabola the Rectangle of the Diameter, and Latus Rectum, is equal to the rectangle of the Segments of the double Ordinate.

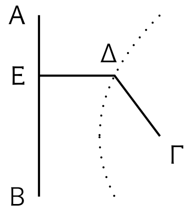

Figure 9 Directrix and focus (dotted curve added)

Apparently "diameter" was used here, rather than "abscissa." The "segments of the double ordinate" are presumably the two halves of a chord, so that each of them is what we are calling an ordinate, and the rectangle contained by them is equal to the square on one of them.

The possibility of defining the conic sections in terms of a *directrix* and *focus* is shown by Pappus [32, VII.312–8, pages 1004–15] and was presumably known to Apollonius. Pappus does not use such technical terms though; there is just a straight line and a point, as in the following, a slight modification of Thomas's translation [42, pages 492–503]:

> If AB be a straight line given in position [Figure 9], and the point Γ be given in the same plane, and ΔΓ be drawn, and ΔE be drawn perpendicular [to AB], and if the ratio of ΓΔ to ΔE be given, then the point Δ will lie on a conic section.

As Heath explains [1, pages xxxvi–xl], Pappus proves this theorem because Euclid did not supply a proof in his treatise on *surface loci.* (This treatise itself is lost to us.) Euclid must have omitted the proof because it was already well known; and Euclid predates Apollonius. Kline summarizes all of this by saying that the focus-directrix property "was known to Euclid and is stated and proved by Pappus" [24, page 96]. Later (on his page 128), Kline gives a precise reference to Pappus: it is Proposition 238, in Hultsch's numbering, of Book VII.

Actually this proposition is a recapitulation, which is incomplete in the extant manuscripts; one must read a few pages earlier in Pappus for more details, as in the selection in Thomas's anthology. In any case, Kline says, "As noted in the preceding chapter, Euclid probably knew" the proposition. According to Boyer however, "It appears that Apollonius knew of the focal properties for central conics, but it is possible that the focus-directrix property for the parabola was not known before Pappus" [8, §XI.12, page 211].

I modified Thomas's translation of Pappus by putting "the ratio of ΓΔ to ΔE" where Thomas has "the ratio ΓΔ : ΔE." Pappus uses no special notation for a ratio as such, but refers merely to λόγος. . . τῆς ΓΔ πρὸς ΔE. The recognition of ratios as individual mathematical objects (namely positive real numbers) distinguishes modern from ancient mathematics, although the beginnings of this recognition can be seen in Pappus, as Descartes observed.

A modern textbook may define the parabola in terms of a directrix and focus, explicitly so called. An example is Nelson, Folley, and Borgman, *Analytic Geometry* [29]. I have this book because my mother used it in college. I perused it at the age of 12 when I wanted to understand how curves could be encoded in equations. Dissatisfaction with such textbooks leads me back to the Ancients. According to Nelson *& al.*,

> The chord of the parabola which contains the focus and is perpendicular to the axis is called the *latus rectum.* Its length is of value in estimating the amount of "spread" of the parabola.

The first sentence here defines the *latus rectum* as a certain straight line that is indeed equal to Apollonius's upright side. The second sentence correctly describes the significance of the *latus rectum.* However, the juxtaposition of the two sentences may mislead somebody, like me, who knows just a little Latin. I suppose Latin was more commonly studied when Nelson *& al.* were writing, and I have the idea that they put the word "spread" in quotation marks to suggest that it is a translation of the Latin *latus.* The Latin adjective *latus, -a, -um* does mean "broad, wide; spacious, extensive" [27]: it is the root of the English noun "latitude." An extensive *latus rectum* does mean a broad or spreading parabola. However, the Latin adjective *latus* is unrelated to the noun *latus, -eris* "side; flank," which is found in English in the adjective "lateral"; and the noun *latus* is what is used in the phrase *latus rectum.*

It is possible that Nelson *& al.* put the word "spread" in quotation marks simply because, writing in 1949, they felt its use as a noun to be colloquial. It is not a noun in the *OED*: this has only two entries for "spread," as a verb and as a participial adjective. The section "Speech–Spring" of the dictionary was first published in September, 1914. The 1976 sixth edition of the *Concise Oxford Dictionary* [41] does list "spread" as a noun, without the tag of being colloquial or being otherwise restricted. Somewhere in between then, the nominal use of "spread" must have occurred only colloquially, and this may have been when Nelson *& al.* were writing.

