FIELDS WITH SEVERAL COMMUTING DERIVATIONS

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Abstract. The existentially closed models of the theory of fields (of arbitrary characteristic) with an assigned finite number of commuting derivations can be given a first-order geometric characterization in several ways. In each case, the existentially closed models are those models that contain points of certain differential varieties, which are determined by certain ordinary varieties.

How can we tell whether a given system of partial differential equations has a solution? An answer given in this paper is that, if we differentiate the equations enough times, and no contradiction arises, then it never will, and the system is soluble. Here, the meaning of 'enough times' can be expressed *uniformly;* this is one way of showing that the theory, m-DF, of fields with a finite number m of commuting derivations has a model-companion, m-DCF. In fact, this theorem is worked out here (as Corollary 4.6, of Theorem 4.5), not in terms of polynomials, but in terms of the varieties that they define, and the function-fields of these: in a word, the treatment is *geometric.*

The model-companion of m-DF$_0$ (in characteristic 0) has been axiomatized before, explicitly in terms of differential polynomials: see § 3. The existence of a model-companion of m-DF (with no specified characteristic) appears to be a new result when $m > 1$ (despite a remark by Saharon Shelah [23, p. 315]: 'I am quite sure that for characteristic p as well, [making m greater than 1] does not make any essential difference').

The theory of model-companions and model-completions was worked out decades ago; perhaps for that very reason, it may be worthwhile to review the theory here, as I do in § 1. I try to give the original references, when I have been able to consult them. In § 2, I review the various known characterizations of existentially closed fields with single derivations. In fact, little of this work is of use in the passage to several derivations; but this near-irrelevance is itself interesting. In § 3, I analyze the error of my earlier attempt, in [12], to axiomatize m-DCF$_0$ in terms of differential forms. Something of value from this earlier work does remain: when we do have m-DCF$_0$, or more generally m-DCF, then we can obtain from it a model-companion of the theory of fields with m derivations whose linear span over the field is closed under the Lie bracket. In § 4, I obtain m-DCF itself.

sect:mt

§1. Model-theoretic background. Let $\mathfrak{M}$ be an arbitrary (first-order) structure; its theory is Th($\mathfrak{M}$). Let T be an arbitrary consistent (first-order) theory; its models compose the class Mod(T). Every class $\boldsymbol{K}$ of structures in some signature has a theory, Th($\boldsymbol{K}$). Then $\boldsymbol{K} \subseteq$ Mod(Th($\boldsymbol{K}$)); in case of equality, $\boldsymbol{K}$ is **elementary.** Always, Th(Mod(T)) = T.

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The structure $\mathfrak{M}$ has the universe M. The structure denoted by $\mathfrak{M}_M$ is the expansion of $\mathfrak{M}$ that has a name for every element of M. Then $\mathfrak{M}$ embeds in $\mathfrak{N}$ if and only if $\mathfrak{M}_M$ embeds in an expansion of $\mathfrak{N}$. However, although the class of structures in which $\mathfrak{M}$ embeds need not be elementary, the class of structures in which $\mathfrak{M}_M$ embeds *is* elementary. The theory of the latter class is the **diagram** of $\mathfrak{M}$, or $\operatorname{diag}(\mathfrak{M})$: it is axiomatized by the quantifier-free sentences in $\operatorname{Th}(\mathfrak{M}_M)$ [17, Thm 2.1.3, p. 24]. The class of structures in which $\mathfrak{M}_M$ embeds *elementarily* is also elementary, and its theory is just $\operatorname{Th}(\mathfrak{M}_M)$. The class of substructures of models of T is elementary, and its theory is denoted by $T_\forall$: this is axiomatized by the universal sentences of T [17, Thm 3.3.2, p. 71].

By a **system over** $\mathfrak{M}$, I mean a finite conjunction of atomic and negated atomic formulas in the signature of $\mathfrak{M}_M$; likewise, a system **over** T is in the signature of T. A structure $\mathfrak{M}$ **solves** a system $\varphi(\boldsymbol{x})$ if $\mathfrak{M} \models \exists \boldsymbol{x}\ \varphi(\boldsymbol{x})$. Note well here that $\boldsymbol{x}$, in boldface, is a *tuple* of variables, perhaps $(x^0, \dots, x^{n-1})$. By an **extension** of a model of T, I mean another model of T of which the first is a substructure. Two systems over a model $\mathfrak{M}$ of T are **equivalent** if they are soluble in the same extensions.

An **existentially closed** model of T is a model of T that solves every system over itself that is soluble in some extension. So a model $\mathfrak{M}$ of T is existentially closed if and only if $T \cup \operatorname{diag}(\mathfrak{M}) \vdash \operatorname{Th}(\mathfrak{M}_M)_\forall$, that is, every extension of $\mathfrak{M}$ is a substructure of an elementary extension ([5, § 7] or [24, § 2]).

A theory is **model-complete** if its every model is existentially closed. An equivalent formulation explains the name: T is model-complete if and only if $T \cup \operatorname{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \models T$ [18, Ch. 2].

Suppose every model of T has an existentially closed extension. Such is the case when T is **inductive,** that is, $\operatorname{Mod}(T)$ is closed under unions of chains [5, Thm 7.12]: equivalently, $T = T_{\forall\exists}$ [8, 3]. Suppose further that we have a uniform first-order way to tell when systems over models of T are soluble in extensions: more precisely, suppose there is a function

$$\varphi(\boldsymbol{x}, \boldsymbol{y}) \longmapsto \widehat{\varphi}(\boldsymbol{x}, \boldsymbol{y}), \tag{1}$$

where $\varphi(\boldsymbol{x}, \boldsymbol{y})$ ranges over the systems over T (with variables analyzed as shown), such that, for every model $\mathfrak{M}$ of T and every tuple $\boldsymbol{a}$ of parameters from M, the system $\varphi(\boldsymbol{x}, \boldsymbol{a})$ is soluble in some extension of $\mathfrak{M}$ just in case $\widehat{\varphi}(\boldsymbol{x}, \boldsymbol{a})$ is soluble in $\mathfrak{M}$. Then the existentially closed models of T compose an elementary class, whose theory T^* is axiomatized by T together with the sentences

$$\forall \boldsymbol{y}\ (\exists \boldsymbol{x}\ \widehat{\varphi}(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow \exists \boldsymbol{x}\ \varphi(\boldsymbol{x}, \boldsymbol{y})). \tag{2}$$

Immediately, T^* is model-complete, so $T^* \cup \operatorname{diag}(\mathfrak{M})$ is complete when $\mathfrak{M} \models T^*$. What is more, $T^* \cup \operatorname{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \models T$ [17, Thm 5.5.1].

In general, T^* is a **model-completion** of T if $T^*{}_\forall \subseteq T \subseteq T^*$ and $T^* \cup \operatorname{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \models T$. Model-completions are unique [16, (2.8)]. We have sketched the proof of part of the following (the rest is [16, (3.5)]):

lem:rob-crit

LEMMA 1.1 (Robinson's Criterion). *Let T be inductive. Then T has a model-completion if and only if a function $\varphi(\boldsymbol{x}, \boldsymbol{y}) \mapsto \widehat{\varphi}(\boldsymbol{x}, \boldsymbol{y})$ exists as in* (1)*. In this case, the model-completion is axiomatized* modulo T *by the sentences in* (2)*.*

If $T_\forall = T^*{}_\forall$ and T^* is model-complete, then T^* is a **model-companion** of T ([1, § 5]; *cf.* [5, § 2]). Model-completions are model-companions, and model-companions are unique [1, Thm 5.3]. If T has a model-companion, then its models are just the existentially closed models of T [5, Prop. 7.10]. Conversely, if T is inductive, and the class of existentially closed models of T is elementary, then the theory of this class is the model-companion of T [5, Cor. 7.13].

sect:one

§2. Fields with one derivation. Let DF be the theory of fields with a derivation D, and let DPF be the theory of models of DF that, for each prime ℓ, satisfy also

$$\forall x\ \exists y\ (\underbrace{1+\dots+1}_{\ell}=0\ \&\ Dx=0 \Rightarrow \underbrace{y\cdots y}_{\ell}=x).$$

So models of DPF are **differentially perfect.** A subscript on the name of one of these theories will indicate a required characteristic for the field. In particular, we have DPF_0, which is the same as DF_0.

Abraham Seidenberg [22] shows the existence of the function in Lemma 1.1 in case T is DPF_p, where p is prime or 0. Consequently:

THEOREM 2.1 (Robinson). DF_0 *has a model-completion, called* DCF_0.

THEOREM 2.2 (Wood [28]). *If p is prime, then* DF_p *has a model-companion,* DCF_p, *which is the model-completion of* DPF_p.