Again, it is also possible that they mistakenly connected *latus rectum* with "latitude." Finally, it is possible that they knowingly and deliberately made the connection, being happy to suggest a false etymology to their students if it would help them. The students would be studying calculus, after all, not classics. In that case though, the students would still be better served with the correct mathematical etymology of *latus rectum.*

Perhaps it was felt that introducing a third dimension to the students' thoughts at this stage would be confusing. The last four of the fifteen chapters of Nelson *& al.* are on solid analytic geometry; but on the blank page before they begin, I see that my mother has written "THE END!" in her copy. Analytic geometry was the last mathematics course that she took at her liberal arts college, and apparently her course covered only plane geometry. Perhaps this was because, indeed, it was thought too hard to teach students to connect an equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

with a certain surface called a hyperbolic paraboloid. In that case though, why not teach the students to visualize cones, even with the aid of solid models, and to understand what a *latus rectum* really is, rather than to manipulate equations?

There is even more that can be said about the Latin words spelled *latus*; I give it here for completeness. In the noun phrase *latus rectum,* the adjective *rectus, -a, -um* "straight, upright" is given the neuter form, because the noun *latus* is (despite appearances) neuter. The plural of *latus rectum* is *latera recta.* The neuter plural of the adjective *latus* would be *lata.* The dictionary writes the adjective as *lātus,* with a long A; but the A in the noun is unmarked and therefore short. As far as I can tell, the adjective is to be distinguished from another Latin adjective with the same spelling (and the same long A), but with the meaning of "carried, borne." This adjective is used for the past participle of the verb *fero, ferre, tulī, lātum.* The past participle appears in English in words like "translate," while *fer-* appears in "transfer." According to the *American Heritage Dictionary* [26],

***lātus* "broad"** comes from the Indo-European root *stel-* and has the English derivatives "latitude" and "dilate";

***lātus* "carried"** comes from an Indo-European root *tel-* and is found in English words like "translate" and "relate," but also "dilatory."

Thus "dilatory" is not to be considered as a derivative of "dilate." A French etymological dictionary [11] implicitly confirms this under the adjacent entries *dilater* and *dilatoire.* The older English etymological dictionary of Skeat [38] does give "dilatory" as a derivative of "dilate." However, under "latitude," Skeat traces *lātus* "broad" to the Old Latin *stlātus,* while under "tolerate" he traces *lātum* "borne" to *tlātum.* In his introduction, Skeat says he has collated his dictionary "with the *New English Dictionary* [as the *Oxford English Dictionary* was originally called] from A to H (excepting a small portion of G)." So Skeat should know what the *OED* has to say about "dilate." In fact the *OED* distinguishes *two* English verbs "dilate," one for each of the Latin adjectives *lātus.* But concerning the meanings of these verbs, the dictionary notes,

> The sense 'prolong' comes so near 'enlarge', 'expand', or 'set forth at length'...that the two verbs were probably not thought of as distinct words.

Thus it is possible for an etymological purist to get carried away, making distinctions without a difference. Still, the adjective or adjectives *lātus* have nothing to do with the noun *latus.*

So much for etymology then. Denoting abscissa by x, and ordinate by y, and *latus rectum* by ℓ, we have for the parabola the modern equation

$$y^2 = \ell x. \tag{$*$}$$

The letters here can be considered as numbers in the modern sense, or just as line segments, or congruence-classes of segments.

8 The hyperbola

The second possibility for a conic section is that the diameter meets the other side of the axial triangle when this side is extended beyond the vertex of the cone. In Figure 10, the diameter FG, crossing one side of the axial triangle ABC at G, crosses the other side, extended, at K. Again $DF^2 = BF \cdot FC$; but the latter product now varies as $KF \cdot FG$. The upright side GH can now be defined so that

$$BF \cdot FC : KF \cdot FG : : GH : GK.$$

We draw KH and extend to L so that FL is parallel to GH, and we extend GH to M so that LM is parallel to FG. Then

$$FL : KF : : FL \cdot FG : KF \cdot FG,$$

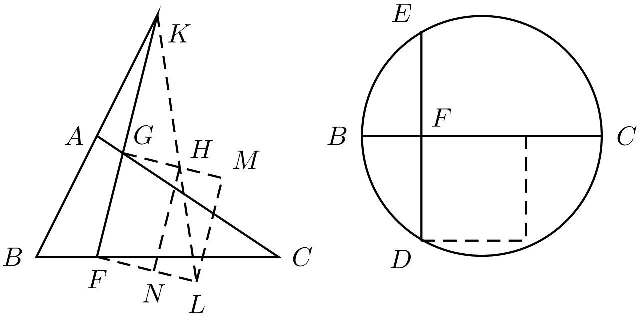

Figure 10 The hyperbola in the cone

but also

$$FL : KF : : GH : GK.$$

Eliminating common ratios from the three proportions yields

$$FL \cdot FG : KF \cdot FG : : BF \cdot FC : KF \cdot FG,$$

and so $FL \cdot FG = BF \cdot FC$. Thus

$$DF^2 = FG \cdot FL.$$

Apollonius calls the conic section here an **hyperbola** (ἡ ὑπερβολή), that is, an *excess,* an *overshooting,* a *throw* (βολή) *beyond* (ὑπέρ), because the square on the ordinate DF is equal to a rectangle FM whose one side is the abscissa FG, and whose other side GM is applied along the upright side. But the applied rectangle FM actually *exceeds* (ὑπερβάλλω) the rectangle contained by the abscissa FG and the upright side GH itself. The *excess* rectangle—MN in the figure—is similar to the rectangle contained by the upright side GH and GK.

Apollonius calls GK the **transverse side** (ἡ πλαγία πλευρά) of the hyperbola. Denoting it by a, and the other segments as before, we have the modern equation

$$y^2 = \ell x + \frac{\ell}{a}x^2. \tag{$\dagger$}$$

9 The ellipse

The last possibility is that the diameter meets the other side of the axial triangle when this side is extended below the base. All of the computations will be as for the hyperbola, except that now, if it is considered as a *directed* segment, the transverse side is negative, and so the modern equation is

$$y^2 = \ell x - \frac{\ell}{a}x^2. \tag{$\ddagger$}$$

In this case Apollonius calls the conic section an **ellipse** (ἡ ἔλλειψις), that is, a *falling short,* because again the square on the ordinate is equal to a rectangle whose one side is the abscissa, and whose other side is applied along the upright side: but this rectangle now *falls short* (ἐλλείπω) of the rectangle contained by the abscissa and the upright side by another rectangle. Again this last rectangle is similar to the rectangle contained by the upright and transverse sides.

Apollonius concludes Book I concludes by showing that every curve given by an equation of one of the forms ($*$), ($\dagger$), and ($\ddagger$) is indeed a conic section. This may be true for us today, by definition; but Apollonius finds a cone of which the given curve is indeed a section.

10 Descartes

We have seen that the terms "abscissa" and "ordinate" are ultimately translations of Greek words that describe certain line segments determined by points on conic sections. For Apollonius, an ordinate and its corresponding abscissa are not required to be at right angles to one another.