The existence of a model-companion or -completion of a theory does not necessarily tell us much about the existentially closed models of the theory. Since it involves *all* systems over a given theory, Robinson's criterion yields the crudest possible axiomatization for a model-completion. In the case of DCF_p, there are two ways of refining the axiomatization—refining in the sense of finding weaker conditions on models of DF_p that are still sufficient for being existentially closed. It suffices to consider either systems in only one variable or systems involving only first derivatives. In the generalization to several derivations, the former refinement seems to be of little use; the latter refinement is of use indirectly, through its introduction of geometric ideas.

subsect:one-var

2.1. Single variables. Though the theory ACF of algebraically closed fields is the model-completion of the theory of fields, its axioms (*modulo* the latter theory) can involve only systems in one variable (indeed, single equations in one variable). A generalization of this observation is the following, which can be extracted from the proof of [20, Thm 17.2, pp. 89–91] (see also [2]):

lem:blum

LEMMA 2.3 (Blum's Criterion). *Say* $T^*{}_\forall \subseteq T \subseteq T^*$.

(i) *The theory T^* is the model-completion of T if and only if the commutative diagram*

$$\begin{tikzcd} \mathfrak{M} & \\ \mathfrak{A} \arrow[u] \arrow[r] & \mathfrak{B} \arrow[ul, dotted] \end{tikzcd}$$

of structures and embeddings can be completed as indicated when $\mathfrak{A}$ and $\mathfrak{B}$ are models of T and $\mathfrak{M}$ is a $|B|^+$-saturated model of T^*.

item:blum2

(ii) *If $T = T_\forall$, it is enough to assume that $\mathfrak{B}$ is generated over $\mathfrak{A}$ by a single element.*

This allows a refinement of Lemma 1.1 in a special case:

LEMMA 2.4. *Suppose $T = T_\forall$. Then Lemma 1.1 still holds when $\varphi(\boldsymbol{x}, \boldsymbol{y})$ is replaced with $\varphi(x, \boldsymbol{y})$ (where x is a single variable).*

From Lemma 2.3, Lenore Blum obtains Theorem 2.5 below in characteristic 0, in which case the first two numbered conditions amount to $K \models$ ACF ([20, pp. 298 ff.] or [2]). If $p > 0$, then DPF_p is not universal, so part (ii) of Blum's criterion does *not* apply; Carol Wood instead uses a primitive-element theorem of Seidenberg [21] to obtain new axioms for DCF_p [29]. These can be combined with Blum's axioms for DCF_0 to yield the following. (Here SCF is the theory of separably closed fields.)

thm:dcf-1

Theorem 2.5 (Blum, Wood). *A model (K, D) of* DF *is existentially closed if and only if*

(i) $K \models$ SCF*;*
(ii) $(K, D) \models$ DPF*;*
(iii) $(K, D) \models \exists x\ (f(x, Dx, \dots, D^{n+1}x) = 0\ \&\ g(x, Dx, \dots, D^n x) \neq 0)$ *whenever f and g are ordinary polynomials over K in tuples $(x^0, \dots, x^{n+1})$ and $(x^0, \dots, x^n)$ of variables respectively such that $g \neq 0$ and $\partial f/\partial x^{n+1} \neq 0$.*

Hence DF *has a model-companion,* DCF.

There is a similar characterization of the existentially closed *ordered* differential fields [25].

sect:first-der

2.2. First derivatives. Alternative simplified axioms for DCF are parallel to those found for the model-companion ACFA of the theory of fields with an automorphism [9, 4]. Suppose $(K, D) \models$ DPF and $K \models$ SCF. Given a system over (K, D), we can ensure that D is applied only to variables or derivatives of variables; then we can replace each derivative with a new variable, obtaining a system

$$\bigwedge_f f = 0\ \&\ g \neq 0\ \&\ D\boldsymbol{x} = \boldsymbol{y}, \tag{3}$$

where $f, g \in K[\boldsymbol{x}, \boldsymbol{y}]$. We can write also $\bigwedge_f f(\boldsymbol{x}, D\boldsymbol{x}) = 0\ \&\ g(\boldsymbol{x}, D\boldsymbol{x}) \neq 0$. Suppose (3) has the solution $(\boldsymbol{a}, \boldsymbol{b})$. Then $K(\boldsymbol{a}, \boldsymbol{b})/K$ is separable [13, Lem. 1.5, p. 1328]. Let V and W be the varieties over K with generic points $\boldsymbol{a}$ and $(\boldsymbol{a}, \boldsymbol{b})$ respectively, let $T_D(V)$ be the twisted tangent bundle of V, and let U be the open subset of W defined by the inequation $g \neq 0$. In characteristic 0, the model (K, D) of DF is existentially closed if and only if, in every such geometric situation, U contains a K-rational point $(\boldsymbol{c}, D\boldsymbol{c})$; this yields the so-called geometric axioms for DCF_0 found with Anand Pillay [14]. In positive characteristic, it is still true that, if $(\boldsymbol{a}, \boldsymbol{b})$ is a generic point of V, then D extends to $K(\boldsymbol{a})$ so that $D\boldsymbol{a} = \boldsymbol{b}$. However, an additional condition is needed to ensure that D extends to all of $K(\boldsymbol{a}, \boldsymbol{b})$; it is enough to require that the projection of $T_D(W)$ onto $T_D(V)$ contain a generic point of W; this yields Piotr Kowalski's geometric axioms for DCF_p [7].

By the usual trick of replacing $g \neq 0$ with $z \cdot g = 1$, we may assume that there is no inequation in (3). In an alternative geometric approach to DCF, we can then consider (3) as a special case of

$$\bigwedge_f f = 0\ \&\ \bigwedge_{i<k} Dx^i = g^i, \tag{4}$$

where $f \in K[x^0, \dots, x^{n-1}]$ and $g^i \in K(x^0, \dots, x^{n-1})$. Suppose this has solution $\boldsymbol{a}$, which is a generic point of a variety V. It is enough to assume that $(a^0, \dots, a^{k-1})$ is a separating transcendence-basis of $K(\boldsymbol{a})/K$. Then we have a dominant, separable rational map $\boldsymbol{x} \mapsto (x^0, \dots, x^{k-1})$ or φ from V onto $\mathbb{A}^k$, and another rational map $\boldsymbol{x} \mapsto (g^0(\boldsymbol{x}), \dots, g^{k-1}(\boldsymbol{x}))$

or ψ from V to $\mathbb{A}^k$. So (K, D) is existentially closed if and only if V always has a K-rational point P such that $D(\varphi(P)) = \psi(P)$ [13, Thm 1.6, p. 1328].

sect:several

§3. Fields with several derivations. Let m-DF be the theory of fields with m commuting derivations. Tracey McGrail [11] axiomatizes the model-completion, m-DCF$_0$, of m-DF$_0$. Alternative axiomatizations arise as special cases in work of Yoav Yaffe [30] and Marcus Tressl [27]. There is a common theme: A differential ideal has a generating set of a special form; in the terminology of Joseph Ritt [15, § I.5, p. 5] (when $m = 1$) and Ellis Kolchin [6, § I.10, pp. 81 ff.], this is a *characteristic set.* There is a first-order way to tell, uniformly in the parameters, whether a given set of differential polynomials is a characteristic set of some differential ideal, and then to tell, if it *is* a characteristic set, whether it has a root. In short, the function $\varphi \mapsto \widehat{\varphi}$ in Robinson's criterion (Lemma 1.1) is defined for sufficiently many systems φ. (Applying Blum's criterion, McGrail and Yaffe consider only systems in one variable, so they must include inequations in these systems; Tressl uses only equations, in arbitrarily many variables.)

I do not give the definition of a characteristic set, as not all ingredients of the definition are needed for the arguments presented in § 4. However, some of the ingredients *are* needed; these are in 4.1.

subsect:fga

3.1. Spaces of derivations. In [12] I attempted to apply the geometric approach described in 2.2 to m-DF$_0$. I worked more generally with DF^m_0, where DF^m is the theory of structures $(K, D_0, \dots, D_{m-1})$ such that $(K, D_i) \models \mathrm{DF}$ for each i, and each bracket $[D_j, D_k]$ is a K-linear combination of the D_i. (This is roughly what Yaffe did too.) In [13, § 2] I made some minor corrections and otherwise adapted the argument to arbitrary characteristic. Nonetheless, in May, 2006, Ehud Hrushovski showed me a counterexample to [12, Thm A, p. 926], a theorem that was an introductory formulation of [12, Thm 5.7, p. 942]. Then I found an error at the end of the proof of the latter theorem. That theorem is simply wrong; the present paper does not so much correct the theorem as replace it.