Descartes generalizes the use of the terms slightly. In one example [10, page 339], he considers a curve derived from a given conic section in such a way that, if a point of the conic section is given by an equation of the form

$$y^2 = \dots x \dots,$$

then a point on the new curve is given by

$$y^2 = \dots x' \dots,$$

where xx' is constant. But Descartes just describes the new curve in words:

> *toutes les lignes droites appliquées par ordre a son diametre estant esgales a celles d'une section conique, les segmens de ce diametre, qui sont entre le sommet & ces lignes, ont mesme proportion a une certaine ligne donnée, que cete ligne donnée a aux segmens du diametre de la section conique, auquels les pareilles lignes sont appliquées par ordre.*
>
> All of the straight lines drawn in an orderly way to its diameter being equal to those of a conic section, the segments of this diameter that are between the vertex and these lines have the same ratio to a given line that this given line has to the segments of the diameter of the conic section to which the parallel lines are drawn in an orderly way.

In particular, the new curve has **ordinates,** namely *les lignes droites appliqués par ordre a son diametre.* These ordinates have corresponding **abscissas,** which are *les segmens de ce diametre, qui sont entre le sommet & ces lignes.* There is still no notion that an arbitrary point might have two coordinates, called abscissa and ordinate respectively. A point determines an ordinate and abscissa only insofar as the point belongs to a given curve with a designated diameter.

The *Wikipedia* (`en.wikipedia.org`) articles “ordinate” and “abscissa” do not explain the origins of these terms (at least as of October 4, 2014). This is unfortunately true of many *Wikipedia* articles on mathematics. Of course, anybody who cares may work to change this (as I did, for example, in adding the section “Origins” to the article “Pappus’s hexagon theorem”).

The *Wikipedia* “ordinate” article does have a reference to the website *Earliest Known Uses of Some of the Words of Mathematics* (maintained by Jeff Miller, `jeff560.tripod.com/mathword.html`); but this site currently provides only a *modern* history of “ordinate” and “abscissa.” It is said that Descartes did not use the latter term, but did use the former; however, the quotation above suggests that he did not even use the term “ordinate” as such, but used only the longer phrase, derived from Apollonius, that was presumably the precursor of this term.

It appears that the use of “ordinate” and “abscissa” as technical terms is due to Leibniz. The articles on Miller’s website refer to Struik in *A Source Book in Mathematics, 1200–1800,* where a footnote [40, page 272, note 1] to an article of Leibniz reads,

> Note the Latin term *abscissa.* This term, which was not new in Leibniz’s day, was made by him into a standard term, as were so many other technical terms. In the article “De linea ex lineis numero infinitis ordinatim ductis inter se concurrentibus formata . . . ,” *Acta Eruditorum 11* (1692), 168–171 (Leibniz, *Mathematische Schriften,* Abth. 2, Band I (1858), 266–269), in which Leibniz discusses evolutes, he presents a collection of technical terms. Here we find *ordinata, evolutio, differentiare, parameter, differentiabilis, functio,* and *ordinata* and *abscissa* together designated as *coordinatae.* Here he also points out that ordinates may be given not only along straight but also along curved lines. The term *ordinate* is derived from *rectae ordinatim applicatae,* “straight lines designated in order,” such as parallel lines. The term *functio* appears in the sentence: “the tangent and some other functions depending on it, such as perpendiculars from the axis conducted to the tangent.”

Presumably Leibniz (1646–1716) knew Books I–IV of Apollonius's *Conics* from Commandino's 1566 Latin translation of them. Arabic manuscripts of the later books were brought to Europe, starting later in the 16th century; but the first proper Latin translation did not appear until Halley's 1710 edition [3, xxi–xxv]. In any case, all we have needed for the present article is Book I.

It was noted in the beginning that the original Greek of Books V–VII of Apollonius has been lost, and Book VIII exists in no language at all. Terminology based on Apollonius survives, thanks apparently to Leibniz; but a proper understanding of the terminology has not generally survived. This is unfortunate, but can change.

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