The developments leading up to the wrong theorem are still of some use. The general situation is as follows. Let $(K, D_0, \dots, D_{m-1}) \models \mathrm{DF}^m$, and let E be the K-linear span of the D_i. Then E is a Lie-ring, as well as a vector-space over K. As a vector-space, E has a dual, E^*; and there is a derivation d from K into E^* given by $D(\mathrm{d}\,x) = Dx$. Then E^* has a basis $(\mathrm{d}\,t^i \colon i < \ell)$ for some t^i in K and some ℓ no greater than m [12, Lem. 4.4, p. 932], and this basis is dual to a basis $(\partial_i \colon i < \ell)$ of E, where $[\partial_i, \partial_j] = 0$ in each case, and d can be given by

$$\mathrm{d}\,x = \sum_{i<\ell} \mathrm{d}\,t^i \cdot \partial_i x \tag{5}$$

[12, Lem. 4.7, p. 934]. We can use these ideas to prove the following.

thm:if-mDF

Theorem 3.1. *If m-DF has a model-companion, then so does* DF^m.

Proof. In characteristic 0, the result is implicit in [12, Thm 5.3 and proof], explicit in [26, § 3]; but the proof works generally. The main point is to find, for any model $(K, D_0, \dots, D_{m-1})$ of DF^m, an extension in which the named derivations are linearly independent over the larger field. As above, the space spanned over K by the D_i has a basis $(\partial_i \colon i < \ell)$ of commuting derivations of K. If $\ell < m$, then let $L = K(\alpha^\ell, \dots, \alpha^{m-1})$, where $(\alpha^\ell, \dots, \alpha^{m-1})$ is algebraically independent over K. Extend the ∂_i to L so that they are 0 at the α^j; then, if $\ell \leqslant k < m$, define ∂_k to be 0 on K and to be δ^j_k at α^j. Then $(\partial_i : i < m)$ is a linearly independent m-tuple of commuting derivations on L; from this, we obtain linearly independent extensions $\tilde{D}_i$ of the D_i to L such that the brackets $[\tilde{D}_j, \tilde{D}_k]$ are the same linear combinations of the $\tilde{D}_i$ that the $[D_j, D_k]$ are of the D_i (by [12, Lem. 5.2, p. 937]—or [26, Lem. 2.1, p. 1930], by a different method—in characteristic 0; generally, [13, Lem. 2.4, p. 1334]). Then $(K, D_i, \dots, D_{m-1}) \subseteq (L, \tilde{D}_0, \dots, \tilde{D}_{m-1})$, and the latter is a model of DF^m. Moreover, $(L, \tilde{D}_0, \dots, \tilde{D}_{m-1})$ is an existentially closed model if and only if $(L, \partial_0, \dots, \partial_{m-1})$ is an existentially closed model of DF^m; so a model-companion of DF^m can be derived from a model-companion of m-DF. $\dashv$

That much stands, and differential forms are convenient for establishing it. The theorem, combined with the results of § 4, will yield a model-companion, DCF^m, of DF^m.

3.2. False steps with differential forms. In [12] I tried also to obtain DCF^m_0 independently as follows. Suppose now we have a separably closed field K, along with a Lie-ring and finite-dimensional space E of derivations of K; as a space, E has a basis $(\partial_i : i < m)$, whose dual is $(\mathrm{d}\, t^i : i < m)$, so that the ∂_i commute. We may assume that $(K, \partial_0, \dots, \partial_{m-1})$ is differentiably perfect [13, Lem. 2.4]. Every system over $(K, \partial_0, \dots, \partial_{m-1})$ is equivalent to a system of the form of (4), generalized to

$$\bigwedge_f f = 0 \mathbin{\&} \bigwedge_{j<k} \bigwedge_{i<m} \partial_i x^j = g^j_i. \tag{6}$$

By means of (5), we can also write this as

$$\bigwedge_f f = 0 \mathbin{\&} \bigwedge_{j<k} \mathrm{d}\, x^j = \sum_{i<m} \mathrm{d}\, t^i \cdot g^j_i. \tag{7}$$

If $\boldsymbol{a}$ is a solution (from some extension), it is enough to assume that $(a^0, \dots, a^\ell)$ is a separating transcendence-basis of $K(\boldsymbol{a})/K$ for some ℓ such that $k \leqslant \ell < n$. That we cannot generally assume $k = \ell$ is an important difference from the case of one derivation; it is what causes the difficulties in the case of several derivations. The solution $\boldsymbol{a}$ to (7) can be understood as follows. First we have the field $K(\boldsymbol{a})$, and then (7) can be be written as

$$\bigwedge_{j<k} \mathrm{d}\, a^j = \sum_{i<m} \mathrm{d}\, t^i \cdot g^j_i(\boldsymbol{a}). \tag{8}$$

A solution of this can be understood as a model $(L, \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$ of m-DF extending $(K, \partial_0, \dots, \partial_{m-1})$ such that $K(\boldsymbol{a}) \subseteq L$ and (8) holds when $\mathrm{d}\, a^j = \sum_{i<m} \mathrm{d}\, t^i \cdot \tilde{\partial}_i a^j$, that is,

$$\bigwedge_{j<k} \bigwedge_{i<m} \tilde{\partial}_i a^j = g^j_i(\boldsymbol{a}). \tag{9}$$

Since the $\tilde{\partial}_i$ commute, it is necessary that

$$\bigwedge_{j<k} \bigwedge_{h<i<m} \tilde{\partial}_h(g^j_i(\boldsymbol{a})) = \tilde{\partial}_i(g^j_h(\boldsymbol{a})) \tag{10}$$

[12, § 1, p. 926]. Any derivative with respect to $\tilde{\partial}_i$ of an element of $K(\boldsymbol{a})$ is a constant plus a linear combination of the derivatives $\tilde{\partial}_i a^j$, where $j < \ell$ (by [13, Fact 1.1 (0, 2)], for example); we know what these derivatives $\tilde{\partial}_i a^j$ are when $j < k$, by (9); so (10) becomes a linear system over $K(\boldsymbol{a})$ in the unknowns $\tilde{\partial}_i a^j$ where $k \leqslant j < \ell$.

If $k = \ell$, then this linear system has no variables, so it is true or false; its truth is a sufficient condition for (8) to have a solution. If $k < \ell$, then the linear system is soluble or not. If it is soluble, then it is possible to extend the ∂_i to derivations $\tilde{\partial}_i$ as required by (9) that commute on $K(a^0, \dots, a^{k-1})$; but these derivations need not commute on all of $K(\boldsymbol{a})$. In [12] I claimed that they could commute, and that the solubility of (10) was sufficient for solubility of (8) in the sense above. I was wrong.

If we take a solution to the linear system, we now have an extension of $K(\boldsymbol{a})$, and we have to check extensibility of the commuting derivations to *this*. That is, we are back in the same kind situation we started with. However, it turns out that there is a bound on the number of times that we need to repeat this process in order to ensure solubility of the original differential system. This is what is shown in § 4; differential forms are apparently not useful for this after all.

3.3. A counterexample. Over a model of DF^2, let (a, b, c) be an algebraically independent triple. The counterexample supplied by Hrushovski is the system

$$\mathrm{d}\,a = \mathrm{d}\,t^0 \cdot c^2 + \mathrm{d}\,t^1 \cdot c, \qquad \mathrm{d}\,b = \mathrm{d}\,t^0 \cdot 2a + \mathrm{d}\,t^1 \cdot c \tag{11}$$

(where c^2 is the square of c; the constants (k, ℓ) of § 3.1 are now $(2, 3)$). Equivalently, by (5), the system comprises the equations

$$\partial_0 a = c^2, \qquad \partial_1 a = c, \qquad \partial_0 b = 2a, \qquad \partial_1 b = c.$$

From these, we compute

$$\partial_1\partial_0 a = 2c \cdot \partial_1 c, \qquad \partial_0\partial_1 a = \partial_0 c, \qquad \partial_1\partial_0 b = 2 \cdot \partial_1 a = 2c, \qquad \partial_0\partial_1 b = \partial_0 c.$$

Equating $\partial_0\partial_1$ and $\partial_1\partial_0$ yields the linear system

$$\partial_0 c - 2c \cdot \partial_1 c = 0, \qquad \partial_0 c = 2c, \tag{12}$$

which has the solution $(\partial_0 c, \partial_1 c) = (2c, 1)$. But then we must have $\partial_1\partial_0 c = 2 \cdot \partial_1 c = 2$, while $\partial_0\partial_1 c = \partial_0 1 = 0$, which means (11) has no solution, contrary to my claim in [12].

For the record, the mistake is at the end of the proof of [12, Thm 5.7, p. 942] and can be seen as follows. Write the system (11) as $\mathrm{d}\,a = \alpha$, $\mathrm{d}\,b = \beta$; then

$$\mathrm{d}\,\alpha = \mathrm{d}\,c \wedge (\mathrm{d}\,t^0 \cdot 2c + \mathrm{d}\,t^1), \qquad \begin{aligned} \mathrm{d}\beta &= \mathrm{d}\,a \wedge \mathrm{d}\,t^0 \cdot 2 + \mathrm{d}\,c \wedge \mathrm{d}\,t^1 \\ &= (\mathrm{d}\,c - \mathrm{d}\,t^0 \cdot 2c) \wedge \mathrm{d}\,t^1. \end{aligned} \tag{13}$$

Since also $\mathrm{d}\beta = \mathrm{d}^2\,b = 0$, we now have a condition on $\mathrm{d}\,c \wedge \mathrm{d}\,t^1$, hence on $\partial_0 c$; in particular, $\partial_0 c = 2c$, which is what we found above. But there is no apparent condition on $\partial_1 c$, so I try introducing a new transcendental, d, for this derivative. By (12) then,

$$\mathrm{d}\,c = \mathrm{d}\,t^0 \cdot 2c + \mathrm{d}\,t^1 \cdot d,$$

which by (13) yields $\mathrm{d}\,\alpha = \mathrm{d}\,t^0 \wedge \mathrm{d}\,t^1 \cdot 2c(1 - d)$. But we must have $\mathrm{d}\,\alpha = 0$, so $d = 1$, contrary to assumption. In short, the next to last sentence of the proof of [12, Thm 5.7] (beginning ‘This ideal is linearly disjoint from’) is simply wrong. (I had not attempted to argue that it was correct.)

sect:resolution

§4. Resolution. For a correct understanding of the existentially closed differential fields, it is better not to introduce differential forms from the beginning, but to allow equations to involve any number of applications of the derivations. In contrast to 2.1, there does not seem to be an advantage now in restricting attention to equations in one variable.

subsect:terms

4.1. Terminology. I shall now avoid working with differential polynomials as such, but shall work instead with the algebraic dependencies that they determine.

Let $(K, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}$. Higher-order derivatives with respect to the ∂_i can be indexed by elements of ω^m: so, for ${\partial_0}^{\sigma(0)} \cdots {\partial_{m-1}}^{\sigma(m-1)} x$, we may write $\partial^\sigma x$. Let $\leqslant$ be the product ordering of ω^m. Then the derivative $\partial^\sigma x$ is **below** $\partial^\tau x$ (and the latter is **above** the former) if $\sigma \leqslant \tau$. (In particular, a derivative is both below and above itself.) If $n \in \omega$, then two elements of $\omega^n \times n$ will be related by $\leqslant$ only if they agree in the last coordinate, so that

$$(\sigma, k) \leqslant (\tau, \ell) \iff \sigma \leqslant \tau \;\&\; k = \ell;$$

we may use the corresponding terminology of 'above' and 'below', so that $\partial^\sigma x_k$ is below $\partial^\tau x_\ell$ if (and only if) $(\sigma, k) \leqslant (\tau, \ell)$.

If $\sigma \in \omega^m$, let the sum $\sum_{i<m} \sigma(i)$ be denoted by $|\sigma|$: this is the **height** of σ or of $\partial^\sigma x$. (Kolchin [6, § I.1, p. 59] uses the word *order*.) If n is a positive integer, let $\omega^m \times n$ be (totally) ordered by $\trianglelefteq$, which is taken from the left lexicographic ordering of ω^{m+1} by means of the embedding

$$(\xi, k) \longmapsto (|\xi|, k, \xi(0), \dots, \xi(m-2))$$

of $\omega^m \times n$ in ω^{m+1}. Then $(\omega^m \times n, \trianglelefteq)$ is isomorphic to $(\omega, \leqslant)$. We may write $(\sigma, k) \triangleleft \infty$ for all (σ, k) in $\omega^m \times n$. Suppose $(x_h \colon h < n)$ is a tuple of indeterminates. By ordering the formal derivatives $\partial^\sigma x_k$ in terms of (σ, k) and $\trianglelefteq$, we have Kolchin's example of an *orderly ranking* of derivatives [6, § I.8, p. 75]. If $(\sigma, k) \triangleleft (\tau, \ell)$, I shall say that the derivative $\partial^\sigma x_k$ is **less** than $\partial^\tau x_\ell$ or is a **predecessor** of $\partial^\tau x_\ell$, and $\partial^\tau x_\ell$ is **greater** than $\partial^\sigma x_k$; likewise for the expressions a^σ_k and a^τ_ℓ, introduced in (15) below. (So, the terms just defined refer to the strict total ordering $\triangleleft$, while 'below' and 'above' refer to the partial ordering $\leqslant$.)

Addition and subtraction on ω induce corresponding operations on ω^m. Then

$$\begin{aligned} \tau &\leqslant \sigma + \tau, \\ \partial^\sigma \partial^\tau x_k &= \partial^{\sigma+\tau} x_k, \\ (\sigma, k) &\trianglelefteq (\sigma + \tau, k), \\ (\sigma, k) \trianglelefteq (\tau, \ell) &\iff (\sigma + \rho, k) \trianglelefteq (\tau + \rho, \ell). \end{aligned} \tag{14}$$

If $i < m$, let $\boldsymbol{i}$ denote the characteristic function of $\{i\}$ in ω^m, so that $\partial^{\boldsymbol{i}} = \partial_i$, and more generally $\partial_i \partial^\sigma = \partial^{\sigma+\boldsymbol{i}}$, and $\partial_i \partial^{\sigma-\boldsymbol{i}} = \partial^\sigma$.

Let L be an extension of K with generators that are indexed by an initial segment of $(\omega^m \times n, \trianglelefteq)$; that is,

$$L = K(a^\xi_h \colon (\xi, h) \triangleleft (\tau, \ell)), \tag{15}$$

where $(\tau, \ell) \in \omega^m \times n$, or possibly $(\tau, \ell) = \infty$, in which case $L = K(a^\xi_h \colon (\xi, h) \in \omega^m \times n)$. It could happen that, in the generating tuple $(a^\xi_h \colon (\xi, h) \triangleleft (\tau, \ell))$ of L/K, the same element of L may appear twice, with different indices. In this, case, when writing a^ξ_h, we may mean not just a particular element of L, but that element together with the pair (ξ, h) of indices. For example, by (14), if $(\sigma + \boldsymbol{i}, k) \triangleleft (\tau, \ell)$, then $(\sigma, k) \triangleleft (\tau, \ell)$; hence we may say that, if $a^{\sigma+\boldsymbol{i}}_k$ is one of the generators of L/K, then so is a^σ_k. Let us say that L, with the tuple of generators given in (15), meets the **differential condition** if there is no obstacle to extending each derivation ∂_i to a derivation D_i on $K(a^\xi_h\colon (\xi+\boldsymbol{i},h)\lhd(\tau,\ell))$ such that

$$D_i a^\sigma_k = a^{\sigma+\boldsymbol{i}}_k \tag{16}$$

whenever $(\sigma+\boldsymbol{i},k)\lhd(\tau,\ell)$. (If the right-hand member of (16) is not defined, then the left need not be defined.) To be precise, if f is a rational function over K in variables $(x^\xi_h\colon (\xi,h)\trianglelefteqslant(\sigma,k))$ for some (σ,k) in $\omega^m\times n$, and D is a derivation of K, then f has a derivative Df, which is the linear function over $K(x^\xi_h\colon (\xi,h)\trianglelefteqslant(\sigma,k))$ given by

$$Df=\sum_{(\xi,h)\trianglelefteqslant(\sigma,k)}\frac{\partial f}{\partial x^\xi_h}\cdot y^\xi_h+f^D.$$

Then the differential condition is that for all such f, if $(\sigma+\boldsymbol{i},k)\lhd(\tau,\ell)$ for some i in m, and if

$$f(a^\xi_h\colon (\xi,h)\trianglelefteqslant(\sigma,k))=0, \tag{17}$$

then $\partial_i f(a^\xi_h,a^{\xi+\boldsymbol{i}}_h\colon (\xi,h)\trianglelefteqslant(\sigma,k))=0$, that is,

$$\sum_{(\eta,g)\trianglelefteqslant(\sigma,k)}\frac{\partial f}{\partial x^\eta_g}(a^\xi_h\colon (\xi,h)\trianglelefteqslant(\sigma,k))\cdot a^{\eta+\boldsymbol{i}}_g+f^{\partial_i}(a^\xi_h\colon (\xi,h)\trianglelefteqslant(\sigma,k))=0. \tag{18}$$

(Note well the assumption that $(\sigma+\boldsymbol{i},k)\lhd(\tau,\ell)$. In (18), each of the $a^{\eta+\boldsymbol{i}}_g$ must exist, even though the coefficient $(\partial f/\partial x^\eta_g)(a^\xi_h\colon (\xi,h)\trianglelefteqslant(\sigma,k))$ might be 0.) So the differential condition is *necessary* for the extensibility of the ∂_i as desired (see for example [13, Fact 1.1 (0)]); sufficiency is part of Lemma 4.1 below.

An extension $(M,D_0,\dots,D_{m-1})$ of $(K,\partial_0,\dots,\partial_{m-1})$ is **compatible** with the extension L of K given in (15) if $L\subseteq M$, and (16) holds whenever $(\sigma+\boldsymbol{i},k)\lhd(\tau,\ell)$.

Borrowing some terminology used for differential polynomials [15, § IX.1, p. 163], let us say that a generator a^σ_k of L/K is a **leader** if it is algebraically dependent over K on its predecessors, that is,

$$a^\sigma_k\in K(a^\xi_h\colon (\xi,h)\lhd(\sigma,k))^{\mathrm{alg}}.$$

Then a^σ_k is a **separable** leader if it is separably algebraic over $K(a^\xi_h\colon (\xi,h)\lhd(\sigma,k))$; otherwise, it is an **inseparable** leader. A separable leader a^σ_k is **minimal** if there is no separable leader strictly below it—no separable leader a^ρ_k such that $\rho<\sigma$.

lem:above LEMMA 4.1. *Suppose $(K,\partial_0,\dots,\partial_{m-1})\models m$-DF, and L is an extension $K(a^\xi_h\colon (\xi,h)\lhd(\tau,\ell))$ of K meeting the differential condition. Then the derivations ∂_i extend to derivations D_i from $K(a^\xi_h\colon (\xi+\boldsymbol{i},h)\lhd(\tau,\ell))$ into L such that (16) holds when $(\sigma+\boldsymbol{i},k)\lhd(\tau,\ell)$. If a^σ_k is a separable leader, and $(\sigma+\boldsymbol{i},k)\lhd(\tau,\ell)$, then*

$$a^{\sigma+\boldsymbol{i}}_k\in K(a^\xi_h\colon (\xi,h)\lhd(\sigma+\boldsymbol{i},k)) \tag{19}$$

(that is, $a^{\sigma+\boldsymbol{i}}_k$ is a rational function over K of its predecessors); in particular, $a^{\sigma+\boldsymbol{i}}_k$ is a separable leader. Therefore generators of L/K that are above separable leaders are themselves separable leaders.

PROOF. The claim follows from the basic properties of derivations, such as are gathered in [13, § 1]. Let B comprise the generators of L/K that are not leaders. Then B is algebraically independent over K, so we are free to extend the ∂_i to D_i on $K(B)$ so that (16) holds whenever it applies. Then these extensions are uniquely determined, and there are further unique extensions of the D_i to the separable closure $K(B)^{\mathrm{sep}}$. In particular, suppose a^σ_k is a separable leader, and $(\sigma + \boldsymbol{i}, k) \lhd (\tau, \ell)$. Then $D_i a^\sigma_k$ is obtained by differentiating the minimal polynomial of a^σ_k over $K(B)$. That is, $D_i a^\sigma_k$ is obtained by differentiating an equation like (17); by the differential condition, $D_i a^\sigma_k$ must be $a^{\sigma+\boldsymbol{i}}_k$ as given by (18); this shows that $a^{\sigma+\boldsymbol{i}}_k$ is a rational function over K of its predecessors.

Finally, in a positive characteristic p, there may be an inseparable leader a^σ_k. Then $(a^\sigma_k)^{p^r} \in K(a^\xi_h \colon (\xi, h) \lhd (\sigma, k))^{\mathrm{sep}}$ for some positive r. If $(\sigma + \boldsymbol{i}, k) \lhd (\tau, \ell)$, then we are free to define $D_i a^\sigma_k$ as $a^{\sigma+\boldsymbol{i}}_k$, provided $D_i((a^\sigma_k)^{p^r}) = 0$. But again this condition is ensured by the differential condition. Indeed, we may suppose (17) shows the separable dependence of $(a^\sigma_k)^{p^r}$ over the predecessors of a^σ_k. That is, we can understand $f(a^\xi_h \colon (\xi, h) \unlhd (\sigma, k))$ as $g((a^\sigma_k)^{p^r})$ for some separable polynomial g over $K(a^\xi_h \colon (\xi, h) \lhd (\sigma, k))$. Then $D_i((a^\sigma_k)^{p^r})$ is obtained from (18), provided we replace the term $(\partial f/\partial x^\sigma_k)(a^\xi_h \colon (\xi, h) \unlhd (\sigma, k)) \cdot a^{\sigma+\boldsymbol{i}}_k$ with $g'((a^\sigma_k)^{p^r}) \cdot D_i((a^\sigma_k)^{p^r})$. But in the present case, the former term is 0. Since, after the replacement, the resulting equation still holds, we must have $D_i((a^\sigma_k)^{p^r}) = 0$. ⊣

4.2. A solubility condition. If $(K, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}$, then this model has an extension whose underlying field is the separable closure of K (as by [12, Lem. 3.4, p. 930] and [13, Lem. 2.4, p. 1334]). We shall need this in a more general form:

lem:comm

Lemma 4.2. *Suppose a field M has two subfields L_0 and L_1, which in turn have a common subfield K. For each i in 2, suppose there is a derivation D_i mapping K into L_i and L_{1-i} into M. Then the bracket $[D_0, D_1]$ is a well-defined derivation on K. Suppose it is the 0-derivation. Suppose also that a is an element of M that is separably algebraic over K. Then each D_i extends uniquely to $K(a)$, and $D_i a \in L_{1-i}(a)$, so $D_{1-i}D_i a$ is also well-defined. Moreover, $[D_0, D_1]a = 0$.*

Proof. The claim follows from standard facts, at least if $L_0 = K = L_1$; but the proof is the same in the general case. Indeed, though the derivations D_0 and D_1 are defined on K, their bracket $[D_0, D_1]$ need not be so, since the compositions $D_0 D_1$ and $D_1 D_0$ need not be so; but if they are, then $[D_0, D_1]$ is a *derivation* on K. A derivation on K extends uniquely to K^{sep}; if the derivation is 0 on K, then it is 0 on K^{sep} [13, Fact 1.1 (2)]. In the present case, as $a \in K^{\mathrm{sep}}$, so $D_i a \in L_{1-i}(a)$, and therefore $D_i a \in {L_{1-i}}^{\mathrm{sep}}$; hence $D_{1-i} D_i a$ is defined. Thus $[D_0, D_1]$ is defined on $K(a)$, where $a \in K^{\mathrm{sep}}$; and if the bracket is 0 on K, then is 0 at a. ⊣

In positive characteristic, the possibility of inseparably algebraic extensions presents a challenge, which however is handled by the following.

thm:first

Theorem 4.3. *Suppose $(K, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}$, and K has an extension $K(a^\xi_h \colon |\xi| \leqslant 2r \;\&\; h < n)$ meeting the differential condition for some positive integers r and n. Suppose further that, whenever a^σ_k is a minimal separable leader, then $|\sigma| \leqslant r$. Then $(K, \partial_0, \dots, \partial_{m-1})$ has an extension $(M, D_0, \dots, D_{m-1})$ compatible with $K(a^\xi_h \colon |\xi| < 2r \;\&\; h < n)$.*

Proof. The claim can be compared to and perhaps derived from a differential-algebraic lemma of Rosenfeld [19, § I.2], at least in characteristic 0. Here I give an independent argument, for arbitrary characteristic. We shall obtain M recursively as

$K(a^\xi_h\colon (\xi,h)\in\omega^m\times n)$, at the same time proving inductively that the ∂_i can be extended to D_i so that (16) holds in all cases.

Let $L = K(a^\xi_h\colon |\xi| < 2r \mathbin{\&} h < n)$; this is $K(a^\xi_h\colon (\xi,h) \trianglelefteq ((2r-1,0,\dots,0),n-1))$. Then by (18), the differential condition requires of the tuple $(a^\xi_h\colon |\xi| = 2r \mathbin{\&} h < n)$ only that it solve some linear equations over L. The hypothesis of our claim is that there *is* a solution, namely $(a^\xi_h\colon |\xi| = 2r \mathbin{\&} h < n)$. We may therefore assume that this tuple is a *generic* solution of these equations. In particular, no entry of this tuple is an inseparable leader. (If, instead of being chosen generically, the entries of $(a^\xi_h\colon |\xi| = 2r \mathbin{\&} h < n)$ were chosen from the field L, then this field would be closed under the desired extensions D_i of ∂_i, and the derivations D_i would commute on the subfield $K(a^\xi_h\colon |\xi|+1 < 2r \mathbin{\&} h < n)$; but they might not commute on all of L.)

Now, as an inductive hypothesis, suppose we have the extension $K(a^\xi_h\colon (\xi,h) \vartriangleleft (\tau,\ell))$ of K meeting the differential condition, so that there are derivations D_i as given by Lemma 4.1; suppose also that

(i) if a^σ_k is a minimal separable leader, then $|\sigma| \leqslant r$;
(ii) if a^σ_k is an inseparable leader, then $|\sigma| < 2r$.

We need to choose a^τ_ℓ in such a way that these conditions still hold for $K(a^\xi_h\colon (\xi,h) \trianglelefteq (\tau,\ell))$. The inductive hypothesis is correct when $|\tau| \leqslant 2r$, and then the desired conclusion follows; so we may assume $|\tau| > 2r$. Hence, if $\tau(i) > 0$, so that $\tau - \boldsymbol{i}$ is defined, then $|\tau - \boldsymbol{i}| \geqslant 2r$, so $a^{\tau-\boldsymbol{i}}_\ell$ is not an inseparable leader.

If $a^{\tau-\boldsymbol{i}}_\ell$ is not a leader at all, for any i in m, then we may let a^τ_ℓ be a new transcendental, and we may define each derivative $D_i a^{\tau-\boldsymbol{i}}_\ell$ as this [13, Fact 1.1 (1)].

In the other case, $a^{\tau-\boldsymbol{i}}_\ell$ is a separable leader for some i. Then $D_i a^{\tau-\boldsymbol{i}}_\ell$ is determined (Lemma 4.1). We want to let a^τ_ℓ be this derivative. However, possibly also $a^{\tau-\boldsymbol{j}}_\ell$ is a separable leader, where $i \neq j$. In this case, we must check that

$$D_j a^{\tau-\boldsymbol{j}}_\ell = D_i a^{\tau-\boldsymbol{i}}_\ell, \tag{20}$$

that is, $[D_i, D_j] a^{\tau-\boldsymbol{i}-\boldsymbol{j}}_\ell = 0$.

There are minimal separable leaders a^π_ℓ and a^ρ_ℓ below $a^{\tau-\boldsymbol{i}}_\ell$ and $a^{\tau-\boldsymbol{j}}_\ell$ respectively. Let ν be $\pi \vee \rho$, the least upper bound of $\{\pi,\rho\}$ with respect to $\leqslant$. Then $\nu \leqslant \tau$. But $|\nu| \leqslant |\pi| + |\rho| \leqslant 2r < |\tau|$; so $\nu < \tau$. Hence $\nu \leqslant \tau - \boldsymbol{k}$ for some k in m, which means a^ν_ℓ is below $a^{\tau-\boldsymbol{k}}_\ell$. Consequently,

(i) a^π_ℓ is below both $a^{\tau-\boldsymbol{i}}_\ell$ and $a^{\tau-\boldsymbol{k}}_\ell$;
(ii) a^ρ_ℓ is below both $a^{\tau-\boldsymbol{j}}_\ell$ and $a^{\tau-\boldsymbol{k}}_\ell$.

If $k = j$, then a^π_ℓ is below $a^{\tau-\boldsymbol{i}-\boldsymbol{j}}_\ell$, so this is a separable leader. As D_i and D_j commute on $K(a^\xi_h\colon (\xi,h) \vartriangleleft (\tau - \boldsymbol{i} - \boldsymbol{j}, \ell))$ by the differential condition, they must commute also at $a^{\tau-\boldsymbol{i}-\boldsymbol{j}}_\ell$ (Lemma 4.2), so (20) is established. The argument is the same if $k = i$. If k is different from i and j, then again the same argument yields $D_j a^{\tau-\boldsymbol{j}}_\ell = D_k a^{\tau-\boldsymbol{k}}_\ell$ and $D_k a^{\tau-\boldsymbol{k}}_\ell = D_i a^{\tau-\boldsymbol{i}}_\ell$, so (20) holds.

In no case did we introduce a new minimal separable leader or an inseparable leader. This completes the induction and the proof. ⊣

In terms of differential polynomials and ideals, the theorem can be understood as follows. Given the hypothesis of the theorem, let S be the set of differential polynomials

$f(\partial^\xi x_h\colon |\xi| < 2r \mathbin{\&} h < n)$, where f ranges over the ordinary polynomials over K such that $f(a^\xi_h\colon |\xi| < 2r \mathbin{\&} h < n) = 0$. Then S includes a characteristic set for the differential ideal that it generates.

We can now characterize the existentially closed models of m-DF by means of the following lemma. The lemma follows from unproved statements in [6, § 0.17, p. 49]; let's just prove it here.

lem:fin Lemma 4.4. *For every m in ω and positive integer n, every antichain of $(\omega^m \times n, \leqslant)$ is finite.*

Proof. The general case follows from the case when $n = 1$, since if S is an antichain of $(\omega^m \times n, \leqslant)$, then

$$S = \bigcup_{j<n} \{(\xi, h) \in S : h = j\},$$

and each component of the union is in bijection with an antichain of $(\omega^m, \leqslant)$. As an inductive hypothesis, suppose every antichain of $(\omega^\ell, \leqslant)$ is finite; but suppose also, if possible, that there is an infinite antichain S of $(\omega^{\ell+1}, \leqslant)$. Then S contains some σ. By inductive hypothesis, the subset

$$\bigcup_{j\leqslant\ell} \bigcup_{i\leqslant\sigma(j)} \{\xi \in S : \xi(j) = i\}$$

of S is a finite union of finite sets, so its complement in S has infinitely many elements τ; but then $\sigma < \tau$, so S was not an antichain. ⊣

thm:dcf Theorem 4.5. *Suppose $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF. Then the following are equivalent:*

item:ec (i) *The model $(K, \partial_0, \dots, \partial_{m-1})$ of m-DF is existentially closed.*

item:< (ii) *For all positive integers r and n, if K has an extension $K(a^\xi_h\colon |\xi| \leqslant 2r \mathbin{\&} h < n)$ meeting the differential condition such that $|\sigma| \leqslant r$ whenever a^σ_k is a minimal separable leader, then the tuple $(a^\xi_h\colon |\xi| < 2r \mathbin{\&} h < n)$ has a specialization $(\partial^\xi b_h\colon |\xi| < 2r \mathbin{\&} h < n)$ for some tuple $(b_h\colon h < n)$ of elements of K.*

Proof. Assume (i) and the hypothesis of (ii). Let S be a (finite) generating set of the ideal of $(a^\xi_h\colon |\xi| < 2r \mathbin{\&} h < n)$ over K. By Theorem 4.3, the system

$$\bigwedge_{f\in S} f(\partial^\xi x_h\colon |\xi| < 2r \mathbin{\&} h < n) = 0$$

has a solution in some extension, hence it has a solution in K itself, which means the conclusion of (ii) holds. So (ii) is necessary for (i).

Every system over $(K, \partial_0, \dots, \partial_{m-1})$ is equivalent to a system of equations. Suppose such a system has a solution $(a_h\colon h < n)$ in some extension $(L, D_0, \dots, D_{m-1})$. Then the extension $K(\partial^\xi a_h\colon (\xi, h) \in (\omega^m \times n))$ has a *finite* set of minimal separable leaders, by Lemma 4.4, since this set is indexed by an antichain of $(\omega^m \times n, \leqslant)$. Hence there is r large enough that all of these minimal separable leaders are also generators of $K(D^\xi a_h\colon |\xi| \leqslant r \mathbin{\&} h < n)$. We may assume also that r is large enough that $|\sigma| \leqslant r$ for every derivative $\partial^\sigma x_k$ that appears in the original system. The hypothesis of (ii) is now satisfied when each a^σ_k is taken as $D^\sigma a_k$. If the conclusion of (ii) follows, then $(b_h\colon h < n)$ is a solution of the original system. Thus, (ii) is sufficient for (i). ⊣

cor:dcf Corollary 4.6. *The theory m-DF has a model-companion, m-DCF.*

Proof. Let $(K, \partial_0, \dots, \partial_{m-1})$ be a model of m-DF, and L an extension $K(a_h^\xi \colon |\xi| \leqslant 2r \mathrel{\&} h < n)$ of K meeting the differential condition, and $|\sigma| \leqslant r$ whenever a_k^σ is a minimal separable leader. As noted in the proof of Theorem 4.3, we may assume that there are no inseparable leaders a_h^ξ such that $|\xi| = 2r$. For the moment, write $(a_h^\xi \colon |\xi| \leqslant 2r \mathrel{\&} h < n)$ simply as $\boldsymbol{a}$. The ideal of $K[\boldsymbol{x}]$ comprising the polynomials that are 0 at $\boldsymbol{a}$ is generated by a set $\{f(\boldsymbol{p}, \boldsymbol{x}) \colon f \in S\}$, where S is a finite subset of $\mathbb{Z}[\boldsymbol{y}, \boldsymbol{x}]$, and $\boldsymbol{p}$ is a (finite) list of parameters from K. In order to develop an axiomatization of m-DCF, suppose there is a formula $\varphi(\boldsymbol{y})$ such that, for all differentially perfect models $(K', \partial_0, \dots, \partial_{m-1})$ of m-DF and all lists $\boldsymbol{q}$ of parameters from K', the sentence $\varphi(\boldsymbol{q})$ holds in that model if and only if

(i) the set $\{f(\boldsymbol{q}, \boldsymbol{x}) \colon f \in S\}$ generates a prime ideal of $K'[\boldsymbol{x}]$, and
(ii) for a generic zero $\boldsymbol{b}$ of the set, the extension $K'(\boldsymbol{b})$, that is, $K'(b_h^\xi \colon |\xi| \leqslant 2r \mathrel{\&} h < n)$, of K' meets the differential condition, and $|\sigma| \leqslant r$ whenever b_k^σ is a minimal separable leader.

Then the sentence

$$\varphi(\boldsymbol{y}) \Rightarrow \exists \boldsymbol{x} \Big(\bigwedge_{f \in S} f(\boldsymbol{y}, \boldsymbol{x}) = 0 \mathrel{\&} \bigwedge_{i<m} \bigwedge_{h<n} \bigwedge_{|\xi+\boldsymbol{i}| \leqslant 2r} x_h^{\xi+\boldsymbol{i}} = \partial_i x_h^\xi \Big) \tag{21}$$

is true in every existentially closed model of m-DF. So now all we have to do is find such a formula $\varphi(\boldsymbol{y})$. In fact, what we find will not be this $\varphi(\boldsymbol{y})$ exactly, but it will be close enough.

Every existentially closed model of m-DF is differentially perfect [13, Lem. 2.4, p. 1334]. The differentially perfect models compose an elementary class, with a theory m-DPF. We shall show that m-DCF is axiomatized by the axioms of m-DPF, along with sentences that are formed roughly as in (21) and that are obtained as $(K, \partial_0, \dots, \partial_{m-1})$ ranges over the differentially perfect models of m-DF (or simply the countable differentiably perfect models).

The point of considering only differentially perfect models $(K, \partial_0, \dots, \partial_{m-1})$ is that now, since the derivations on K can be extended to L as above, the extension L/K is separable [13, Lem. 1.5, p. 1328]. Hence there is a set A of indices (ξ, h) such that $(a_h^\xi \colon (\xi, h) \in A)$ is a separating transcendence basis of L/K. Changing its earlier meaning, let us now use $\boldsymbol{a}$ to denote the basis $(a_h^\xi \colon (\xi, h) \in A)$. Let B consist of the indices (ξ, h) that are not in A, although still $|\xi| \leqslant 2r$ and $h < n$; also, write $\boldsymbol{b}$ for $(a_h^\xi \colon (\xi, h) \in B)$. Now we can write

$$(a_h^\xi \colon |\xi| \leqslant 2r \mathrel{\&} h < n) = (a_h^\xi \colon (\xi, h) \in A \cup B) = (\boldsymbol{a}, \boldsymbol{b}).$$

For each (η, g) in B, there is an irreducible polynomial over K that expresses the separable algebraic dependence of a_g^η on $\boldsymbol{a}$. More precisely, there is a list $\boldsymbol{p}$ of parameters from K, and for each (η, g) in A, there is an element f_g^η of $\mathbb{Z}[\boldsymbol{y}, x_h^\xi \colon (\xi, h) \in A \cup \{(\eta, g)\}]$ such that $f_g^\eta(\boldsymbol{p}, \boldsymbol{a}, x_g^\eta)$ is irreducible and separable, and $f_g^\eta(\boldsymbol{p}, \boldsymbol{a}, a_g^\eta) = 0$.

Let V be the variety over K with generic point $(\boldsymbol{a}, \boldsymbol{b})$. The tangent space $T(V)_{(\boldsymbol{a},\boldsymbol{b})}$ consists of all points $(\boldsymbol{a}', \boldsymbol{b}')$ such that the zero-derivation D on K extends to $K(\boldsymbol{a}, \boldsymbol{b})$, that is, to L, so that $D(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}', \boldsymbol{b}')$. In particular, the tangent space is defined by the linear functions $Df_g^\eta(\boldsymbol{p}, a_h^\xi, \boldsymbol{0}, z_h^\xi \colon (\xi, h) \in A \cup \{(\eta, g)\})$, which we may write as $Df_g^\eta(\boldsymbol{p}, \boldsymbol{a}, a_g^\eta, \boldsymbol{0}, \boldsymbol{z}, z_g^\eta)$. For each i in m, this tangent space has the translation $T_i(V)_{(\boldsymbol{a},\boldsymbol{b})}$ (part of the twisted tangent bundle mentioned in 2.2) comprising those $(\boldsymbol{a}', \boldsymbol{b}')$ such that

∂_i extends to D_i so that $D_i(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}', \boldsymbol{b}')$. In particular, the affine space $T_i(V)_{(\boldsymbol{a},\boldsymbol{b})}$ is defined by the functions $\partial_i f_g^\eta(\boldsymbol{p}, \boldsymbol{a}, a_g^\eta, \partial_i \boldsymbol{p}, \boldsymbol{z}, z_g^\eta)$. Therefore, whenever $f \in K(x_h^\xi : (\xi, h) \in A \cup B)$ and $f(\boldsymbol{a}, \boldsymbol{b}) = 0$, then $\partial_i f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{z}, \boldsymbol{w})$ is a linear combination over L of the functions $\partial_i f_g^\eta(\boldsymbol{p}, \boldsymbol{a}, a_g^\eta, \partial_i \boldsymbol{p}, \boldsymbol{z}, z_g^\eta)$. This will allow the formula $\varphi(\boldsymbol{y})$ to ensure that the differential condition is met.

A complication is that, for some (ξ, h) in A, we may have $|\xi| = 2r$. But by Theorem 4.3, or more precisely its proof, since we have assumed there is no separable leader a_h^ξ such that $|\xi| = 2r$, we may assume further that there are a_h^ξ chosen generically when $|\xi| = 2r + 1$ such that $(K, \partial_0, \dots, \partial_{m-1})$ has an extension compatible with $K(a_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n)$. By lengthening $\boldsymbol{p}$ as necessary, we may assume that there is a finite subset S of $\mathbb{Z}[\boldsymbol{y}, x_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n]$ that contains all of the polynomials f_g^η and is such that $\{f(\boldsymbol{p}, x_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n) : f \in S\}$ generates the ideal of polynomials over K that are 0 at $(a_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n)$. By the existence of an extension of $(K, \partial_0, \dots, \partial_{m-1})$ compatible with $K(a_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n)$, each polynomial $\partial_i f_g^\eta(\boldsymbol{p}, x_h^\xi, \partial_i \boldsymbol{p}, x_h^{\xi+\boldsymbol{i}} : (\xi, h) \in A \cup \{(\eta, g)\})$ is a certain linear combination of the polynomials $f(\boldsymbol{p}, x_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n)$, where $f \in S$. Conversely, this conclusion is enough to ensure that $K(a_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n)$ meets the differential condition.

We can lengthen $\boldsymbol{p}$ and enlarge S further, if necessary, to ensure that S contains polynomials showing that each leader a_h^ξ is algebraically dependent on its predecessors. We can now write a formula $\varphi(\boldsymbol{y})$ that is satisfied in $(K, \partial_0, \dots, \partial_{m-1})$ by $\boldsymbol{p}$ and is such that, if $\boldsymbol{q}$ satisfies $\varphi(\boldsymbol{y})$ in an arbitrary differentially perfect model $(K', \partial_0, \dots, \partial_{m-1})$ of m-DF, and $(c_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n)$ is a generic point of a component of the zero-set of $\{f(\boldsymbol{q}, x_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n) : f \in S\}$, then:

(i) each polynomial $f_g^\eta(\boldsymbol{q}, \boldsymbol{c}, z_g^\eta)$ is irreducible and separable, where $\boldsymbol{c} = (c_h^\xi : (\xi, h) \in A)$;
(ii) if a_h^ξ is a leader, then so is c_h^ξ, and these are alike separable or not;
item:mt (iii) the transcendence-degrees of $K(a_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n)/K$ and $K'(c_h^\xi : |\xi| \leqslant 2r + 1 \ \&\ h < n)/K'$ are the same, so that c_h^ξ is a leader only if a_h^ξ is;
(iv) the extension $K'(c_h^\xi : (\xi, h) \in A \cup B)$ satisfies the differential condition.

Here the possibility of ensuring the field-theoretic condition (iii) is an instance of the general model-theoretic result that Morley rank is definable in strongly minimal theories [10, Lem. 6.2.20, p. 225]. It is not necessary to ensure that the set $\{f(\boldsymbol{q}, x_h^\xi : |\xi| \leqslant 2r+1 \ \&\ h < n) : f \in S\}$ generates a prime ideal; it is enough that this is so when $\boldsymbol{q}$ is $\boldsymbol{p}$. The sentence in (21) is now true in all existentially closed models of m-DF, and the set of all such sentences, together with m-DPF, axiomatizes m-DCF as described. ⊣

By Theorem 3.1, DF^m now also has a model-companion.

4.3. Differential forms again. The condition in Theorem 4.3 can be adjusted to yield the following:

thm:second THEOREM 4.7. *Suppose* $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, *and K has an extension* $K(a_h^\xi : |\xi| \leqslant |\mu| \ \&\ h < n)$ *meeting the differential condition for some μ in ω^m and some positive integer n. Suppose further that, if a_k^σ is a minimal separable leader, then $\sigma \leqslant \mu$. Then* $(K, \partial_0, \dots, \partial_{m-1})$ *has an extension compatible with* $K(a_h^\xi : |\xi| < |\mu| \ \&\ h < n)$.

Proof. The proof is as for Theorem 4.3, *mutatis mutandis.* What needs adjusting is the choosing of a_ℓ^τ in case both $a_\ell^{\tau-\boldsymbol{i}}$ and $a_\ell^{\tau-\boldsymbol{j}}$ are separable leaders. Again we have minimal separable leaders a_ℓ^π and a_ℓ^ρ below $a_\ell^{\tau-\boldsymbol{i}}$ and $a_\ell^{\tau-\boldsymbol{j}}$ respectively. Since we may assume $|\mu| < |\tau|$, there is some k in m such that $\mu(k) < \tau(k)$. If $k = j$, then $\pi(j) \leqslant \mu(j) < \tau(j)$, so $\pi(j) \leqslant (\tau - \boldsymbol{j})(j) = (\tau - \boldsymbol{i} - \boldsymbol{j})(j)$. Then $\pi \leqslant \tau - \boldsymbol{i} - \boldsymbol{j}$, so a_ℓ^π is below $a_\ell^{\tau-\boldsymbol{i}-\boldsymbol{j}}$. Now we can proceed as before. ⊣

As Theorem 4.3 yields Theorem 4.5, so Theorem 4.7 yields a characterization of the existentially closed models of m-DF. Moreover, Theorems 4.3 and 4.7 can be combined in the following way:

thm:third

Theorem 4.8. *Suppose $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, and K has an extension $K(a_h^\xi : |\xi| \leqslant 2r \;\&\; h < n)$ meeting the differential condition for some positive integers n and r. Suppose further that, for each k in m, either $|\sigma| \leqslant r$ whenever a_k^σ is a minimal separable leader, or else there is some τ in ω^m such that $|\tau| = 2r$, and $|\sigma| \leqslant |\tau|$ whenever a_k^σ is a minimal separable leader. Then $(K, \partial_0, \dots, \partial_{m-1})$ has an extension compatible with $K(a_h^\xi : |\xi| < 2r \;\&\; h < n)$.*

Proof. Combine the proofs of Theorems 4.3 and 4.7. ⊣

There is a corresponding first-order characterization of the models of m-DCF, parallel to Theorem 4.5 and Corollary 4.6.

4.4. Another sufficient condition. If $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, and $K(a_h^\xi : |\xi| \leqslant |\pi| \;\&\; h < n)$ is an extension L of K meeting the differential condition, this by itself is not enough to ensure that $(K, \partial_0, \dots, \partial_{m-1})$ has an extension compatible with L. However, if such an extension does exist, then its existence can be shown by means of Theorem 4.3, provided $|\pi|$ can be made large enough: this is Theorem 4.10 below, which relies on the existence of bounds as in the following.

Lemma 4.9. *For all positive integers m and n, for all sequences $(a_i : i \in \omega)$ of positive integers, there is a bound on the length of strictly increasing chains*

$$S_0 \subset S_1 \subset S_2 \subset \cdots \tag{22}$$

of antichains S_k of $(\omega^m \times n, \leqslant)$, where also $S_k \subseteq \{(\xi, h) : |\xi| \leqslant a_k\}$.

Proof. Divide and conquer. First reduce to the case when $n = 1$. Indeed, suppose the claim does hold in this case. Suppose also, as an inductive hypothesis, that the claim holds when $n = \ell$. Now fix m and the sequence $(a_i : i \in \omega)$ or rather $(a(i) : i \in \omega)$, and consider arbitrary chains as in (22), where $n = \ell + 1$. Analyze each S_k as $S_k' \cup S_k''$, where

$$S_k' = \{(\xi, h) \in S_k : h < \ell\}, \qquad S_k'' = \{(\xi, h) \in S_k : h = \ell\}.$$

For each k such that S_{k+1} exists, at least one of the inclusions $S_k' \subseteq S_{k+1}'$ and $S_k'' \subseteq S_{k+1}''$ is strict; also, by our assumption, there is an upper bound $f(k)$ on those r such that

$$S_k'' \subset S_{k+1}'' \subset \cdots \subset S_{r-1}''. \tag{23}$$

The function f depends only on m and $(a_i : i \in \omega)$), not on the choice of chain in (22).

Let $k(0) = 0$, and if $k(i)$ has been chosen, let $k(i + 1)$ be the least r, if it exists, such that $S_{k(i)}' \subset S_r'$. Here $k(i)$ does depend on the chain. But if r is maximal in (23), and S_r' exists, then $S_k' \subset S_r'$. Hence $k(i + 1) \leqslant f(k(i))$. Since the function f is not necessarily increasing, we derive from it the increasing function g, where $g(k) = \max_{i\leqslant k} f(i)$. Then $x \leqslant y \Longrightarrow g(x) \leqslant g(y)$, so

$$k(r) \leqslant f(k(r-1)) \leqslant g(k(r-1)) \leqslant g \circ g(k(r-2)) \leqslant \cdots \leqslant \overbrace{g \circ \cdots \circ g}^{r}(0) = g^r(0). \quad (24)$$

In particular, $S_{k(r)} \subseteq \{(\xi, h) : |\xi| \leqslant a(g^r(0))\}$. The sequence $(a(g^i(0)) : i \in \omega)$ does not depend on the original chain. Hence the inductive hypothesis applies to the chain

$$S'_{k(0)} \subset S'_{k(1)} \subset \cdots , \quad (25)$$

showing that there is s (independent of the original chain) such that $k(s)$ is defined, and $r \leqslant s$ for all entries $S'_{k(r)}$ in (25). Hence also, by (24), if S'_r is an entry in (25), then $r \leqslant k(s) \leqslant g^s(0)$.

Now suppose S'_r is the final entry in (25). Then $S''_r \subset S''_{r+1} \subset \cdots$; but if S''_t is an entry of this chain, then $t < f(r) \leqslant g(r) \leqslant g(g^s(0)) = g^{s+1}(0)$.

Therefore the original chain in (22) has a final entry S_t, where $t < g^{s+1}(0)$. Thus the claim holds when $n = \ell + 1$. By induction, the claim holds for all positive n, provided it holds when $n = 1$.

It remains to show that, for all positive m, for all sequences $(a_i : i \in \omega)$, there is a bound on the length of chains

$$S_0 \subset S_1 \subset S_2 \subset \cdots \quad (26)$$

of antichains S_k of $(\omega^m, \leqslant)$, where $S_k \subseteq \{\xi : |\xi| \leqslant a_k\}$. The claim is trivially true when $m = 1$. Suppose it is true when $m = \ell$. Now let $m = \ell + 1$, and suppose we have a chain as in (26). We may assume that S_0 contains some σ. If $i < m$ and $j \in \omega$, let

$$S_k^{i,j} = \{\xi \in S_k : \xi(i) = j\}.$$

Then the inductive hypothesis applies to chains of the form

$$S_{k(0)}^{i,j} \subset S_{k(1)}^{i,j} \subset S_{k(2)}^{i,j} \subset \cdots .$$

Moreover, if $\tau \in S_k$, then $\tau(i) \leqslant \sigma(i)$ for some i in m (since σ is also in S_k, and this is an antichain). Hence

$$S_k = \bigcup_{i<m} \bigcup_{j \leqslant \sigma(i)} S_k^{i,j},$$

a union of no more than $|\sigma| + m$-many sets, hence no more than $a_0 + m$-many sets. So the proof can proceed as in the reduction to $n = 1$: for each k such that S_{k+1} exists, one of the inclusions $S_k^{i,j} \subseteq S_{k+1}^{i,j}$ is strict, and so forth. ⊣

thm:s

Theorem 4.10. *Suppose m, r, and n are positive integers. Then there is a positive integer s, where $r \leqslant s$, such that, if $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, and $K(a_h^\xi : |\xi| \leqslant s \;\&\; h < n)$ meets the differential condition, then $(K, \partial_0, \dots, \partial_{m-1})$ has an extension that is compatible with $K(a_h^\xi : |\xi| \leqslant r \;\&\; h < n)$.*

Proof. Suppose $K(a_h^\xi : |\xi| \leqslant 2^t r \;\&\; h < n)$ meets the differential condition for some t. When $u \leqslant t$, let $K_u = K(a_h^\xi : |\xi| \leqslant 2^u r \;\&\; h < n)$, and let S_u be the set of minimal separable leaders of K_u. Then we have an increasing chain $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_t$. By the preceding lemma, there is a value of t, depending only on m, r, and n, large enough that this chain cannot be strictly increasing. Then $S_u = S_{u+1}$ for some u less than this t. Then K_{u+1} satisfies the hypothesis of Theorem 4.3. So $(K, \partial_0, \dots, \partial_{m-1})$ has an extension compatible with $K(a_h^\xi\colon |\xi| < 2^{u+1}r \;\&\; h < n)$, and *a fortiori* with $K(a_h^\xi\colon |\xi| \leqslant r \;\&\; h < n)$. In short, the desired s is $2^t r$. ⊣

This theorem yields yet another first-order characterization of the models of m-DCF.

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